

# A Computational Solution of MHD Stokes Problem for a Heat Generating Fluid with Hall Currents

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**Abstract** In this paper, the finite element method is employed in order to study the MHD free convection heat fluid flow past an impulsively started vertical infinite plate when a strong magnetic field is imposed in a plane which makes an angle  $\alpha$  with the normal to the plate. Numerical calculations have been carried out on the velocity and temperature distributions which are depicted graphically. The results obtained are discussed extensively in terms of the different entering into the problem.

**Keywords** Computer Program, Finite Element Method, Hall Currents, Modeling, MHD

## 1. Introduction

In recent years, considerable interest has been given to the theory of rotating fluids due to its application in cosmic and geophysical sciences. In an ionized gas where the density is low and/or the magnetic field is very strong, the conductivity normal to the magnetic field is reduced due to the free spiraling of electrons and ions about the magnetic lines of force before suffering collisions and a current is induced in a direction normal to both the electric and magnetic fields. The effect of Hall currents on the Stokes problem has provided a new dimension in its application to MHD power generators and extended problem of ionized plasma flow in the presence of special configuration of magnetic field. One of the most important applications of this problem is in the study of coronal plasma flow in the configuration of plasma sheet formation in the active region of the sun or in the magnetic tail region. The Stokes problem under transversely applied magnetic field was studied by Rossow[1] whereas the corresponding MHD problem for a vertical impulsively started infinite plate was studied by Soundalgekar et al[2]. When the gas or fluid is partially ionized, the Hall current becomes significant and plays an important role in the development of the flow field. Although many improvements have been made in tackling this problem, we have chosen a computational solution based on the finite element analysis. Pop[3] studied the effect of Hall currents on hydromagnetic flow near an accelerated plate when the magnetic field is imposed normal to the plant. The

problem of Hall current has also been considered by a number of authors[4-7]. It is now proposed to study, by the finite element method, the MHD free convection of a heat generating fluid flow past an impulsively started vertical plate taking into account the Hall currents when a strong magnetic field is imposed in a plane which makes an angle  $\alpha$  with the normal to the plate.

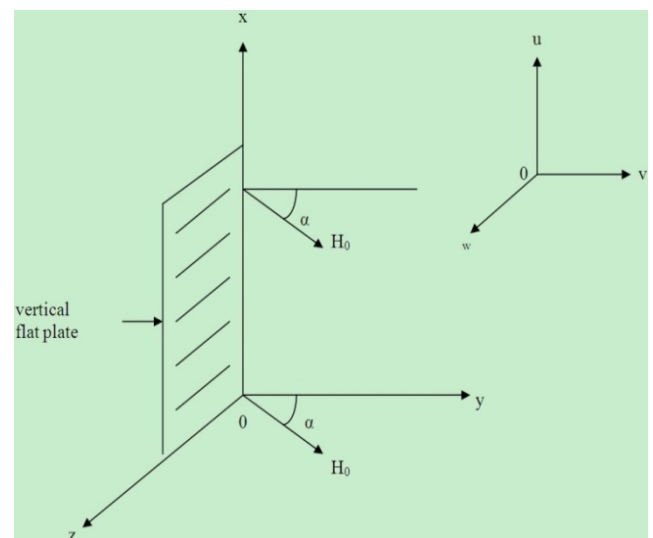


Figure 1. The flow configuration with the coordinate system used

## 2. Mathematical Analysis

Consider the MHD free-convection heat generating fluid flow past an infinite vertical plate started impulsively in its own plane. The  $x'$ -axis is taken along the plate in the vertically upward direction and the  $y'$ -axis is taken normal to the plate. Initially, the temperatures of the plate and the fluid are assumed to be the same. At time  $t' > 0$ , the plate starts

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moving impulsively in its own plane with velocity  $U_0$  and its temperature is instantaneously raised or lowered to  $T_w'$  which is maintained at a constant value. The fluid is permeated by a strong magnetic field  $H = (0, \lambda H_0, \sqrt{(1-\lambda^2)H_0})$ , where  $\lambda = \cos\alpha$ ,  $\alpha$  being the angle made by  $H$  with the normal to the plate. The flow configuration, together with the coordinate system used, is shown in Figure 1.

Since the plate is infinite in length, all variables are functions of  $y'$  and  $t'$  only. It is assumed that the induced magnetic field is negligible. This assumption is justified when the magnetic Reynolds number is very small (Shercliff,[8]).

The equation of conservation of electric charge  $\nabla \cdot J = 0$  gives  $j_y' = \text{constant}$  where  $J = (j_x', j_y', j_z')$ . This constant is zero since  $j_y' = 0$  at the plate, which is electrically non-conducting. Thus,  $j_y' = 0$  everywhere in the flow.

Next, we consider the case of a short circuit problem in which the applied electrical field  $E = 0$ . When the strength of the magnetic field is very large, the generalized Ohm's law in the absence of an electric field (Cowling,[9]) is given by:

$$J + \frac{\omega_e \tau_e}{H_0} J \times H = \sigma \left( \mu_e Q \times H + \frac{1}{en_e} \nabla p_e \right) \quad (1)$$

where  $\sigma$ ,  $\mu_e$ ,  $\omega_e$ ,  $\tau_e$ ,  $e$ ,  $n_e$ ,  $P_e$  and  $Q$  are defined in nomenclature. Neglecting the electron pressure, the thermoelectric pressure and the ion slip, we have:

$$j_x' = \frac{\sigma \mu_e H_0 \lambda}{1 + m^2 \lambda^2} (m \lambda u' - w') \quad (2)$$

$$j_z' = \frac{\sigma \mu_e H_0 \lambda}{1 + m^2 \lambda^2} (u' + m \lambda w') \quad (3)$$

where  $m = \omega_e \tau_e$  is the Hall parameter.

By applying the usual boundary layer approximation, the flow is governed by the following equations:

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial y'^2} + g \beta (T' - T_\infty') - \frac{\sigma \mu_e^2 H_0^2 \lambda}{\rho (1 + m^2 \lambda^2)} (u' + m \lambda w') \quad (4)$$

$$\frac{\partial w'}{\partial t'} = \nu \frac{\partial^2 w'}{\partial y'^2} + \frac{\sigma \mu_e^2 H_0^2 \lambda}{\rho (1 + m^2 \lambda^2)} (m \lambda u' - w') \quad (5)$$

$$\frac{\partial T'}{\partial t'} = \frac{k}{\rho C_p} \frac{\partial^2 T'}{\partial y'^2} + Q^* + \frac{\nu}{C_p} \left( \left( \frac{\partial u'}{\partial y'} \right)^2 + \left( \frac{\partial w'}{\partial y'} \right)^2 \right) \quad (6)$$

In equations (4)-(6), the inertia terms are neglected as we are considering only fully developed flow. Hence, our results are true for short times after the motion and temperature jump at the wall.

The initial boundary conditions of the problem are given by:

$$\begin{aligned} t' \leq 0: u' = 0, w' = 0, T' = T_\infty' \quad \text{everywhere} \\ t' > 0: u' = U_0, w' = 0, T' = T_w' \quad \text{at } y' = 0 \\ u' \rightarrow 0, w' \rightarrow 0, T' \rightarrow T_\infty' \quad \text{as } y' \rightarrow \infty \end{aligned} \quad (7)$$

We now introduce the following non-dimensional quantities as defined in the nomenclature:

$$\begin{aligned} t = \frac{t' U_0^2}{\nu}, \quad y = \frac{y' U_0}{\nu}, \quad u = \frac{u'}{U_0}, \quad w = \frac{w'}{U_0}, \quad \theta = \frac{T' - T_\infty'}{T_w' - T_\infty'} \\ \text{Pr} = \frac{\mu C_p}{k}, \quad \text{Gr} = \frac{\nu g \beta (T_w' - T_\infty')}{U_0^3}, \quad M^2 = \frac{\sigma \mu_e^2 H_0^2 \nu}{\rho U_0^2} \quad (8) \\ \delta = \frac{Q \nu^2}{k U_0^2}, \quad E = \frac{U_0^2}{C_p (T_w' - T_\infty')}, \quad M_1 = \frac{M^2 \lambda^2 (1 - i m \lambda)}{1 + m^2 \lambda^2} \end{aligned}$$

Using the above formulation, equations (4) – (6) reduce to :

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial y^2} - M_1 q + \text{Gr} \theta \quad (9)$$

$$\text{Pr} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} + \text{Pr} E \left( \frac{\partial q}{\partial y} \cdot \frac{\partial \bar{q}}{\partial y} \right) - \delta \theta \quad (10)$$

Where  $q = u + iw$  and  $\bar{q}$  denotes the complex conjugate of  $q$ .

The corresponding boundary conditions are:

$$\begin{cases} q(y, t) = 0, \quad \theta(y, t) = 0 & t \leq 0 \\ q(0, t) = 1, \quad \theta(0, t) = 1 & t > 0 \\ q(\infty, t) = 0, \quad \theta(\infty, t) = 0 & t > 0 \end{cases} \quad (11)$$

For simplicity in the calculations, we consider  $m=0$  and  $\lambda=1$  which reduce  $M_1$  to a real number  $M^2$ . In view of this, Equations (9) and (10) can be rewritten as:

$$\frac{\partial^2 u}{\partial y^2} - M_1 u + \text{Gr} \theta - \frac{\partial u}{\partial t} = 0 \quad (12)$$

$$\frac{\partial^2 w}{\partial y^2} - M_1 w - \frac{\partial w}{\partial t} = 0 \quad (13)$$

$$\frac{\partial^2 \theta}{\partial y^2} + \text{Pr} E \left( \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) - \delta \theta - \text{Pr} \frac{\partial \theta}{\partial t} = 0 \quad (14)$$

The above system of Equations (12 - 14) with boundary conditions (11) has been solved numerically by a computer program using the finite element method in steps 1, 2 and 3.

### Step 1:

We solve Equation (14) with the help of boundary conditions (11). Constructing the quasi - variational equivalent of Equation (14), we have:

$$0 = \int_{\Omega} \phi_i \left( \frac{\partial^2 \theta}{\partial y^2} + \text{Pr} E \left( \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) - \delta \theta - \text{Pr} \frac{\partial \theta}{\partial t} \right) dy \quad (15)$$

where  $\phi_i$  denotes the test function and  $\Omega$  is the domain of the flow.

Consider an  $N$  elements mesh and a two parameter (semi discrete) Galerkin approximation [10,11] of the form:

$$\theta(y, t) = \sum_{j=1}^N C_j(t) \phi_j(y) \quad (16)$$

with

$$\begin{cases} \phi_l^{(i)}(y) = \frac{(y_{i+l} - y)}{(y_{i+l} - y_i)} \\ \phi_2^{(i)}(y) = \frac{(y - y_i)}{(y_{i+l} - y_i)} \end{cases} \quad (17)$$

where  $i = 1, 2, 3, \dots, N$  and  $y_i$  and  $y_{i+1}$  are respectively the lower and upper coordinates of the element  $i$ .

Using Equations (16 - 17), Equation (15) reduces to:

$$[A] \left\{ \frac{\partial C}{\partial t} \right\} + [B] \{C\} = \{P\} \quad (18)$$

where

$$\begin{cases} A_{ij} = \int_{\Omega} \text{Pr} \phi_i \phi_j dy \\ B_{ij} = \int_{\Omega} \left( \frac{\partial \phi_i}{\partial y} \cdot \frac{\partial \phi_j}{\partial y} + \delta \phi_i \phi_j \right) dy \\ P_i = \text{Pr} E \int_{\Omega} \phi_i (u_y^2 + w_y^2) dy \end{cases} \quad (19)$$

Using the  $\Theta$  - family of approximation developed by Reddy [11], Equation (18) reduces to:

$$\left[ \hat{A} \right] \{C\}_{n+1} = \left[ \hat{B} \right] \{C\}_n + \left\{ \hat{P} \right\} \quad (20)$$

where

$$\begin{cases} \left[ \hat{A} \right] = [A] + \Theta \Delta t [B] \\ \left[ \hat{B} \right] = [A] - (1 - \Theta) \Delta t [B] \\ \left[ \hat{P} \right] = \Delta t [\Theta \{P\}_{n+1} + (1 - \Theta) \{P\}_n] \end{cases} \quad (21)$$

The initial value  $C_0$  is obtained by the Galerkin method from a 32 elements mesh and is given by:

$$C_0 = [1, 0, 0, \dots, 0]^T \quad (22)$$

For  $t > 0$ ,

$$\{C\}_{n+1} = \left[ \hat{A}^{-1} \right] \left[ \hat{B} \right] \{C\}_n + \left[ \hat{A}^{-1} \right] \left\{ \hat{P} \right\} \quad (23)$$

### Step 2:

We solve Equation (12) with the help of boundary conditions (11). Constructing the quasi - variational statement of Equation (12), we have:

$$0 = \int_{\Omega} \Psi_i \left\{ \frac{\partial^2 u}{\partial y^2} - M_1 u + Gr \theta - \frac{\partial u}{\partial t} \right\} dy \quad (24)$$

where  $\Psi_i$  is the test function and  $\Omega$  is the region of the flow.

Consider a two parameter (semi-discrete) Galerkin approximation [10,11] of the form:

$$u(y, t) = \sum_{j=1}^N d_j(t) \Psi_j(y) \quad (25)$$

with

$$\begin{cases} \Psi_1^{(i)}(y) = \frac{(y_{i+1} - y)}{(y_{i+1} - y_i)} \\ \Psi_2^{(i)}(y) = \frac{(y - y_i)}{(y_{i+1} - y_i)} \end{cases} \quad (26)$$

where  $i=1, 2, 3, \dots, N$  and  $y_i$  and  $y_{i+1}$  are respectively the lower and upper coordinates of the element  $i$ .

Using Equations (25-26), Equation (24) reduces to:

$$\left[ \hat{A} \right] \left\{ \frac{\partial d}{\partial t} \right\} + \left[ \hat{B} \right] \{d\} = \left\{ \hat{P} \right\} \quad (27)$$

where

$$\begin{cases} \bar{A}_{ij} = \int_{\Omega} \Psi_i \Psi_j dy \\ \bar{B}_{ij} = \int_{\Omega} \left( \frac{\partial \Psi_i}{\partial y} \cdot \frac{\partial \Psi_j}{\partial y} + M_1 \Psi_i \Psi_j \right) dy \\ \bar{P}_i = Gr \int_{\Omega} \Psi_i \theta dy \end{cases} \quad (28)$$

Using the  $\Theta$  - family of operators developed by Reddy [11], Equation (27) takes the form:

$$\{d\}_{n+1} = \left[ \hat{A}^{-1} \right] \left[ \hat{B} \right] \{d\}_n + \left[ \hat{A}^{-1} \right] \left\{ \hat{P} \right\} \quad (29)$$

where

$$d_0 = [1, 0, 0, \dots, 0]^T \quad (30)$$

### Step 3:

We solve Equation (13) with the help of boundary conditions (11). Constructing the quasi - variational statement of Equation (13), we have:

$$0 = \int_{\Omega} \phi_i \left\{ \frac{\partial^2 w}{\partial y^2} - M_1 w - \frac{\partial w}{\partial t} \right\} dy \quad (31)$$

where  $\phi_i$  is the test function and  $\Omega$  is the region of the flow.

Consider a two parameter (semi-discrete) Galerkin approximation [10,11] of the form:

$$w(y, t) = \sum_{j=1}^N f_j(t) \phi_j(y) \quad (32)$$

with

$$\begin{cases} \phi_1^{(i)}(y) = \frac{(y_{i+1} - y)}{(y_{i+1} - y_i)} \\ \phi_2^{(i)}(y) = \frac{(y - y_i)}{(y_{i+1} - y_i)} \end{cases} \quad (33)$$

where  $i=1, 2, 3, \dots, N$  and  $y_i$  and  $y_{i+1}$  are respectively the lower and upper coordinates of the element  $i$ .

Using Equations (32-33), Equation (31) reduces to:

$$\left[ \bar{A} \right] \left\{ \frac{\partial f}{\partial t} \right\} + \left[ \bar{B} \right] \{f\} = \left\{ \bar{P} \right\} \quad (34)$$

where

$$\begin{cases} \bar{A}_{ij} = \int_{\Omega} \phi_i \phi_j dy \\ \bar{B}_{ij} = \int_{\Omega} \left( \frac{\partial \phi_i}{\partial y} \cdot \frac{\partial \phi_j}{\partial y} + M_1 \phi_i \phi_j \right) dy \\ \bar{P}_i = 0 \end{cases} \quad (35)$$

Using the  $\Theta$  - family of operators developed by Reddy [11], Equation (34) takes the form:

$$\{f\}_{n+1} = \left[ \bar{A}^{-1} \right] \left[ \bar{B} \right] \{f\}_n + \left[ \bar{A}^{-1} \right] \left\{ \bar{P} \right\} \quad (36)$$

where

$$f_0 = [0, 0, 0, \dots, 0]^T \quad (37)$$

The numerical values of the temperature and velocity fields have been computed from Equations (23), (29) and

(36) and are shown in Figures.2-5.

### 3. Discussion of Results

To study the behavior of the velocity and temperature profiles, curves are drawn for various values of the parameters that describe the flow. Numerical calculations have been carried out for the velocity and temperature distributions for  $m=0$  and  $\lambda=1$ , reducing  $M_1$  to  $M^2$ . The results obtained are displayed in Figures 2-5. The method used is unconditionally stable and independent of the time step  $\Delta t$ . The velocity profiles are examined for the cases  $Gr > 0$  and  $Gr < 0$ .  $Gr > 0$  ( $= +5$ ) is used for the case when the flow is in the presence of cooling of the plate by free convection currents.  $Gr < 0$  ( $= -5$ ) is used for the case when the flow is in the presence of heating of the plate by free convection currents. Figures 2-4 show the velocity distribution for the two cases from which we observe that the velocity ( $u$ ) decreases away from the plate.

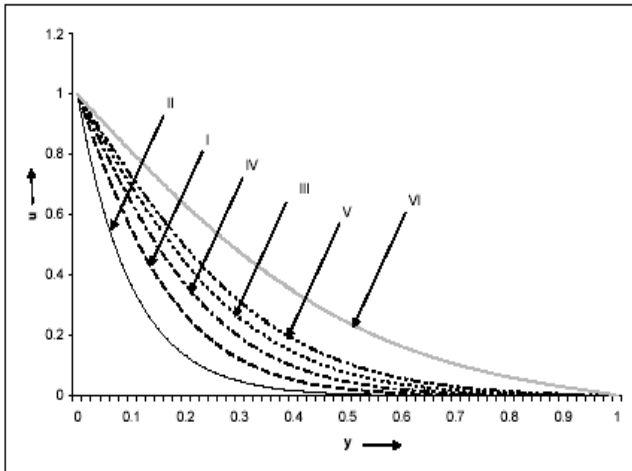


Figure 2. Primary velocity ( $u$ ) distribution for  $Gr = 5$

Series	M	Pr	E	$\delta$	t
I	2.00	0.73	0.10	0.20	0.002
II	5.00	0.73	0.10	0.20	0.002
III	2.00	1.50	0.10	0.20	0.002
IV	2.00	0.73	0.50	0.20	0.002
V	2.00	0.73	0.10	0.50	0.002
VI	2.00	0.73	0.10	0.20	0.005

From Figure 2, for the case when  $Gr > 0$  (in the presence of cooling of the plate by free convection currents), we observe that:

- The primary velocity ( $u$ ) decreases due to an increase in the magnetic parameter  $M$ .
- The primary velocity ( $u$ ) increases due to an increase in the Prandtl number  $Pr$ , the Eckert number  $E$ , the heat source parameter  $\delta$  and the time  $t$ .

From Figure 3, for the case when  $Gr < 0$  (in the presence of heating of the plate by free convection currents), we observe that:

- The primary velocity ( $u$ ) decreases due to an increase in the magnetic parameter  $M$ , the Prandtl number  $Pr$ , the

Eckert number  $E$  and the heat source parameter  $\delta$ .

- The primary velocity ( $u$ ) increases due to an increase in the time  $t$ .

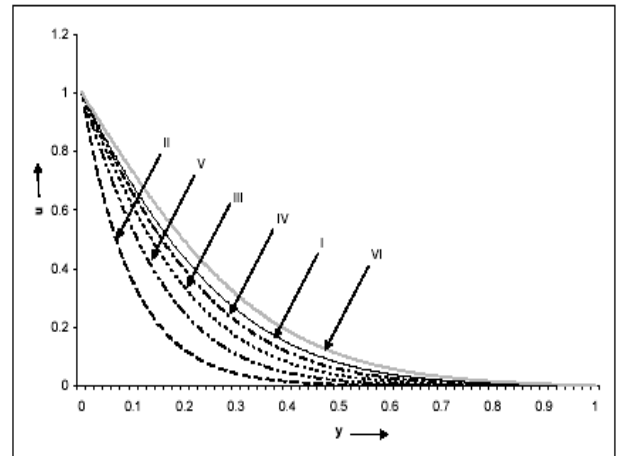


Figure 3. Primary velocity ( $u$ ) distribution for  $Gr = -5$

Series	M	Pr	E	$\delta$	t
I	2.00	0.73	0.10	0.20	0.002
II	5.00	0.73	0.10	0.20	0.002
III	2.00	1.50	0.10	0.20	0.002
IV	2.00	0.73	0.50	0.20	0.002
V	2.00	0.73	0.10	0.50	0.002
VI	2.00	0.73	0.10	0.20	0.005

From Figure 4, for the case when  $Gr = \pm 5$ , we observe that:

- The secondary velocity ( $w$ ) decreases due to an increase in the magnetic parameter  $M$  and the time parameter  $t$ .
- There is an insignificant change in the secondary velocity ( $w$ ) due to an increase in the Prandtl number  $Pr$ , the Eckert number  $E$  and the heat source parameter  $\delta$ .

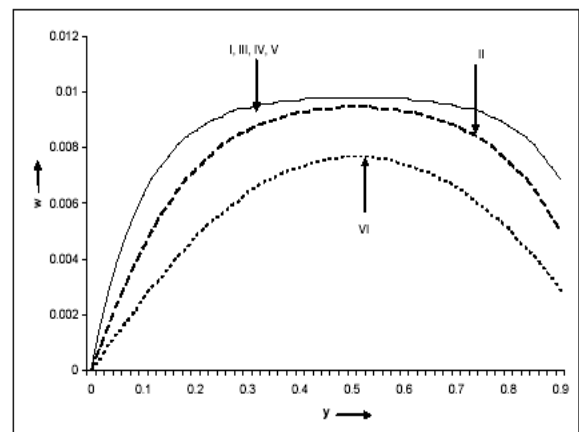
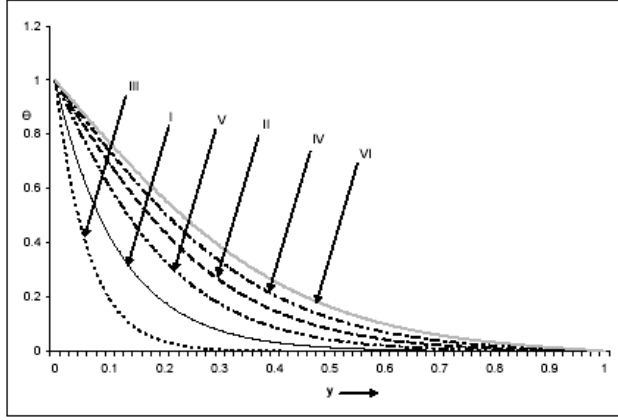


Figure 4. Secondary velocity ( $w$ ) distribution for  $Gr = \pm 5$

Series	M	Pr	E	$\delta$	t
I	2.00	0.73	0.10	0.20	0.002
II	5.00	0.73	0.10	0.20	0.002
III	2.00	1.50	0.10	0.20	0.002
IV	2.00	0.73	0.50	0.20	0.002
V	2.00	0.73	0.10	0.50	0.002
VI	2.00	0.73	0.10	0.20	0.005



**Figure 5.** Temperature ( $\Theta$ ) distribution for  $Gr = \pm 5$

Series	M	Pr	E	$\delta$	t
I	2.00	0.73	0.10	0.20	0.002
II	5.00	0.73	0.10	0.20	0.002
III	2.00	1.50	0.10	0.20	0.002
IV	2.00	0.73	0.50	0.20	0.002
V	2.00	0.73	0.10	0.50	0.002
VI	2.00	0.73	0.10	0.20	0.005

From Figure 5, for the case when  $Gr = \pm 5$ , we observe that:

- The temperature ( $\Theta$ ) decreases far away from the plate. The decrease is greater for a Newtonian fluid than it is for a non-Newtonian fluid ( $\Theta$  decreases with  $Pr$ ).
- There is an increase in the temperature profile ( $\Theta$ ) due to an increase in the magnetic parameter  $M$ , the Eckert number  $E$ , the heat source parameter  $\delta$  and the time parameter  $t$ .

## Nomenclature

$g$	the gravitational acceleration
$\beta$	the volumetric coefficient of thermal expansion
$\nu$	the kinematic coefficient of viscosity of the fluid
$k$	the thermal conductivity
$\rho$	the density of the fluid
$Q^*$	the heat generation of the type $Q(T_\infty' - T')$
$C_p$	the specific heat at constant pressure
$T$	the temperature in the boundary layer
$T_\infty'$	the temperature in the free stream
$Pr$	the Prandtl number
$Gr$	the Grashof number
$M^2$	the magnetic field parameter
$\delta$	the heat source parameter
$\sigma$	the electric conductivity
$\mu_e$	the magnetic permeability
$\omega_e$	the cyclotron frequency

$\tau_e$	the electron collision time
$e$	the electric charge
$n_e$	the number density of electrons
$P_e$	the electron pressure
$Q$	the velocity of the fluid
$u$	the non-dimensional primary velocity
$w$	the non-dimensional secondary velocity
$\Theta$	the non-dimensional temperature

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