

# Exponential-Gamma Additive Failure Rate Model

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**Abstract** The addition of hazard functions of Exponential model and Gamma model with shape 2 is developed. The probability model is considered and an attempt is made to present the distributional properties, estimation of parameters and testing of hypothesis about the proposed model. The findings are described.

**Keywords** LFRD, EGAFRM, Percentiles

## 1. Introduction

In reliability studies, combinations of components forming series, parallel, k out of n systems are quite popular. The survival probabilities of such systems are evaluated either by the system as a whole or through the survival probabilities of the components that define the system. It is well known that in a series system of a finite number of components with independent life time random variables, the system reliability is equal to the product of the component reliabilities. If  $f(x)$ ,  $F(x)$ ,  $h(x)$  respectively indicate the failure density, failure probability, failure rate of a component with life time random variable 'X', then we know that the reliability is given by

$$R(x) = 1 - F(x) = \frac{\int_0^x h(x) dx}{e^0}$$

If a series system has two components with independent but non-identical life patterns explained by two distinct random variables say  $X_1, X_2$ , with respective failure densities, failure probabilities, failure rates as  $f_1(x), f_2(x); F_1(x), F_2(x); h_1(x), h_2(x)$  then the system reliability is given by

$$R(x) = e^{-\int_0^x [h_1(x) + h_2(x)] dx} \quad (1.1)$$

From the above expression we can get the failure density and failure rate of the series system whose reliability is given by (1.1). Such models are already studied in the past with different choices of  $h_1(x)$  and  $h_2(x)$ . One such situation is the popular linear failure rate distribution [LFRD]. In this model  $h_1(x)$  is taken as a constant failure rate model,  $h_2(x)$  is

taken as an increasing failure rate (IFR) model with specific choices of exponential for  $h_1(x)$  and Weibull with shape 2 for  $h_2(x)$ . The failure density, the cumulative distribution function, the reliability and the failure rate of LFRD model are given by

$$f(x; \alpha, \beta) = (\alpha + \beta x) \exp\left\{-\alpha x - \frac{\beta}{2} x^2\right\} \quad (1.2)$$

$$F(x; \alpha, \beta) = 1 - \exp\left\{-\alpha x - \frac{\beta}{2} x^2\right\} \quad (1.3)$$

$$x > 0, \alpha, \beta > 0$$

$$\bar{F}(x; \alpha, \beta) = \exp\left\{-\alpha x - \frac{\beta}{2} x^2\right\} \quad (1.4)$$

$$h(x; \alpha, \beta) = \alpha + \beta x \quad (1.5)$$

A hazard rate given in (1.5) is in the form of a straight line equation justifying the name "Linear Failure Rate" for this distribution. A number of researchers made an extensive study on LFRD model. Some recent works in this regard are Bain (1974), Balakrishnan and Malik (1986), Ananda Sen and Bhattacharya (1995), Mohie El-Din *et al.* (1997), Mahmoud *et al.* (2006), M.E. Ghitany and Kotz (2007), ABD EL -Baset A. Ahmad (2008), Khedhairi (2008), Sarhan and Zaindin (2009), Sarhan and Kundu (2009), Mahmoud and Al-Nagar (2009), Mazen and Zaindin (2010). Kantam and Priya (2011) have worked out an additive life testing model combining a CFR and DFR model with DFR generated from a Weibull model of shape parameter  $< 1$ . The rest of the paper is organized as follows:

The distributional properties of our proposed model are given in Section 2. The ML estimations is discussed in Section 3. Discrimination of our model from exponential using likelihood ratio criterion is given in Section 4. Summary and Conclusions are given in Section 5.

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## 2. Distributional Properties

we consider the hazard function of the exponential distribution with parameter ' $\lambda$ ' and a gamma distribution with shape parameter 2 and scale parameter  $v$ . Then their respective failure rate functions are

$$\text{i.e. } h_1(x) = \lambda \quad \forall x, \quad x > 0 \quad (2.1)$$

$$h_2(x) = \frac{v^2 x}{1 + vx} \quad (2.2)$$

The corresponding reliabilities are

$$e^{-\lambda x} \text{ and } e^{-vx} (1 + vx).$$

The series system reliability is

$$R(x) = e^{-x(\lambda+v)} (1 + vx), \quad x > 0, \quad \lambda, v > 0 \quad (2.3)$$

We consider the failure density corresponding to (2.3) as our exponential gamma additive failure rate model (EGA FRM). The probability density function, the CDF, failure rate of EGA FRM are respectively given by

$$f(x) = e^{-x(\lambda+v)} [\lambda(1 + vx) + v^2 x], \quad x > 0; \quad \lambda, v > 0 \quad (2.4)$$

$$F(x) = 1 - e^{-x(\lambda+v)} (1 + vx), \quad x > 0; \quad \lambda, v > 0 \quad (2.5)$$

$$h(x) = \lambda + \frac{v^2 x}{1 + vx}, \quad x > 0; \quad \lambda, v > 0 \quad (2.6)$$

The mean, median, mode and variance of EGA FRM are

$$\text{Mean} = E(x) = \frac{\lambda + 2v}{(\lambda + v)^2} \quad (2.7)$$

The median is calculated by taking

$$F(x) = \frac{1}{2} \text{ and is given by } x=0.5731 \quad (2.8)$$

$$\text{Mode} = x = \frac{v(\lambda + v)^2 - \lambda}{(\lambda + 1)v} \quad (2.9)$$

The variance is given by

$$v(x) = \frac{\lambda^2 + 4\lambda v + 2v^2}{(\lambda + v)^4} \quad (2.10)$$

The central moments and skewness of EGA FRM are given by

$$\mu_2 = \frac{\lambda^2 + 4\lambda v + 2v^2}{(\lambda + v)^4} \quad (2.11)$$

$$\mu_3 = \frac{2}{(\lambda + v)^6} [2(\lambda + v)^3 - \lambda^3] \quad (2.12)$$

$$\beta_1 = \frac{4[2(\lambda + v)^3 - \lambda^3]^2}{(\lambda^2 + 4\lambda v + 2v^2)^3} \quad (2.13)$$

## 3. Maximum Likelihood Estimation

Let  $x_1, x_2, \dots, x_n$  is a random sample of size ' $n$ ' drawn from the EGA FRM with pdf  $f(x; \lambda, v)$  then the likelihood function is given by

$$L = \prod_{i=1}^n f(x_i; \lambda, v) \quad (3.1)$$

$$\Rightarrow L = \prod_{i=1}^n e^{-x_i(\lambda+v)} [\lambda(1 + vx_i) + v^2 x_i] \quad (3.2)$$

$$\Rightarrow \log L = \sum_{i=1}^n [-x_i(\lambda + v) + \log [\lambda(1 + vx_i) + v^2 x_i]] \quad (3.3)$$

The MLEs of  $\lambda, v$  can be obtained by solving the following likelihood equations

$$\begin{aligned} \frac{\partial \log L}{\partial \lambda} &= 0 \\ \Rightarrow \sum_{i=1}^n \left\{ -x_i + \frac{1 + vx_i}{\lambda(1 + vx_i) + v^2 x_i} \right\} &= 0 \end{aligned} \quad (3.4)$$

and

$$\frac{\partial \log L}{\partial v} = 0 \Rightarrow \sum_{i=1}^n \left\{ \frac{2vx_i - vx_i^2(\lambda + v)}{\lambda(1 + vx_i) + v^2 x_i} \right\} = 0 \quad (3.5)$$

The equations (3.4) and (3.5) have to be solved through iteration only with some well known numerical methods and get the MLEs of  $\lambda$  and  $v$  say  $\hat{\lambda}^*$  and  $\hat{v}^*$  respectively. However, by using a simple successive method, the ML equations (3.4) and (3.5) can be further simplified and get the following estimators (not ML estimators) for  $\lambda, v$  say  $\hat{\lambda}, \hat{v}$  are obtained

$$\hat{\lambda} = \frac{n + \sum_{i=1}^n x_i - \sum_{i=1}^n x_i^2}{3 \sum_{i=1}^n x_i - \sum_{i=1}^n x_i^2} \quad (3.6)$$

$$\hat{v} = \frac{2 \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2} - 1 \quad (3.7)$$

Accordingly the exact variances of the MLEs are not mathematically tractable. However, the asymptotic variance, covariance of the estimates of the parameters are obtained using the following elements of the information matrix:

$$I_{11} = -E \left( \frac{\partial^2 \log L}{\partial \lambda^2} \right) \\ = -E \left[ \sum_{i=1}^n \left\{ \frac{-(1 + vx_i)^2}{(\lambda(1 + vx_i) + v^2 x_i)^2} \right\} \right] \quad (3.8)$$

$$I_{12} = I_{21} = -E \left( \frac{\partial^2 \log L}{\partial \lambda \partial v} \right) \\ = -E \left[ \sum_{i=1}^n \left\{ \frac{3v^2 x_i^2 + 2vx_i}{(\lambda(1 + vx_i) + v^2 x_i)^2} \right\} \right] \quad (3.9)$$

$$I_{22} = -E \left( \frac{\partial^2 \log L}{\partial v^2} \right) \\ = -E \left[ \sum_{i=1}^n \left\{ \frac{(\lambda x_i + 2vx_i)^2 - (\lambda(1 + vx_i) + v^2 x_i) 2x_i}{(\lambda(1 + vx_i) + v^2 x_i)^2} \right\} \right] \quad (3.10)$$

The estimated information matrix elements are

$$\hat{I}_{11} = - \frac{\partial^2 \log L}{\partial \lambda^2} \Big|_{\lambda=\lambda^*} \\ \hat{I}_{12} = \hat{I}_{21} = - \frac{\partial^2 \log L}{\partial \lambda \partial v} \Big|_{\lambda=\lambda^*, v=v^*} \\ \hat{I}_{22} = - \frac{\partial^2 \log L}{\partial v^2} \Big|_{v=v^*}$$

The estimated asymptotic dispersion matrix of the MLEs

$$\text{is given by the inverse of } \begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix}$$

#### 4. Discrimination between EGAFRM and Exponential Model

We know that the exponential distribution is having a number of preferable properties to be handled for problems of statistical inference. We, therefore are interested in assessing whether exponential distribution is an alternative to our model. In other words given a sample we are interested in studying whether the sample clearly discriminates between our model from that of exponential. Let us designate our distribution EGAFRM as a null population say  $P_0$ . We call exponential distribution as alternate population say  $P_1$ . We propose a null hypothesis  $H_0$ : “A given sample belongs

to the population  $P_0$ ” against an alternative hypothesis  $H_0$  : “the sample belongs to population  $P_1$ ”.

Consider a sample from  $P_0$ . Let  $L_1$ ,  $L_0$  respectively stand for the likelihood function of the sample with population  $P_1$  and  $P_0$ . Both  $L_1$  and  $L_0$  contain the respective parameters of the population. The given sample is used to get the parameters of  $P_1$ ,  $P_0$ , so that for the given sample the value

of  $\frac{L_1}{L_0}$  is now estimated. If  $H_0$  is true,  $\frac{L_1}{L_0}$  must be small,

therefore for accepting  $H_0$  with a given degree of confidence

$\frac{L_1}{L_0}$  is compared with a critical value with the help of the

percentiles in the sampling distribution of  $\frac{L_1}{L_0}$ . But the

sampling distribution of  $\frac{L_1}{L_0}$  is not analytical, we therefore

resorted to the empirical sampling distribution through simulation. We have generated random samples of size 5(1)10, 15, 20, 25, 30 from the population  $P_0$  with various parameter combinations ( $\lambda=1,2; v=1,2$ ) and got the value of  $L_1$ ,  $L_0$  along with the estimates of respective parameters for

each sample. The percentiles of  $\frac{L_1}{L_0}$  at various

probabilities are computed and are given in Table 4.1.

In testing of hypothesis we know that the power of a test statistic is the complementary probability of accepting a false hypothesis at a given level of significance. Let us conventionally fix 5% level of significance. So that the percentiles in Table 4.1 under the column 0.05 shall become the critical values. We generate a random sample of sizes 5(1)10,15,20,25, 30 from the population  $P_1$  namely exponential. At this sample we find the estimates of the parameters of  $P_1$  and  $P_0$  using the respective probability models. Accordingly we got the estimates of  $L_1$ ,  $L_0$  for the sample from  $P_1$ .

Over repeated simulation runs we got the proportion of

values of  $\frac{L_1}{L_0}$  that fall below the respective critical values

of Table 4.1. These proportions would give the value of  $\beta$ , the probability of type II error namely  $\beta$  so that the power  $1 - \beta$  would be the power. Various power values are given in Table 4.2. We conclude that as long as  $n$  is less than 10, a given sample can distinguish between the populations  $P_1$  and  $P_0$  only with a probability of 10%. Hence exponential can be a reasonable alternative to our model in small samples.

**Table 4.1.** Percentiles of  $L_1/L_0$  for various values of  $\lambda$  and  $\nu$ 

$n$	$\lambda=1, \nu=1$					
	<b>0.990</b>	<b>0.975</b>	<b>0.950</b>	<b>0.050</b>	<b>0.025</b>	<b>0.010</b>
5	13.9964	<b>7.4704</b>	<b>3.2650</b>	0.2737	0.1163	0.0038
6	21.6020	<b>6.4687</b>	<b>3.7101</b>	0.2128	0.1159	0.0027
7	66.5826	<b>9.3778</b>	<b>4.9507</b>	0.1719	0.1146	0.0025
8	75.0588	<b>10.7142</b>	<b>5.3353</b>	0.1362	0.1121	0.0021
9	79.2645	<b>16.3024</b>	<b>7.3426</b>	0.1117	0.0928	0.0020
10	83.4073	<b>24.1984</b>	<b>9.2781</b>	0.0891	0.0813	0.0018
15	64.0254	<b>37.2427</b>	<b>20.0810</b>	0.0381	0.0299	0.0009
20	209.2082	<b>68.3498</b>	<b>42.0277</b>	0.0284	0.0168	0.0005
25	322.0724	<b>162.9943</b>	<b>98.5263</b>	0.0256	0.0110	0.0003
30	1048.1422	<b>389.7236</b>	<b>210.8025</b>	0.0244	0.0086	0.0002
$\lambda=1, \nu=2$						
$n$	<b>0.990</b>	<b>0.975</b>	<b>0.950</b>	<b>0.050</b>	<b>0.025</b>	<b>0.010</b>
5	3.4394	<b>2.3524</b>	<b>2.0234</b>	0.2735	0.2688	0.2652
6	3.7813	<b>2.8412</b>	<b>2.4287</b>	0.2131	0.2086	0.2054
7	4.5536	<b>3.2306</b>	<b>2.8194</b>	0.1700	0.1652	0.1609
8	4.9990	<b>3.8551</b>	<b>3.3099</b>	0.1384	0.1302	0.1271
9	5.0829	<b>4.4550</b>	<b>3.9317</b>	0.1092	0.1029	0.0989
10	5.8007	<b>5.1577</b>	<b>4.4141</b>	0.0876	0.0812	0.0781
15	11.9528	<b>10.4432</b>	<b>8.3963</b>	0.0338	0.0281	0.0252
20	24.4373	<b>19.7675</b>	<b>14.1949</b>	0.0184	0.0131	0.0093
25	47.2504	<b>36.8431</b>	<b>25.7201</b>	0.0113	0.0059	0.0034
30	99.4277	<b>72.7438</b>	<b>50.0531</b>	0.0074	0.0033	0.0013
$\lambda=2, \nu=1$						
$n$	<b>0.990</b>	<b>0.975</b>	<b>0.950</b>	<b>0.050</b>	<b>0.025</b>	<b>0.010</b>
5	2.7616	<b>2.4455</b>	<b>2.2853</b>	0.2719	0.2659	0.2628
6	3.1786	<b>2.8940</b>	<b>2.7845</b>	0.2127	0.2074	0.2023
7	3.7943	<b>3.4898</b>	<b>3.2923</b>	0.1713	0.1644	0.1593
8	4.3771	<b>4.1062</b>	<b>3.8875</b>	0.1408	0.1313	0.1257
9	5.2401	<b>4.9137</b>	<b>4.5738</b>	0.1126	0.1045	0.0984
10	6.2404	<b>5.8552</b>	<b>5.5040</b>	0.0897	0.0833	0.0775
15	15.9772	<b>14.2342</b>	<b>12.9624</b>	0.0397	0.0299	0.0265
20	38.3442	<b>33.7745</b>	<b>29.4773</b>	0.0278	0.0167	0.0108
25	97.3105	<b>79.5516</b>	<b>67.9939</b>	0.0251	0.0096	0.0043
30	251.6332	<b>197.9550</b>	<b>163.6982</b>	0.0238	0.0076	0.0019
$\lambda=2, \nu=2$						
	<b>0.990</b>	<b>0.975</b>	<b>0.950</b>	<b>0.050</b>	<b>0.025</b>	<b>0.010</b>
5	2.4016	<b>2.2721</b>	<b>2.0702</b>	0.2702	0.2655	0.2626
6	2.9761	<b>2.8032</b>	<b>2.5752</b>	0.2114	0.2064	0.2016
7	3.5305	<b>3.3076</b>	<b>3.0460</b>	0.1682	0.1633	0.1588
8	4.2378	<b>3.9290</b>	<b>3.5972</b>	0.1367	0.1278	0.1235
9	5.2667	<b>4.8739</b>	<b>4.3043</b>	0.1080	0.1023	0.0966
10	6.0044	<b>5.5127</b>	<b>5.0037</b>	0.0874	0.0801	0.0760
15	15.3850	<b>13.5629</b>	<b>11.0645</b>	0.0350	0.0272	0.0248
20	36.2126	<b>30.4878</b>	<b>22.2955</b>	0.0195	0.0137	0.0093
25	84.5300	<b>66.8563</b>	<b>46.9586</b>	0.0141	0.0066	0.0035
30	211.5102	<b>152.7887</b>	<b>99.0498</b>	0.0110	0.0043	0.0014

**Table 4.2.** Power of Likelihood Ratio criterion

	$\lambda=1, v=1$	
	<b>0.975</b>	<b>0.950</b>
5	0.9710	0.7940
6	0.9740	0.6860
7	0.9750	0.8380
8	0.9760	0.8160
9	0.9770	0.8810
10	0.9780	0.9100
15	0.9800	0.9270
20	0.9820	0.9630
25	0.9840	0.9770
30	0.9870	0.9850
$\lambda=1, v=2$		
<b>n</b>	<b>0.975</b>	<b>0.950</b>
5	0.5560	0.4940
6	0.6620	0.5610
7	0.6430	0.5980
8	0.7030	0.6540
9	0.7250	0.6910
10	0.7630	0.7270
15	0.8240	0.8010
20	0.8780	0.8440
25	0.8940	0.8640
30	0.9200	0.8940
$\lambda=2, v=1$		
<b>n</b>	<b>0.975</b>	<b>0.950</b>
5	0.6150	0.5890
6	0.6750	0.6660
7	0.7050	0.6900
8	0.7680	0.7510
9	0.7860	0.7720
10	0.8240	0.8140
15	0.9090	0.9000
20	0.9490	0.9400
25	0.9690	0.9670
30	0.9820	0.9790
$\lambda=2, v=2$		
<b>n</b>	<b>0.975</b>	<b>0.950</b>
5	0.5600	0.5260
6	0.6360	0.6000
7	0.6660	0.6460
8	0.7250	0.7050
9	0.7640	0.7320
10	0.7850	0.7720
15	0.8870	0.8660
20	0.9200	0.9030
25	0.9470	0.9300
30	0.9670	0.9520

## 5. Summary and Conclusions

A combination of Exponential model and Gamma model is developed on lines of the well known linear failure rate model. Estimating equations by ML method are also derived. Its validity as a specified model in the presence of a simpler Exponential model as an alternative is established using a likelihood ratio criterion. In some situations the proposed model stood robust against Exponential.

## Acronyms

LFRD - Linear Failure Rate Distribution

EGAFRM - Exponential Gamma Additive Failure Rate Model

CFR - Constant Failure Rate

DFR - Decreasing Failure Rate

IFR - Increasing Failure Rate

MLE - Maximum Likelihood Estimator

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