

Estimation of Hill Growth Model Parameters by Linearization

Wonu Nduka^{1,*}, Biu Emmanuel Oyinebifun²

¹Ignatius Ajuru University of Education, Rivers State, Nigeria

²University of Port Harcourt, Choba, Rivers, Nigeria

Abstract This study is focused on the estimation of the Hill growth model parameters which is a non-linear model. In estimating these parameters, secondary data (amount of transmitted voltage against the time of different values) was used for illustration. Firstly, logarithm transformation was applied to Hill growth model to make it linear. Then, an iteration was used to run the linear regression model using Microsoft Excel Solver and Ordinary Least Square (OLS) estimates. The iterations were run using upper asymptote (or the initial parameter at $t = 0$) and the growth range parameter starting from $(-0.2, -0.1, 0, 0.1, \dots)$ and $(-0.75, -0.50, -0.25, 0, \dots)$ respectively. The iteration was run until the R-square convergence at 86.5%; indicating the estimated parameters are appropriate for the fitted model. The result was confirmed by two model criteria: Bayesian information criterion (BIC) and Akaike information criterion (AIC) used. Hence, the identified Hill growth

$(y_i = 3.7 + 41.46 \left(\frac{x_i}{9.75 + x_i} \right)^{18.17})$ is adequate and can be used for forecasting of the amount of transmitted voltage against time.

Keywords Hill Growth Model, Linearization, Model Selection Criterion, Linear Regression Model

1. Introduction

Regression is a statistical model that can be used to explain the relationship between exogenous and endogenous variables (Ijomah *et al.*, 2018). Hill model is a non-linear regression model with the four parameters β_0 , β_1 , β_2 and β_3 . A technique of finding a non-linear relationship between the dependent variable and a set of (or several) independent variable(s) called Non-linear regression analysis. In this way, non-linear regression is a function which models observational data by a non-linear combination of the model parameters and depends on one or more independent variables. As a result, many situations require non-linear function just like the simple and multiple linear regression functions that seem adequate for modelling a wide variety of relationships between the response variable and independent variables (Seber & Wild, 2003; Roush and Branton, 2005; Ijomah *et al.*, 2018). In particularly, non-linear regression functions have served and will continue to serve as useful models for describing various physical and biological systems e.g. Hill growth model (or Hill model).

Hill model is an S-shaped curve, often referred to as sigmoidal growth model (sigmoidal curve) which has many

applications in agriculture, engineering field, signal detection theory also applicable in biochemistry, forestry (height distribution) and most importantly it is used in prediction. Numerous useful families of non-linear regression functions exist such as Richards, Logistics, Weibull, Gomperts, S. Shapes curves etc. These models are referred to as Sigmoidal Growth Models and are all useful in growth analysis. This study only considers Hill Model in growth analysis and its applications.

The data used in this study is an experiment used to determine the amount of transmitted voltage against time collected from Department of Electrical/Electronic Engineering, University of Port-Harcourt. In some cases, it is possible to transform a non-linear regression function to a linear regression function using some appropriate transformations of the exogenous variable Y_i , the parameters β_0 , β_1 , β_2 , and β_3 , the endogenous variable X_i or any combination of these. If the assumptions for simple or multiple linear regression are satisfied in the transformed variables, then the result can be applied to the transformed problem. By the use of the transformed results, the original problem can obtain its results.

The Hill model of growth with four parameters is expressed as

$$y_i = \beta_0 + \beta_1 \left(\frac{x_i}{\beta_2 + x_i} \right)^{\beta_3} \quad (1.1)$$

where y_i is the i^{th} observation

* Corresponding author:

ndukawonu@gmail.com (Wonu Nduka)

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$\beta_0, \beta_1, \beta_2, \beta_3$ are the parameters

β_0 = represents upper asymptote when the time approaches positive infinite (or the initial parameter at time equal to zero; $t = 0$)

β_1 = represents the shape parameter related to the initial time

β_2 = represents growth range

β_3 = represents the growth rate (or shape parameter)

x_i = represents time (Rudolf *et al.*, 2012)

Over the years, forecasting a non-linear model has been a major problem. On the course of solving this problem, many statistical models have been formed and transformed; the major tool for solving the problem is regression analysis (simple, multiple, linear or non-linear) for more accurate results, non-linear regression (growth models) has been developed for prediction, which includes: Hill (case study), Gomperts, Richards, Logistic, Weibull, Brody, Robertson etc.

The custom statistical techniques in estimating nonlinear models require initial values to begin the optimization. The non-linear model expression must be written, the parameter names declared an initial parameter value specified. In most cases, the quality of the final solution depends upon the position of this initial value or starting value; after the iteration approach end. The problem of the initial parameter is solved (or reduced) by transformation to linear and OLS estimation before the iteration approach begins.

This paper aims to estimate the Hill growth model coefficients/parameters using the equation of a line (simple linear regression). The objectives are as follows: (1) to derive the Hill Model to an equation of a line using transformation techniques (Logarithm) and its properties. (2) to choose the two initial values for the upper asymptote parameter (β_0) and growth range parameter (β_2) increasing by 0.1 and 0.25 rate respectively (3) to identify a suitable Hill Model and its parameters, using OLS estimation.

There is a need to provide an alternative method of choosing the initial values and fitting the Hill growth model. This will provide an alternative way of a solution to this model and enable it to produce the desired level of forecasting. This paper only considers Hill model as a growth model neglecting other growth models such as Richards, Gompertz, Weibull, Brody, Robertson, Bertalanffy etc (Dagogo *et al.*, 2018). Hence, an alternative way of solving Hill Model is explored in this study, where an iterative process, choosing initial values and OLS estimation is used in building the nonlinear model considered.

2. Review Works on Hill Growth Model

The Hill model was first introduced by Archobald Vivian Hill who published the model first in its currently known form on 22th January 1910. This equation possesses three of the most important features of scientific models that is information, understanding and memorizing knowledge; it can be used to find an acceptable solution to practical

problems, which we shall see in this review, and it can serve as a starting point for the development of a particular model and more detailed models. The Hill model is the first milestone in quantitative pharmacology research, being the first formula that reacts a reversible association (as an effect) to the variable concentration of one of two associating substances, as long as the other substance is present in a constant and relatively low concentration. Thereby, the Hill growth model was the first exact quantitative model in pharmacology that react to changes (or receptor model). The Hill equation (model) and its properties show that it is a good case study for understanding the development of quantitative models in life sciences.

Carla *et al.*, (2018), applied the Hills growth model to the treatment regimens that is the anti-microbial Pharmacokinetics (PK) and the antimicrobial Pharmacodynamic (PD) against the pathogen causing the disease. Their study aimed to use a model to analyze the relationship between the antimicrobial drug concentration and the pathogen populace. Their experiment captured the relationship between the antimicrobial drug concentration and its effect on the growth rate of the exposed pathogen population (Wen *et al.*, 2018).

Schmitt *et al.*, (2013) researched on, muscle biomechanics, a vital and broad field. Given the research, it was known that in biology microscopic muscle models can predict muscle characteristics and functioning of biological muscles quite well; unfortunately, they require a large number of parameters like the Hill model.

Zoran (2011) studied the relationship between the Hill function (model) and the more complicated Adair-Klotz model. He carried out an experiment using the dose-response curve, the result of the experiment showed that the curve (dose-response) obtained from Hill model agrees well with the dose-response curve obtained from Adair-Klotz Model describe strongly co-operative binding. Hence, the alternative technique suggested in the paper will help to construct a Hill model that its results indicate an improved analysis of growth performance, efficiency and the prediction of future or past growth rate.

3. Materials and Methods

- Materials

A data set was considered in this project for the illustration of the fitted growth model. The data used in this study is an experiment used to determine the amount of transmitted voltage against time collected from the Department of Electrical/Electronic Engineering, University of Port-Harcourt (Appendix). The Microsoft Excel solver is used in the analysis of this study, however, there are others possibility for solving this kind of analysis such as Wolfram Mathematical, MATLAB, Minitab, Gretl statistical software's etc. In the software's, this problem (for all four parameters at once) can be fitted with a simple build-in command nonlinear model-fitting algorithm.

- Methods

The method of nonlinear least square estimation was applied by taking the derivatives concerning the parameters ($\beta_0, \beta_1, \beta_2, \beta_3$) with additive and multiplicative error terms. A nonlinear regression model is similar to linear regression model both seek to graphically track a particular response from a set of predictor variables. Non-linear models such as Hill growth model in this work are developed because the function is created a series of approximations (iterations) that may stem from trial and error. Many researchers and statisticians use several established methods such as the Gauss-Newton method and the Levenberg-Marquardt method. This research work used the modified version of the Ordinary Least Square method that is

- 1) Input the arbitrary value for (β_0^0 , and β_2^0) as the initial guess values for the iteration process.
- 2) Input the data and initial guess values on the Excel solver; then run the iteration to obtain the results.

- Model Specification

The Hill Growth Model with Multiplicative error term (Rudolf *et al.*, 2012), then Equation (1.1) can be expressed as

$$y_i = \beta_0 + \beta_1 \left(\frac{x_i}{\beta_2 + x_i} \right)^{\beta_3} \times e_i \quad (3.1)$$

where e_i are the error terms.

Re-write Equation (3.1) as

$$y_i - \beta_0 = \beta_1 \left(\frac{x_i}{\beta_2 + x_i} \right)^{\beta_3} \times e_i \quad (3.2)$$

By applied transformation technique to Equation (3.2) (or taking the natural logarithm of both sides); we have

$$\ln(y_i - \beta_0) = \ln \beta_1 + \beta_3 \ln \left(\frac{x_i}{\beta_2 + x_i} \right) + \ln e_i \quad (3.3)$$

Equation (3.3) is in simple linear regression form, with $\epsilon_i = \ln e_i$ is the residual sum of squares of the model, where the number of parameters has been reduced from four to two parameters.

Note: $(y_i - \beta_0) = Y$, $\ln \left(\frac{x_i}{\beta_2 + x_i} \right) = X$ and $\ln e_i = \epsilon_i$

By applying the non-linear least square method, using step 1 and 2 above algorithms and

Let ($\beta = \beta_0^0$ and β_2^0) be the initial parameters as follow; $\beta_0^0 = (-0.2, -0.1, 0, 0.1, \dots)$ and $\beta_2^0 = (-0.75, -0.50, -0.25, 0, \dots)$; where β_0^0 is increase by 0.1 and β_2^0 is increased by 0.25.

Note: One mathematical property of the Hill growth model is as follow;

$$\text{Asymptotes, if for } \beta_1 = 0; y_i = \beta_0 + 0 \left(\frac{x_i}{\beta_2 + x_i} \right)^{\beta_3} = \beta_0$$

- Model Selection Criteria

(1) R-Square (R^2):

The R-Square statistic measures the success of the regression in predicting the values of the dependent variable (Nduka & Ogoke, 2016). It assumes that every independent variable in the model help to explain variation in the dependent (y) and thus gives the percentage of explained variation if all independents in the model affect the dependent variable (y). The R^2 statistic is defined as

$$R^2 = \frac{SSE}{SST} \quad (3.5)$$

where $SST = \sum_{i=1}^n (y_i - \bar{y})^2$ is the total sum of squares

$SSE = \sum_{i=1}^n (y_i - \hat{y}_i)^2$ is the regression sum of squares y_i

and \hat{y}_i are the original and modelled data values.

(2) Adjusted R-Square (R^2_{adj}):

In the least-squares regression, increasing the number of regressors in the model leads to an increase in R-Square. Hence R-Square alone cannot be employed as a meaningful comparison of the model. The adjusted R-Square (R^2_{adj}) tells us the percentage of variation explained by only those independent variables that do not belong to the model (Nduka & Ogoke, 2016).

The adjusted R-square is defined as

$$(R^2_{adj}) = 1 - \frac{SSE/(n-p)}{SST/(n-1)} \quad (3.6)$$

where n is the sample size, p is the model parameter, SST and SSE are defined in Equation (3.5) above.

(3) Bayesian information criterion (BIC):

Bayesian information criterion (BIC) is a criterion for model selection among a finite set of models: the model with the lowest BIC is preferred (Henry, 2010; Arimie *et al.*, 2018). The BIC is expressed as

$$BIC = n \ln \left(\frac{SSE}{n} \right) + p \ln(n) \quad (3.7)$$

where

RSS (or SSE) is the residual sum of squares of the model.
n is the number of data points (or the number of observations)

p is the number of free parameters to be estimated:

\ln is the natural Logarithm

(4) Akaike Information Criterion (AIC)

The residual sum of squares from the regression model was also used to calculate AIC, Arimie *et al.*, (2018):

$$AIC = n \ln \left(\frac{SSE}{n} \right) + 2p \quad (3.8)$$

where all parameters are defined as in Equation (3.7).

Similarly, the model with the lowest AIC is preferred.

Note: $MSE = \frac{SSE}{n}$

4. Analysis and Results

The data in Appendix was used to build a suitable Hill growth model and its parameter estimates.

Firstly, figure 4.1 scatter plot of the amount of transmitted voltage against time with a non-linear model called exponential model with its R^2 value of 86.7%. This plot showed a slightly S-shaped curve and non-linear relationship between the amounts of transmitted voltage against time. Hence, a suitable Hill growth model can be identified.

The results of the ordinary least square technique after logarithm transformation and the steps below are in Appendices are summarized in Table 4.1, 4.2 and 4.3.

Step 1: suggest the initial arbitrary value for $(\beta_0^0, \text{and } \beta_2^0)$ as the initial guess values for the iteration process to start.

Step 2: Then, input the data sets and the initial guess values on the Excel solver; then run the iteration to obtain the results OLS.

Step 3: Continue the process (iterations) by increasing and decreasing arbitrary value for $(\beta_0^0, \text{and } \beta_2^0)$ until the model selection criterion convergence; therefore stop the process.

Table 4.1 below shows the values of the intercept and other parameters estimates with R^2 , Mean Square Error, AIC and BIC for various iterations. When β_0 increase by 0.1 and β_2 increase by 0.25.

We observe that the R-Square values obtained increased slightly after thirteen (13) iterations. Therefore, we increase β_0 from 0.1 to 0.3 and increase β_2 from 0.25 to 0.75 in Table 4.2.

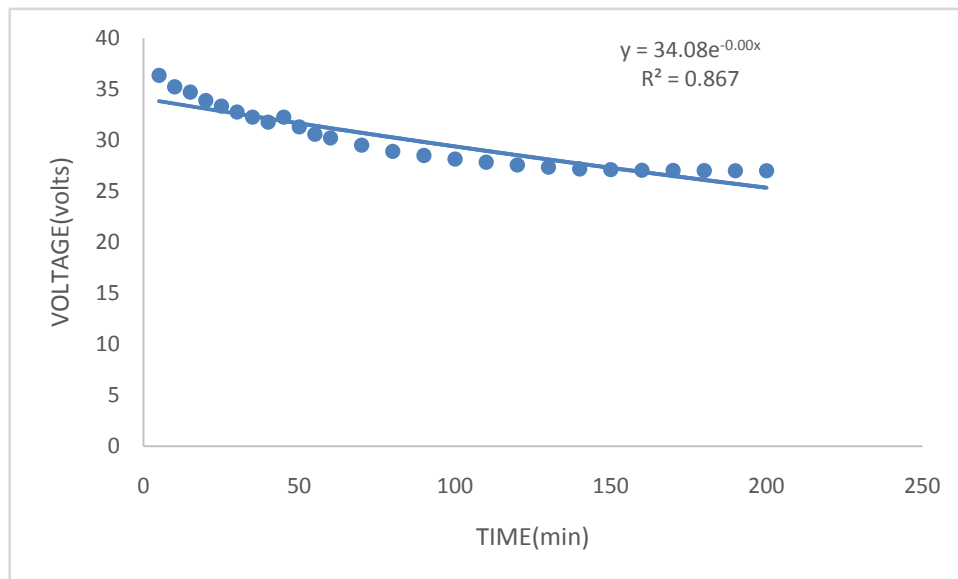


Figure 4.1. Scatter plot of the amount of transmitted voltage against time

Table 4.1. Hill model parameters estimates when $(\beta_0$ increase by 0.1); $(\beta_2$ increase by 0.25)

β_2	β_0	β_1	$\ln\beta_1$	(β_3)	R^2	$R^2(adj)$	MSE	AIC	BIC
-0.75	-0.2	7.99×10^{-16}	-34.76	63.19	57.0	55.79	3.98	39.9	42.42
-0.50	-0.1	2.76×10^{-32}	-72.65	101.13	59.2	57.27	3.84	38.98	41.49
0.50	0	1.16×10^{-68}	-186.42	214.94	60.00	58.00	3.72	38.15	40.67
0.75	0.1	6.20×10^{-68}	-154.75	-126.50	64.00	62.50	3.37	35.52	38.10
1.00	0.2	2.50×10^{42}	97.63	-69.58	66.20	64.80	3.16	33.90	36.49
1.25	0.3	2.40×10^{37}	86.09	-58.19	67.20	65.87	3.07	33.16	35.67
1.50	0.4	1.06×10^{34}	78.35	-50.60	68.24	66.84	2.98	32.39	34.90
1.75	0.5	4.05×10^{31}	72.78	-45.18	69.19	67.87	2.89	31.59	34.10
2.00	0.6	5.95×10^{29}	68.56	-41.10	70.00	68.80	2.80	30.72	33.29
2.25	0.7	2.10×10^{28}	65.25	-37.90	70.90	69.71	2.72	30.02	32.53
2.50	0.8	1.47×10^{27}	62.56	-35.39	71.70	70.51	2.65	29.33	31.85
2.75	0.9	1.60×10^{26}	60.34	-33.32	72.50	71.39	2.57	28.54	31.06
3.00	1.0	2.70×10^{25}	58.47	-31.59	73.33	72.19	2.50	27.82	30.34

Similarity, Table 4.2 below shows the values of the intercept and other parameters estimates with R^2 , Mean Square Error, AIC and BIC for various iterations. When β_0 increase by 0.3 and β_2 increase by 0.75.

We observe that the R-Square values are closed to converge and close to 1 after seven (7) iterations. Therefore,

we decrease β_0 from 0.3 to 0.25 and decrease β_2 from 0.75 to 0.10 in Table 4.3.

Likewise, Table 4.3 below shows the values of the intercept and other parameters estimates with R^2 , Mean Square Error, AIC and BIC for various iterations. When β_0 increase by 0.25 and β_2 increase by 0.1.

Table 4.2. Hill parameters estimates when (β_0 increase 0.3); (β_2 increase 0.75)

β_2	β_0	β_1	$\ln\beta_1$	(β_3)	R^2	$R^2(adj)$	MSE	AIC	BIC
3.75	1.3	3.50×10^{23}	54.22	-27.77	75.40	74.39	2.31	25.70	28.28
4.50	1.6	1.80×10^{22}	51.25	-25.22	77.00	76.00	2.13	23.60	26.17
5.25	1.9	1.90×10^{21}	49.01	-23.39	79.01	78.13	1.96	21.49	24.01
6.00	2.1	3.59×10^{20}	47.33	-22.02	80.50	79.70	1.82	19.57	22.08
6.75	2.4	8.12×10^{19}	45.85	-20.94	81.93	81.10	1.69	17.64	20.16
7.45	2.7	2.34×10^{19}	44.6	-20.10	83.10	82.40	1.58	15.89	18.41
8.20	3.0	8.44×10^{18}	43.58	-19.42	84.26	83.60	1.47	14.02	16.53

Table 4.3. Hill parameters estimates when (β_0 increase 0.1); (β_2 increase 0.25)

β_2	β_0	β_1	$\ln\beta_1$	(β_3)	R^2	$R^2(adj)$	MSE	AIC	BIC
8.45	3.1	4.23×10^{20}	43.19	-19.21	83.90	83.90	1.44	13.48	15.99
8.70	3.2	2.58×10^{20}	42.86	-19.01	84.35	84.35	1.41	12.91	15.43
8.95	3.3	1.03×10^{20}	42.55	-18.83	84.70	84.70	1.37	12.18	14.70
9.20	3.4	6.69×10^{19}	42.24	-18.65	85.60	85.00	1.34	11.61	14.13
9.45	3.5	2.94×10^{19}	41.94	-18.48	85.30	85.30	1.31	11.02	13.53
9.75	3.6	4.30×10^{19}	41.46	-18.29	86.50	86.00	1.28	10.42	12.93
9.95	3.7	2.09×10^{19}	41.37	-18.17	86.50	86.00	1.27	10.41	12.46

We observe the convergence of R^2 (**86.5%**) and R^2 (adj) (**86.0%**) at the 26th and 27th iteration. Note that R^2 of 86.5% is similarly (or the same) to the exponential model with 86.7%.

The estimated Hill Growth model parameters are $\beta_0 = 3.6$, $\beta_2 = 9.75$, $\beta_1 = 1.0 \times 10^{18}$ and $\beta_3 = -18.29$

By substitution into Equation (3.4): $\ln(y_i - \beta_0) = \ln\beta_1 + \beta_3 \ln\left(\frac{x_i}{\beta_2 + x_i}\right) + e_i$

$$\ln(y_i - 3.7) = 41.46 - 18.17 \ln\left(\frac{x_i}{9.75 + x_i}\right)$$

Hence, the identified Hill growth model is

$$y_i = 3.7 + 41.46 \left(\frac{x_i}{9.75 + x_i}\right)^{18.17}$$

5. Conclusions

The Hills Growth Model is nonlinear, then logarithmic transformation was applied to make the linear model paved way for its estimation of parameters. The study was able to show an alternative method of estimating the Hill Growth Model parameters using a real data set. A suitable Hill growth model was determined using Mean Square Error, R-Squared, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The results of the sum of squared residual, R-Squared, AIC and BIC showed that the identified Hill Growth Model is adequate and can be used for forecasting of the amount of transmitted voltage against time.

Appendices

Table 1. Voltage (volts) and Time (minute)

TIME(min)	VOLTAGE(volts)
5	36.36
10	35.24
15	34.72
20	33.90
25	33.34
30	32.77
35	32.26
40	31.77
45	32.26
50	31.30
55	30.58
60	30.22
70	29.51
80	28.90
90	28.50
100	28.15
110	27.84
120	27.56
130	27.34
140	27.18
150	27.11
160	27.05
170	27.02
180	27.01
190	27.00
200	27.00

Hill Growth Model Parameter Estimation**First Iteration with $\beta_0 = -0.2$ and $\beta_2 = -0.75$** **SUMMARY
OUTPUT**

<i>Regression Statistics</i>	
Multiple R	0.758667
R Square	0.575575
Adjusted R Square	0.557891
Standard Error	1.095423
Observations	26

ANOVA

	<i>df</i>	<i>SS</i>	<i>MS</i>	<i>F</i>	<i>Significance F</i>
Regression	1	129.5934	129.5934	32.54716	7.07E-06
Residual	24	95.56109	3.981712		
Total	25	225.1545			

	<i>Coefficients</i>	<i>Standard Error</i>	<i>t Stat</i>	<i>P-value</i>	<i>Lower 95%</i>	<i>Upper 95%</i>	<i>Lower 95.0%</i>	<i>Upper 95.0%</i>
Intercept	-34.7618	11.33618	-3.06645	0.005297	-58.1585	-11.3651	-58.1585	-11.3651
X	63.19544	11.07718	5.705012	7.07E-06	40.33326	86.05761	40.33326	86.05761

Second Iteration with $\beta_0 = -0.1$ and $\beta_2 = -0.50$

Regression Statistics	
Multiple R	0.767997832
R Square	0.59220671
Adjusted R Square	0.572729865
Standard Error	1.96165031
Observations	26

ANOVA					
	<i>df</i>	<i>SS</i>	<i>MS</i>	<i>F</i>	<i>Significance F</i>
Regression	1	132.800785	132.80078	34.5109933	4.64153E-06
Residual	24	92.35372656	3.8480719		
Total	25	225.1545115			

	<i>Coefficients</i>	<i>Standard Error</i>	<i>t Stat</i>	<i>P-value</i>	<i>Lower 95%</i>	<i>Upper 95%</i>	<i>Lower 95.0%</i>
Intercept	-72.65516388	17.47397891	-4.157906	0.00035328	-108.719684	-36.5906	-108.7197
X	101.1305804	17.21487089	5.8746058	4.6415E-06	65.60083311	136.6603	65.600833

<i>Observation</i>	<i>Predicted y</i>	<i>Residuals</i>	<i>Standard Residuals</i>
1	39.71214765	-3.452147651	-1.796107
2	33.79807862	1.341921376	0.698184
3	31.96267789	2.657322109	1.3825696
4	31.0685083	2.731491697	1.421159
5	30.53930589	2.700694106	1.4051354
6	30.18949413	2.480505868	1.2905744
7	29.94107708	2.218922916	1.1544763
8	29.75555043	1.914449572	0.9960628
9	29.611171515	2.548284845	1.3258389
10	29.49693751	1.703062487	0.8860809
11	29.40321999	1.076780012	0.5602344
12	29.32525331	0.794746691	0.4134962
13	29.20297463	0.207025369	0.1077126
14	29.11145788	-0.311457885	-0.162047
15	29.04039181	-0.640391808	-0.333187
16	28.98361037	-0.93361037	-0.485745
17	28.93719997	-1.197199971	-0.622887
18	28.898557	-1.438557002	-0.748462
19	28.86588206	-1.62588206	-0.845925
20	28.8378917	-1.757891697	-0.914608
21	28.81364586	-1.803645865	-0.938413
22	28.79244026	-1.842440262	-0.958597
23	28.77373679	-1.853736795	-0.964475
24	28.75711728	-1.84711728	-0.961031
25	28.7422518	-1.842251803	-0.958499
26	28.7288766	-1.8288766	-0.95154

Continuous until

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•
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26th Iteration with $\beta_0 = 3.6$ and $\beta_2 = 9.70$

SUMMARY OUTPUT

<i>Regression Statistics</i>	
Multiple R	0.952851
R Square	0.865924
Adjusted R Square	0.860921
Standard Error	0.858357
Observations	26

ANOVA

	<i>Df</i>	<i>SS</i>	<i>MS</i>	<i>F</i>	<i>Significance F</i>
Regression	1	167.0972	167.0972	226.7947	2.1E-13
Residual	23	16.94588	1.28777		
Total	24	184.0431			

	<i>Coefficients</i>	<i>Standard Error</i>	<i>t Stat</i>	<i>P-value</i>	<i>Lower 95%</i>	<i>Upper 95%</i>	<i>Lower 95.0%</i>	<i>Upper 95.0%</i>
Intercept	41.46189	1.217413	36.20125	8.75E-22	41.55348	46.5903	41.55348	46.5903
X	-18.2951	1.436195	-15.0597	2.1E-13	-24.5997	-18.6577	-24.5997	-18.6577

RESIDUAL OUTPUT

<i>Observation</i>	<i>Predicted 32.46</i>	<i>Residuals</i>	<i>Standard Residuals</i>
1	33.49552	-2.15552	-2.56522
2	31.32414	-0.50414	-0.59997
3	29.86586	0.134138	0.159635
4	28.81894	0.621057	0.739103
5	28.03084	0.839157	0.998658
6	27.41614	0.943858	1.12326
7	26.92329	0.946715	1.12666
8	26.51931	1.840689	2.190554
9	26.18216	1.217835	1.449313
10	25.89653	0.78347	0.932386
11	25.65144	0.66856	0.795635
12	25.25266	0.357345	0.425267
13	24.94205	0.057952	0.068967
14	24.69328	-0.09328	-0.11101
15	24.48957	-0.23957	-0.2851
16	24.31967	-0.37967	-0.45184
17	24.17583	-0.51583	-0.61387
18	24.05247	-0.61247	-0.72888
19	23.9455	-0.6655	-0.792
20	23.85187	-0.64187	-0.76388
21	23.76923	-0.61923	-0.73693
22	23.69575	-0.57575	-0.68518
23	23.62998	-0.51998	-0.61881
24	23.57077	-0.47077	-0.56025
25	23.51719	-0.41719	-0.49649

and

27th iteration with $\beta_0 = 3.7$ and $\beta_2 = 9.95$

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.891909
R Square	0.865931
Adjusted R Square	0.860105
Standard Error	0.861675
Observations	26

ANOVA					
	df	SS	MS	F	Significance F
Regression	1	166.7672	166.7672	222.0233	2.63E-13
Residual	23	17.27587	1.257125		
Total	24	184.0431			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	-41.37284607	1.248621	35.59611	1.28E-21	41.8631	47.02904	41.8631	47.02904
X	-18.178797	1.468395	-14.9004	2.63E-13	-24.9173	-18.8421	-24.9173	-18.8421

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