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Another Two-Parameter Poisson-Sujatha Distribution

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Abstract In this paper another two-parameter Poisson-Sujatha distribution by compounding Poisson distribution with another two-parameter Sujatha distribution which includes geometric distribution and Poisson-Sujatha distribution as particular cases, has been proposed. Its moments based statistical constants including coefficient of variation, skewness, kurtosis and index of dispersion have been obtained. Maximum likelihood estimation has been explained for estimating its parameters. Goodness of fit of the proposed distribution has been explained with five over-dispersed count datasets from various fields of knowledge and fit has been compared with Poisson-Lindley distribution, Poisson-Sujatha distribution, a generalization of Poisson-Sujatha distribution and two-parameter Poisson-Sujatha distribution.

Keywords Sujatha distribution, Poisson-Sujatha distribution, Another two-parameter Sujatha distribution, Moments based measures, Maximum likelihood estimation, Goodness of Fit

1. Introduction

Poisson distribution is the common distribution for modeling count data in statistics for equi-dispersed (mean equal to variance) data. But the equality of the mean and the variance of Poisson distribution makes it unsuitable for modeling count data which are under-dispersed (mean greater than variance) or over-dispersed (mean less than variance). In recent years, several researchers have proposed Poisson mixture of lifetime distributions which are useful for over-dispersed or under-dispersed. Some one parameter over-dispersed Poisson mixed distributions are Poisson-Lindley distribution (PLD), a Poisson mixture of Lindley distribution of Lindley (1958) proposed by Sankaran (1970), Poisson-Sujatha distribution (PSD), a Poisson mixture of Sujatha distribution of Shanker (2016 a) introduced by Shanker (2016 b), among others.

The probability density function (pdf) of Sujatha distribution having scale parameter θ and introduced by Shanker (2016a) is

$$f_1(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x} \quad ; x>0,\ \theta>0 \quad (1.1)$$

Statistical properties including shapes of the density, moments and moments based measures, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviation, stress-strength reliability, along with the estimation of parameter and applications for modeling lifetime data from biomedical science and engineering of Sujatha distribution are available in Shanker (2016 a). Kaliraja and Perarasan (2019) studied a stochastic model on the generalization of Sujatha distribution for the effects of two types of exercise on plasma growth hormone.

Shanker (2016 b) obtained Poisson-Sujatha distribution (PSD) by compounding Poisson distribution with Sujatha distribution. The PSD is defined by its probability mass function (pmf)

$$P_1(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\frac{x^2+(\theta+4)x+\left(\theta^2+3\theta+4\right)}{(\theta+1)^{x+3}} \quad ; \quad (1.2)$$
$$x=0,1,2...,\ \theta>0$$

Statistical properties including shapes of pmf, moments and moments based measures, over-dispersion, unimodality and increasing hazard rate, estimation of parameters and applications o model over-dispersed data have been discussed by Shanker (2016 b). Wesley et al (2018) proposed a zero-modified Poisson-Sujatha distribution to model over-dispersed count data and discussed its several important properties and applications.

Shanker *et al* (2017) have introduced a generalization of Sujatha distribution (AGSD) having pdf and cdf given by

$$f_2(x;\theta,\alpha)=\frac{\theta^3}{\theta^2+\theta+2\alpha}\left(1+x+\alpha x^2\right)e^{-\theta x} \quad ; \quad (1.3)$$
$$x>0,\ \theta>0,\alpha>0$$

Shanker *et al* (2017) have discussed important statistical

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properties including shapes of the density, moments and moments based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability, along with estimation of parameters using maximum likelihood estimation and applications of AGSD for modeling lifetime data from engineering and medical sciences. It can be easily verified that at $\alpha = 1$, the pdf of AGSD reduces to the corresponding pdf of Sujatha distribution. Also, at $\alpha = 0$, the pdf of AGSD reduces to Lindley distribution introduced by Lindley (1958).

Shanker and Shukla (2019) introduced a generalization of Poisson-Sujatha distribution (AGPSD) by compounding Poisson distribution with AGSD (1.3) and obtained the pmf in the form

$$P_2(x;\theta,\alpha) = \frac{\theta^3}{\theta^2+\theta+2\alpha}\frac{\alpha x^2 + (\theta+3\alpha+1)x + \left(\theta^2+3\theta+2\alpha+2\right)}{(\theta+1)^{x+3}}; \tag{1.3}$$

$x = 0.1.2....,\theta > 0, \alpha > 0$

Shanker and Shukla (2020) proposed a two-parameter Poisson-Sujatha distribution (TPPSD) defined by its pmf

$$P_3(x;\theta,\alpha) = \frac{\theta^3}{\left(\alpha\theta^2+\theta+2\right)}\frac{x^2+(\theta+4)x+\left\{\alpha\theta^2+(2\alpha+1)\theta+(\alpha+3)\right\}}{(\theta+1)^{x+3}}; \tag{1.4}$$

$x = 0,1,2,...,(\theta,\alpha) > 0$.

It should be noted that TPPSD is a Poisson mixture of a two-parameter Sujatha distribution (TPSD) introduced by Mussie and Shanker (2018) and defined by its pdf

$$f_3(x;\theta,\alpha) = \frac{\theta^3}{\alpha\theta^2+\theta+2}\left(\alpha+x+x^2\right)e^{-\theta x}; \tag{1.5}$$

$x > 0, \theta > 0, \alpha \ge 0$

Recently, Mussie and Shanker (2019) proposed another two-parameter Sujatha distribution (ATPSD) defined by its pdf

$$f_4(x;\theta,\alpha) = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left(1+\alpha x+\alpha x^2\right)e^{-\theta x}; \tag{1.6}$$

$x > 0, \theta > 0, \alpha \ge 0$

where θ is a scale parameter and is α is a shape parameter. It can be easily verified that (1.3) reduces to exponential distribution and Sujatha distribution for $\alpha = 0$ and $\alpha = 1$ respectively.

The main motivation for proposing ATPPSD are (i) Sujatha distribution is a better model than both exponential and Lindley distribution for modeling lifetime data, and PSD being a Poisson mixture of Sujatha distribution gives better fit than both Poisson and Poisson-Lindley distribution (PLD), (ii) TPSD gives much better fit than exponential, Lindley and Sujatha distribution, TPPSD being a Poisson mixture of TPSD provides better fit over PLD, PSD and other discrete distributions, and (iii) have a comparative study of ATPPSD with other two-parameter generalizations of Poisson-Sujatha distributions including TPPSD and AGPSD.

Keeping these points in mind, another two-parameter Poisson-Sujatha distribution (ATPPSD), a Poisson mixture of ATPSD has been proposed and its moments and moments based measures have been obtained and their behaviors have been studied. Maximum likelihood estimation of ATPPSD has been discussed for the estimation of its parameters. Its applications have been discussed with five examples of observed count datasets from various fields of knowledge.

2. Another Two-Parameter Poisson-Sujatha Distribution

A random variable X is said to follow another two-parameter Poisson-Sujatha distribution (ATPPSD) if $X \mid \lambda \sim P(\lambda)$ and $\lambda \mid \theta,\alpha \sim ATPSD(\theta,\alpha)$.

That is,

$P(X = x \mid \lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}; x = 0,1,2,..., \lambda > 0$, and

$$f(\lambda \mid \theta,\alpha) = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left(1+\alpha\lambda+\alpha\lambda^2\right)e^{-\theta\lambda};$$

$\lambda > 0, \theta > 0, \alpha \ge 0$

The pmf of unconditional random variable X can be obtained as

$$P_4(x;\theta,\alpha) = P(X = x) = \int_0^\infty P(X = x \mid \lambda) f(\lambda \mid \theta,\alpha)\,d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left(1+\alpha\lambda+\alpha\lambda^2\right)e^{-\theta\lambda}\,d\lambda \tag{2.1}$$

$$= \frac{\theta^3}{\left(\theta^2+\alpha\theta+2\alpha\right)x!}\int_0^\infty e^{-(\theta+1)\lambda}\left(\lambda^x+\alpha\lambda^{x+1}+\alpha\lambda^{x+2}\right)d\lambda$$

$$= \frac{\theta^3}{\left(\theta^2+\alpha\theta+2\alpha\right)x!}\left[\frac{\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{\alpha\Gamma(x+2)}{(\theta+1)^{x+2}}+\frac{\alpha\Gamma(x+3)}{(\theta+1)^{x+3}}\right]$$

$$= \frac{\theta^3}{\left(\theta^2+\alpha\theta+2\alpha\right)}\frac{\alpha x^2+\alpha(\theta+4)x+\left\{\theta^2+(\alpha+2)\theta+(3\alpha+1)\right\}}{(\theta+1)^{x+3}} \tag{2.2}$$

$x = 0,1,2,....,\theta > 0, \alpha \ge 0$

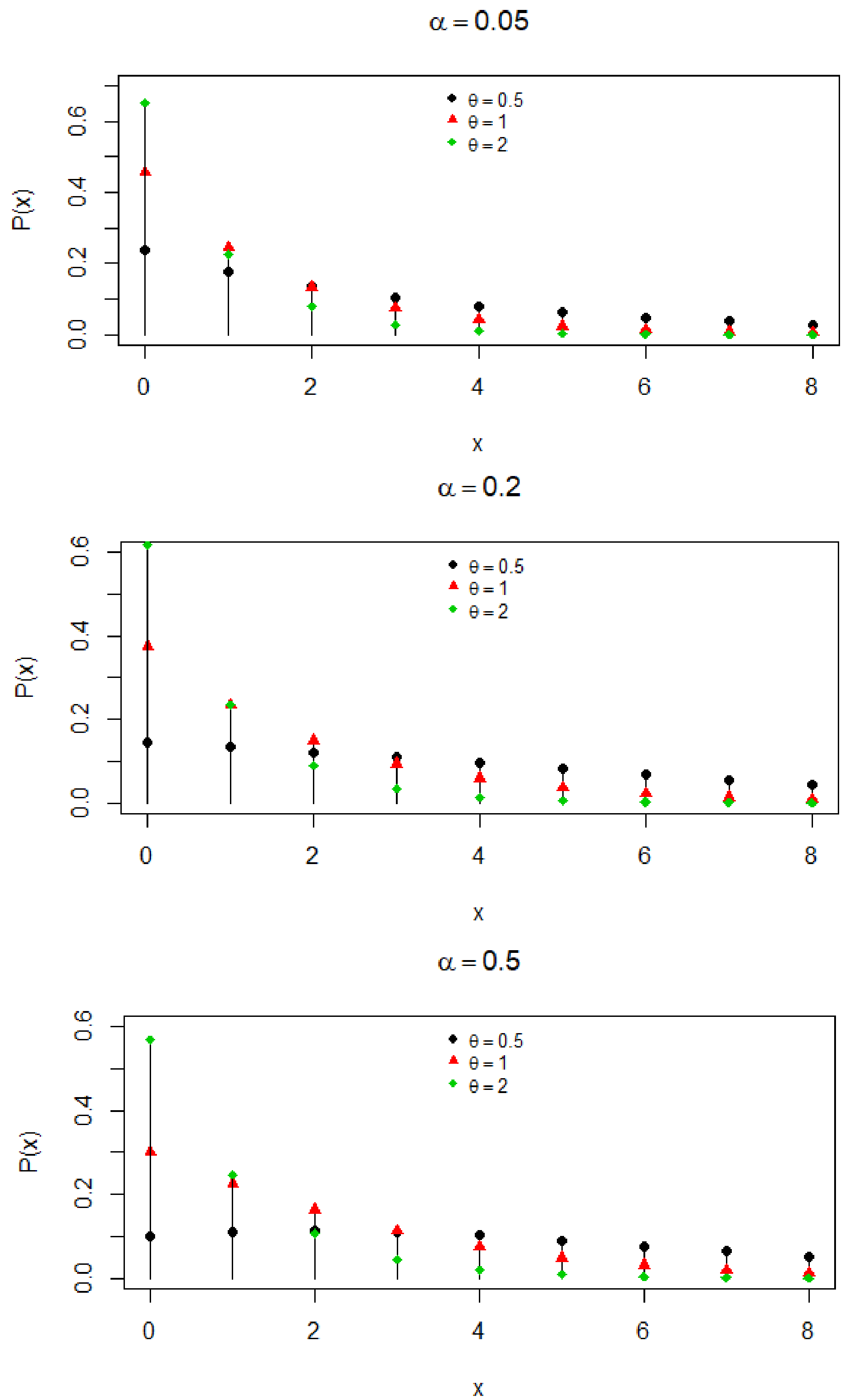
We would call this another two-parameter Poisson-Sujatha distribution (ATPPSD) because for $\alpha = 1$, it reduces to one parameter PSD given in (1.2). Also at $\alpha = 0$, it reduces to geometric distribution.

It can be easily shown that ATPPSD is unimodal and has increasing hazard rate. Since

$$\frac{P_4(x+1;\theta,\alpha)}{P_4(x;\theta,\alpha)}=\frac{1}{\theta+1}\left[1+\frac{(2x+\theta+5)\alpha}{x^2+(\theta+4)x+\left\{\alpha\theta^2+(2\alpha+1)\theta+(\alpha+3)\right\}}\right]$$

is decreasing function in x, $P_4(x;\theta,\alpha)$ is log-concave. Now using the results of relationship between log-concavity, unimodality and increasing hazard rate (IHR) of discrete distributions available in Grandell (1997), it can concluded that ATPPSD has an increasing hazard rate and unimodal.

The behavior of the pmf of ATPPSD for varying values of parameters θ and α are shown in figure 1.



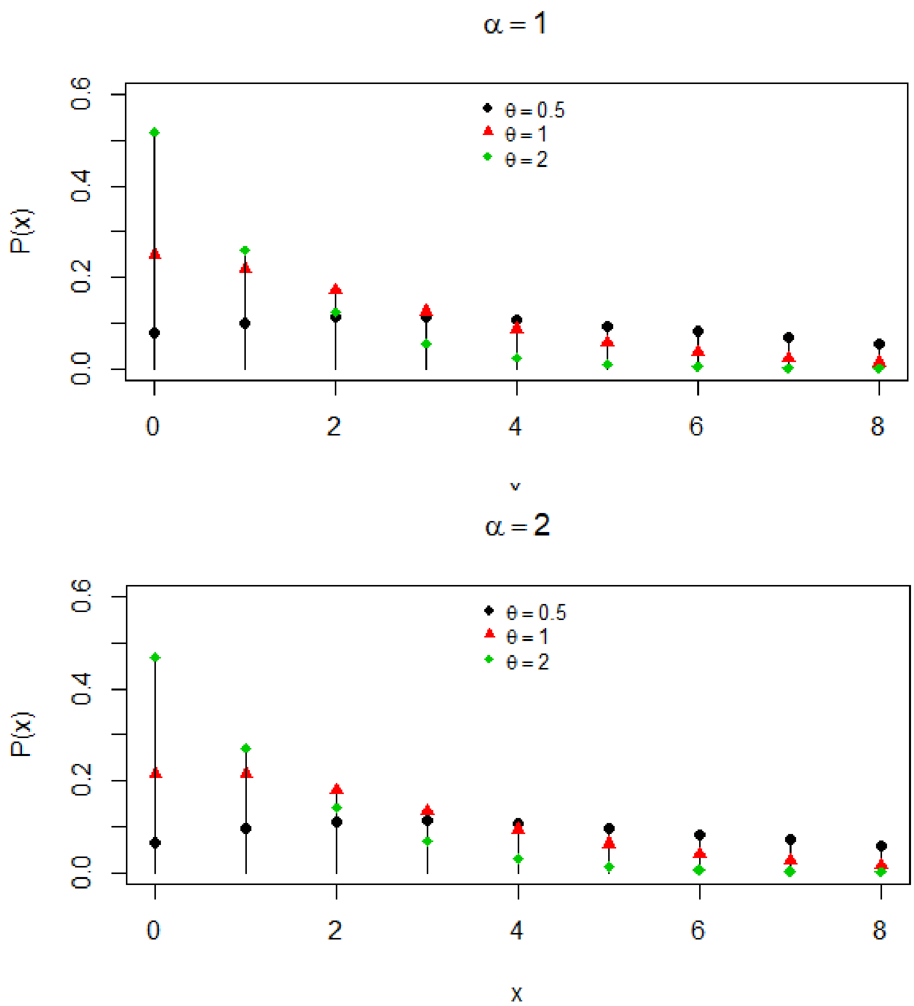


Figure 1. Behaviour of pmf of ATPPSD for varying values of parameters θ and α

3. Moments Based Measures

The r th factorial moment about origin $\mu_{(r)}{}'$ of ATPPSD can be obtained as

$\mu_{(r)}{}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right]$, where $X^{(r)} = X(X-1)(X-2)...(X-r+1)$. Using (2.1), the r th factorial moment about origin $\mu_{(r)}{}'$ of ATPPSD can be obtained as

$$\mu_{(r)}{}' = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty \left[\sum_{x=0}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!}\right]\left(1+\alpha\lambda+\alpha\lambda^2\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty \left[\lambda^r \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\left(1+\alpha\lambda+\alpha\lambda^2\right)e^{-\theta\lambda}d\lambda$$

Taking $x-r=y$ within the bracket, we get

$$\mu_{(r)}' = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty \left[\lambda^r \sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!}\right]\left(1+\alpha\lambda+\alpha\lambda^2\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty \lambda^r\left(1+\alpha\lambda+\alpha\lambda^2\right)e^{-\theta\lambda}d\lambda$$

After some tedious algebraic simplification, a general expression for the rth factorial moment about origin $\mu_{(r)}'$ of ATPPSD can be expressed as

$$\mu_{(r)}' = \frac{r!\left\{\theta^2+\alpha(r+1)\theta+\alpha(r+1)(r+2)\right\}}{\theta^r\left(\theta^2+\alpha\theta+2\alpha\right)}\ ;r=1,2,3,.... \tag{3.1}$$

It can be easily verified that at $\alpha=0$ and $\alpha=1$, the expression (3.1) reduces to the corresponding expression of geometric distribution and PSD. Substituting $r=1,2,3,\text{and } 4$ in (3.1), the first four factorial moments about origin of ATPPSD can be obtained as

$$\mu_{(1)}' = \frac{\theta^2+2\alpha\theta+6\alpha}{\theta\left(\theta^2+\alpha\theta+2\alpha\right)},\ \mu_{(2)}' = \frac{2\left(\theta^2+3\alpha\theta+12\alpha\right)}{\theta^2\left(\theta^2+\alpha\theta+2\alpha\right)}$$

$$\mu_{(3)}' = \frac{6\left(\theta^2+4\alpha\theta+20\alpha\right)}{\theta^3\left(\theta^2+\alpha\theta+2\alpha\right)},\ \mu_{(3)}' = \frac{24\left(\theta^2+5\alpha\theta+30\alpha\right)}{\theta^4\left(\theta^2+\alpha\theta+2\alpha\right)}.$$

Now using the relationship between factorial moments about origin and moments about origin, the first four moment about origin of the ATPPSD are obtained as

$$\mu_1' = \frac{\theta^2+2\alpha\theta+6\alpha}{\theta\left(\theta^2+\alpha\theta+2\alpha\right)}$$

$$\mu_2' = \frac{\theta^3+(2\alpha+2)\theta+12\alpha\theta+24\alpha}{\theta^2\left(\theta^2+\alpha\theta+2\alpha\right)}$$

$$\mu_3' = \frac{\theta^4+(2\alpha+6)\theta^3+(24\alpha+6)\theta^2+96\alpha\theta+120\alpha}{\theta^3\left(\theta^2+\alpha\theta+2\alpha\right)}$$

$$\mu_4' = \frac{\theta^5+(2\alpha+14)\theta^4+(48\alpha+36)\theta^3+(312\alpha+24)\theta^2+840\alpha\theta+720\alpha}{\theta^4\left(\theta^2+\alpha\theta+2\alpha\right)}$$

Using the relationship between moments about mean and the moments about origin, the moments about mean of ATPPSD are obtained as

$$\mu_2 = \frac{\theta^5+(3\alpha+1)\theta^4+\left(2\alpha^2+12\alpha\right)\theta^3+\left(12\alpha^2+16\alpha\right)\theta^2+24\alpha^2\theta+12\alpha^2}{\theta^2\left(\theta^2+\alpha\theta+2\alpha\right)^2}$$

$$\mu_3 = \frac{\left\{\begin{array}{l}\theta^8+(4\alpha+3)\theta^7+\left(5\alpha^2+25\alpha+2\right)\theta^6+\left(2\alpha^3+42\alpha^2+66\alpha\right)\theta^5\\ +\left(20\alpha^3+148\alpha^2+60\alpha\right)\theta^4+\left(84\alpha^3+216\alpha^2\right)\theta^3+\left(168\alpha^3+72\alpha^2\right)\theta^2\\ +144\alpha^3\theta+48\alpha^3\end{array}\right\}}{\theta^3\left(\theta^2+\alpha\theta+2\alpha\right)^3}$$

$$\mu_4=\frac{\left\{\begin{array}{l}\theta^{11}+(5\alpha+10)\theta^{10}+\left(9\alpha^2+72\alpha+18\right)\theta^9+\left(7\alpha^3+158\alpha^2+314\alpha+9\right)\theta^8\\+\left(2\alpha^4+140\alpha^3+940\alpha^2+600\alpha\right)\theta^7+\left(44\alpha^4+1012\alpha^3+2576\alpha^2+384\alpha\right)\theta^6\\+\left(368\alpha^4+3544\alpha^3+3096\alpha^2\right)\theta^5+\left(1560\alpha^4+6232\alpha^3+1224\alpha^2\right)\theta^4\\+\left(3600\alpha^4+5184\alpha^3\right)\theta^3+\left(4512\alpha^4+1728\alpha^3\right)\theta^2+2880\alpha^4\theta+720\alpha^4\end{array}\right\}}{\theta^4\left(\theta^2+\alpha\theta+2\alpha\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2), and index of dispersion (γ) of ATPPSD are thus given by

$$C.V=\frac{\sigma}{\mu_1'}=\frac{\sqrt{\theta^5+(3\alpha+1)\theta^4+\left(2\alpha^2+12\alpha\right)\theta^3+\left(12\alpha^2+16\alpha\right)\theta^2+24\alpha^2\theta+12\alpha^2}}{\theta^2+2\alpha\theta+6\alpha}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{\left\{\begin{array}{l}\theta^8+(4\alpha+3)\theta^7+\left(5\alpha^2+25\alpha+2\right)\theta^6+\left(2\alpha^3+42\alpha^2+66\alpha\right)\theta^5\\+\left(20\alpha^3+148\alpha^2+60\alpha\right)\theta^4+\left(84\alpha^3+216\alpha^2\right)\theta^3+\left(168\alpha^3+72\alpha^2\right)\theta^2\\+144\alpha^3\theta+48\alpha^3\end{array}\right\}}{\left\{\theta^5+(3\alpha+1)\theta^4+\left(2\alpha^2+12\alpha\right)\theta^3+\left(12\alpha^2+16\alpha\right)\theta^2+24\alpha^2\theta+12\alpha^2\right\}^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^{\,2}}=\frac{\left\{\begin{array}{l}\theta^{11}+(5\alpha+10)\theta^{10}+\left(9\alpha^2+72\alpha+18\right)\theta^9+\left(7\alpha^3+158\alpha^2+314\alpha+9\right)\theta^8\\+\left(2\alpha^4+140\alpha^3+940\alpha^2+600\alpha\right)\theta^7+\left(44\alpha^4+1012\alpha^3+2576\alpha^2+384\alpha\right)\theta^6\\+\left(368\alpha^4+3544\alpha^3+3096\alpha^2\right)\theta^5+\left(1560\alpha^4+6232\alpha^3+1224\alpha^2\right)\theta^4\\+\left(3600\alpha^4+5184\alpha^3\right)\theta^3+\left(4512\alpha^4+1728\alpha^3\right)\theta^2+2880\alpha^4\theta+720\alpha^4\end{array}\right\}}{\left\{\theta^5+(3\alpha+1)\theta^4+\left(2\alpha^2+12\alpha\right)\theta^3+\left(12\alpha^2+16\alpha\right)\theta^2+24\alpha^2\theta+12\alpha^2\right\}^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\theta^5+(3\alpha+1)\theta^4+\left(2\alpha^2+12\alpha\right)\theta^3+\left(12\alpha^2+16\alpha\right)\theta^2+24\alpha^2\theta+12\alpha^2}{\theta\left(\theta^2+\alpha\theta+2\alpha\right)\left(\theta^2+2\alpha\theta+6\alpha\right)}.$$

It can be easily verified that at $\alpha=0$ and $\alpha=1$ expressions of these statistical constants of ATPPSD reduce to the corresponding expressions for geometric distribution and PSD.

The behaviors of coefficient of variation (C.V), coefficient of skewness (C.S), coefficient of kurtosis (C.K) and index of dispersion (I.D) of ATPPSD for varying values of parameters θ and α have been explained through graphs and presented in figure 2.

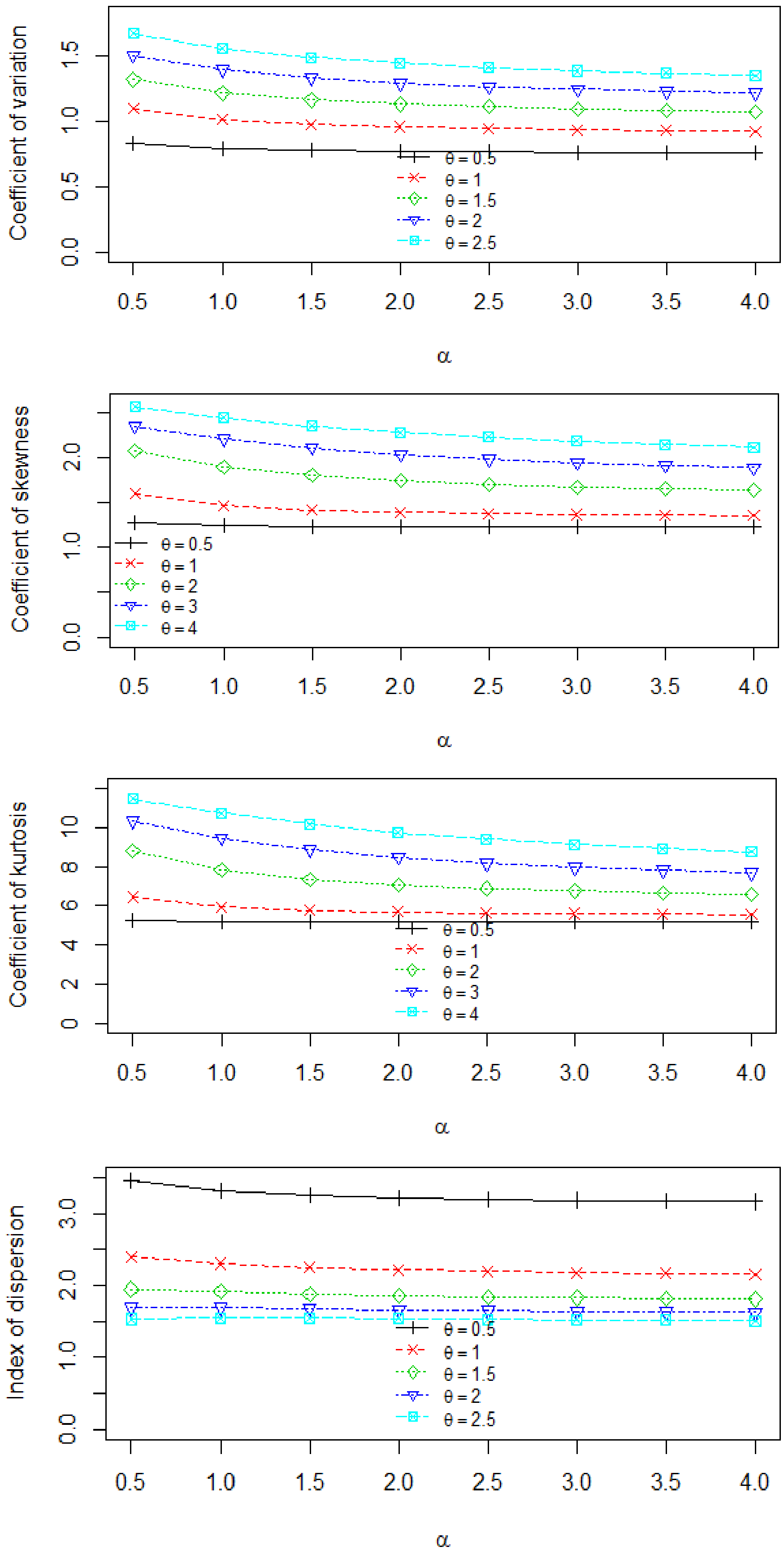


Figure 2. Behaviors of coefficient of variation (C.V), coefficient of skewness (C.S), coefficient of kurtosis (C.K) and index of dispersion (I.D) of ATPPSD for varying values of parameters θ and α

4. Maximum Likelihood Estimation of Parameters

Suppose $(x_1, x_2, ..., x_n)$ be a random sample of size n from ATPPSD and f_x be the observed frequency in the sample corresponding to $X = x$ $(x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of ATPPSD is given by

$$L = \left(\frac{\theta^3}{\theta^2 + \alpha\theta + 2\alpha}\right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k}(x+3)f_x}} \prod_{x=1}^{k}\left[\alpha x^2 + \alpha(\theta+4)x + \left\{\theta^2 + (\alpha+2)\theta + (3\alpha+1)\right\}\right]^{f_x}$$

The log - likelihood function is thus obtained as

$$\log L = n\left\{3\log\theta - \log\left(\theta^2 + \alpha\theta + 2\alpha\right)\right\} - \sum_{x=1}^{k}(x+3)f_x \log(\theta+1)$$
$$+ \sum_{x=1}^{k} f_x \log\left[\alpha x^2 + \alpha(\theta+4)x + \left\{\theta^2 + (\alpha+2)\theta + (3\alpha+1)\right\}\right]$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha})$ of parameters (θ, α) of ATPPSD is the solutions of the following log - likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{3n}{\theta} - \frac{n(2\theta+\alpha)}{\theta^2 + \alpha\theta + 2\alpha} - \frac{n(\bar{x}+3)}{\theta+1} + \sum_{x=1}^{k} \frac{(\alpha x + 2\theta + \alpha + 2)\, f_x}{\left[\alpha x^2 + \alpha(\theta+4)x + \left\{\theta^2 + (\alpha+2)\theta + (3\alpha+1)\right\}\right]} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{-n(\theta+2)}{\theta^2 + \alpha\theta + 2\alpha} + \sum_{x=1}^{k} \frac{\left\{x^2 + (\theta+4)x + (\theta+3)\right\} f_x}{\left[\alpha x^2 + \alpha(\theta+4)x + \left\{\theta^2 + (\alpha+2)\theta + (3\alpha+1)\right\}\right]} = 0,$$

where $\bar{x}$ is the sample mean. These two log likelihood equations do not seem to be solved directly because they do not have closed forms. Therefore, to find the maximum likelihood estimates of parameters, an iterative method such as Fisher Scoring method, Bisection method, Regula Falsi method or Newton-Raphson method can be used. In this paper Newton-Raphson method has been used using R-software.

Table 1. Observed and expected number of European corn-borer available in Gosset (1908)

Number of yeast cells per square	Observed frequency	PLD	PSD	AGPSD	TPPSD	ATPPSD
0	213	234.0	233.2	219.9	238.5	221.0
1	128	99.4	99.6	115.3	95.8	114.4
2	37	40.5	41.0	44.3	38.7	43.5
3	18	16.0	16.3	14.3	15.7	14.5
4	3	6.2	6.7	1.2	6.4	4.5
5	1	2.4	2.3	0.4	2.6	1.3
6	0	1.5	0.9	4.7	2.3	0.8
Total	400	400.0	400.0	400.0	400.0	400.0
ML Estimates		$\hat{\theta} = 1.9502$	$\hat{\theta} = 2.3731$	$\hat{\theta} = 3.9945$ $\hat{\alpha} = 56.988$	$\hat{\theta} = 1.5484$ $\hat{\alpha} = 40.9494$	$\hat{\theta} = 3.4626$ $\hat{\alpha} = 4983.24$
χ^2		11.04	10.86	2.91	14.5	2.91
d.f.		2	2	1	1	1
p-value		0.0040	0.0044	0.0880	0.0000	0.0880
$-2\log L$		905.23	904.88	893.21	909.64	894.65
AIC		907.23	906.88	897.21	913.64	898.65

Table 2. Observed and expected number of European corn-borer available in Mc Guire *et al* (1957)

Number of Corn-borer per plant	Observed frequency	Expected frequency				
		PLD	PSD	AGPSD	TPPSD	ATPPSD
0	188	194.0	193.6	187.0	186.8	186.8
1	83	79.5	79.6	87.2	87.8	87.8
2	36	31.3	31.6	33.5	33.1	33.1
3	14	12.0	12.1	11.3	11.2	11.2
4	2	4.5	4.5	3.5	3.5	3.5
5	1	2.7	2.6	1.5	1.6	1.6
Total	324	324.0	324.0	324.0	324.0	324.0
ML Estimates		$\hat{\theta} = 2.0432$	$\hat{\theta} = 2.4717$	$\hat{\theta} = 3.6824$ $\hat{\alpha} = 16.4716$	$\hat{\theta} = 3.31692$ $\hat{\alpha} = 0.094800$	$\hat{\theta} = 3.31671$ $\hat{\alpha} = 10.53961$
χ^2		1.29	1.16	0.41	0.55	0.55
d.f.		2	2	1	1	1
p-value		0.5247	0.5599	0.5220	0.4583	0.4583
$-2\log L$		714.09	713.83	711.38	711.65	711.65
AIC		716.09	715.83	715.38	715.65	715.65

Table 3. Accidents to 647 women working on high explosive shells in 5 weeks, available in Sankaran (1970)

No of Accidents	Observed Frequency	Expected frequency				
		PLD	PSD	AGPSD	TPPSD	ATPPSD
0	447	439.5	439.8	439.8	442.2	442.2
1	132	142.8	142.1	142.1	139.3	139.3
2	42	45.0	45.0	45.0	44.4	44.4
3	21	13.9	13.9	13.9	14.2	14.2
4	3	4.2	4.2	4.2	4.5	4.5
≥ 5	2	1.3	2.0	2.0	2.4	2.4
Total	647	647.0	647.0	647.0	647.0	647.0
ML Estimates		$\hat{\theta} = 2.729$	$\hat{\theta} = 3.168063$	$\hat{\theta} = 3.1047$ $\alpha = 0.81286$	$\hat{\theta} = 2.6496$ $\alpha = 3.4025$	$\hat{\theta} = 2.6570$ $\alpha = 0.3004$
χ^2		4.82	2.75	4.87	4.34	4.34
d.f.		3	3	2	2	2
p-value		0.1855	0.4318	0.0875	0.1141	0.1141
$-2\log L$		1185.21	1185.21	1185.21	1184.78	1184.78
AIC		1187.21	1187.21	1189.21	1188.78	1188.78

Table 4. Observed and Expected number of European red mites on Apple leaves, available in Bliss (1953)

Number of Red mites per leaf	Observed frequency	Expected frequency				
		PLD	PSD	AGPSD	TPPSD	ATPPSD
0	70	67.2	66.4	67.3	69.1	69.1
1	38	38.9	39.2	38.7	37.4	37.4
2	17	21.2	21.8	21.2	20.3	20.3
3	10	11.1	11.4	11.2	11.0	11.0
4	9	5.7	5.7	5.7	5.8	5.8
5	3	2.8	2.8	2.9	3.0	3.0
6	2	1.4	1.3	1.4	1.6	1.6
7	1	0.9	0.6	0.7	0.8	0.8
8	0	0.8	0.8	0.9	1	1
Total	150	150.0	150.0	150.0	150.0	150.0

Number of Red mites per leaf	Observed frequency	Expected frequency				
		PLD	PSD	AGPSD	TPPSD	ATPPSD
ML estimates		$\hat{\theta}=1.26010$	$\hat{\theta}=1.6533$	$\hat{\theta}=1.4043$ $\hat{\alpha}=0.2316$	$\hat{\theta}=1.3640$ $\hat{\alpha}=3.2989$	$\hat{\theta}=1.3640$ $\hat{\alpha}=0.3031$
χ^2		2.49	3.41	2.99	2.43	2.43
d.f		4	4	3	3	3
p-value		0.5595	0.4916	0.3931	0.4880	0.4880
$-2\log L$		445.02	445.27	444.95	444.53	444.53
AIC		447.02	447.27	448.95	448.53	448.53

Table 5. Observed and Expected number of households according to the number of male migrants aged 15 years and above, available in Shukla and Yadav (2006)

Number of migrants	Observed frequency	Expected frequency				
		PLD	PSD	AGPSD	TPPSD	ATPPSD
0	242	240.1	239.9	239.9	240.8	240.8
1	97	98.8	98.7	98.8	97.8	97.8
2	35	39.0	39.3	39.3	39.0	39.0
3	19	15.0	15.1	15.1	15.1	15.1
4	6	5.6	5.6	5.6	5.7	5.7
5	3	2.0	2.0	2.0	2.1	2.1
6	0	0.7	0.7	0.7	0.7	0.7
7	0	0.3	0.2	0.2	0.3	0.3
8	0	0.4	0.5	0.4	0.5	0.5
Total	402	402.0	402.0	402.0	402.0	402.0
ML estimates		$\hat{\theta}=2.0329$	$\hat{\theta}=2.4665$	$\hat{\theta}=2.4835$ $\hat{\alpha}=1.0589$	$\hat{\theta}=2.3668$ $\hat{\alpha}=1.2916$	$\hat{\theta}=2.3665$ $\hat{\alpha}=0.7737$
χ^2		1.46	1.52	1.53	1.43	1.43
d.f		3	3	2	2	2
p-value		0.6915	0.6776	0.4653	0.4891	0.4891
$-2\log L$		892.37	892.25	892.25	892.22	892.22
AIC		894.37	894.25	896.25	896.22	896.22

5. Goodness of Fit

In this section five count datasets which are over-dispersed have been considered for testing the goodness of fit of ATPPSD and the fit has been compared with PLD, PSD, AGPSD and TPPSD. The goodness of fit of all these distributions is based on maximum likelihood estimation. The first dataset is regarding the number of European corn-borer available in Gosset (1908) , the second dataset is regarding the number of European corn-borer available in Mc Guire *et al* (1957), the third dataset is regarding the accidents to 647 women working on high explosive shells in 5 weeks, available in Sankaran (1970), the fourth dataset is regarding the number of European red mites on Apple leaves, available in Bliss (1953) and the fifth dataset is regarding the observed number of households according to the number of male migrants aged 15 years and above, available in Shukla and Yadav (2006). The maximum likelihood estimates, chi-squares, value of $-2\log L$ and Akaike information criterion (AIC) for the considered distributions for the given datasets have been computed and presented in the respective table. The AIC has been calculated using the formula $AIC=-2\log L+2k$, where k is the number of parameters involved in the distribution. In table 1, AGPSD and ATPPSD give almost the same fit. In table 2, AGPSD gives the best fit, whereas TPPSD and ATPPSD gives the second best fit. In table 3, PSD gives the best fit whereas TPPSD and ATPPSD gives the second best fit. TPPSD and ATPPSD gives the same fit in table 4 and 5, respectively. Therefore, we can say that ATPPSD is competing well with AGPSD, and TPPSD for count datasets.

6. Concluding Remarks

In this paper, another two-parameter Poisson Sujatha distribution (ATPPSD) which includes geometric distribution and Poisson-Sujatha distribution (PSD) proposed by Shanker as a special case has been proposed.. Its unimodality, increasing hazard rate, moments and moments based measures including coefficients of variation, skewness, kurtosis and index of dispersion has been obtained and their behaviors have been explained graphically for varying values of parameters. The method of maximum likelihood estimation has been discussed. The applications of the proposed distribution has been explained through two examples of count data from ecology and the goodness of fit of the distribution has been found quite satisfactory over PLD, PSD, AGPSD, TPPSD.

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