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Construction of Bivariate Distribution by Mixing Positively Dependent and Negatively Dependent Distributions

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Abstract In this paper, a new method for constructing bivariate distributions with given marginals is proposed, based on a mixing two bivariate distributions. A new bivariate distribution family is introduced by adding an appropriate term with independence class of distribution. During this construction process, the model is not complicated. By choosing a base distribution from the same marginals we derive a new distribution around the independent class. We note that the new distribution has additional parameter which would provide additional flexibility in applications. The joint probability density, joint reliability and reversed hazard rate functions of the new bivariate distribution are obtained. Furthermore, it is established that construction of bivariate distributions by this method allows for some flexibility in the values of Spearman's correlation coefficient.

Keywords Dependence, Bivariate Distribution, Spearman's Rho, Fréchet Bounds

1. Introduction

Univariate models are insufficient to explain random phenomena. Today, data such as drought, wind speed and rainfall are measured together with the variables that may affect them. With the development of technology, the construction of continuous bivariate distribution functions with given marginals has become an importance. When creating new bivariate distributions, models that can express high correlation are generally tried to be obtained. [2] introduced a method which based on the choice of pairs of order statistics of the marginal distributions. [8] studied on construction of continuous bivariate distributions that possesses the Positive Quadrant Dependence property. [13] introduced a generalization of Farlie-Gumbel-Morgentern (FGM) distribution family. They extend the maximal correlation coefficient for FGM family. Furthermore, [14] introduced bivariate and multivariate generalization of quadratic transmutation distribution family. Proposal of [14] draws our attention in particular. Because the transition from univariate case to bivariate or multivariate cases is not so easy. While in univariate case the real line is the complement of the probabilities, at least in the bivariate case these supplements are on the quadrant. There are some issues to overcome for the case of positive values of the transmutation parameter. However, marginals of this model are univariate transmuted distributions and it cannot detect independency. They proposed quadratic rank transmuted bivariate distribution as $H(x,y)=(1+\lambda)F(x,y)-\lambda F(x,y)^2$, where $\lambda\in[-1,1]$. Here, for $\lambda=0$, $H(x,y)$ gives the base distribution $F(x,y)$. However, if the base distribution is taken from independency case, i.e., $F(x,y)=F(x)G(y)$, then $H(x,y)$ can be written as in the eq. (2) of [8] follows: $H(x,y)=F(x)G(y)+\lambda F(x)G(y)\big(1-F(x)G(y)\big)$. For $\lambda>0$, $w(x,y)=\lambda F(x)G(y)\big(1-F(x)G(y)\big)$ cannot meet the conditions (4) and (5) proposed by [8]. Inspired by these studies, the contribution of the article is to propose a simpler but more useful model than the model introduced by [14]. Proposed model also includes both positive and negative values of the parameter as in FGM. Thus, the model gains some flexibility in modeling both positive and negative dependence. Furthermore, proposed model can detect independency. After giving the necessary conditions to construct a new distribution, Spearman's rank correlation coefficient is calculated on two illustrative examples and the usefulness of this family is discussed. Furthermore, some reliability properties are studied for this family.

Let $H(x,y)$ denote the bivariate distribution function of (X,Y) having continuous marginal cdfs $F(x)$ and $G(y)$. Also, let $\mathcal{F}(F,G)$ be the distribution family where the respective marginal are F and G. Then, according to the eq. (2) and the condition (3) of [8], we have the function

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$w(x,y) = F(x)G(y)\bar{H}(x,y)$ where $\bar{H}$ denotes survival function. This function meets the conditions (3)-(5) given by [8]. Hence, first mixture component distribution is $H_1(x,y) = F(x)G(y) + F(x)G(y)\bar{H}(x,y)$. Similar work of [8] given by [15] introduces some conditions for negatively dependent families. According to the eq. (2.1) and the condition (2.1) of [15], we have $w(x,y) = -F(x)G(y)\bar{H}(x,y)$. Except for the condition (2.3) of [15], this function meets the conditions (2.2) and (2.4) given by [15]. Distribution properties for the second mixture component $F(x)G(y) - F(x)G(y)\bar{H}(x,y)$ have not yet been provided. To overcome this issue, we have the following theorem.

Theorem 1. Let $H(x,y)$ be a distribution function belongs to the distribution family $\mathcal{F}(F,G)$ which is differentiable on $\mathbb{R}^2$ and $h(x,y) = \frac{\partial^2 H(x,y)}{\partial x \partial y}$ denote the joint probability density function. Then

(i) $F(x,y) = F(x)G(y)[2 - F(x) - G(y) + H(x,y)]$ is a distribution function,

(ii) $F(x,y) = F(x)G(y)[F(x) + G(y) - H(x,y)]$ is a distribution function if $H(x,y) \le F(x)G(y)$, for all $(x,y) \in \mathbb{R}^2$ (or $H(x,y) = F(x)G(y)$, for all $(t,v) \in \mathbb{R}^2$).

Proof. (i) Multivariate distribution function must satisfy following properties:

(P1)

$$\lim_{x\to\infty} F(x,y) = \lim_{x\to\infty} F(x)G(y)[1+\bar{H}(x,y)] = G(y)$$
$$\lim_{y\to\infty} F(x,y) = \lim_{y\to\infty} F(x)G(y)[1+\bar{H}(x,y)] = F(x)$$
$$\lim_{x\wedge y\to\infty} F(x,y) = \lim_{x\wedge y\to\infty} F(x)G(y)[1+\bar{H}(x,y)] = 1.$$

(P2) $\frac{\partial F(x,y)}{\partial x} \ge 0$ and $\frac{\partial F(x,y)}{\partial y} \ge 0$. For the simplicity $f_x = \frac{dF(x)}{dx}$ and $g_y = \frac{dG(y)}{dy}$.

$$\begin{aligned}\frac{\partial F(x,y)}{\partial x} &= f_x G(y)[1+\bar{H}(x,y)] + F(x)G(y)\frac{\partial \bar{H}(x,y)}{\partial x}\\ &\ge f_x G(y)[1 - F(x) + \bar{H}(x,y)]\\ &\ge 0.\end{aligned}$$

Obviously, $\frac{\partial F(x,y)}{\partial y} \ge 0$.

(P3) $\frac{\partial^2 F(x,y)}{\partial x \partial y} \ge 0$. For the simplicity, let $f_0 = f(x)g(y)$, $H_0 = F(x)G(y)$ and $h_{xy} = \frac{\partial^2 H(x,y)}{\partial x \partial y}$. Then

$$\begin{aligned}\frac{\partial^2 F(x,y)}{\partial x \partial y} &= f_0[1+\bar{H}(x,y)] + f_x G(y)\frac{\partial \bar{H}(x,y)}{\partial y} + g_y F(x)\frac{\partial \bar{H}(x,y)}{\partial x} + H_0 h_{xy}\\ &= f_0[1+\bar{H}(x,y)] - f_0 G(y)Pr(X > x|Y=y) - f_0 F(x) Pr(Y > y|X = x) + H_0 h_{xy}\\ &\ge f_0[1 - F(x) - G(y) + \bar{H}(x,y)] + H_0 h_{xy}.\end{aligned}$$

Now, by noting that positively dependence implies $h_{xy} \ge f_0$, then we have $\frac{\partial^2 F(x,y)}{\partial x \partial y} \ge f_0[\bar{H}_0(x,y) + \bar{H}(x,y)] \ge 0$.

Also, negatively dependence implies both $Pr(X > x|Y=y) \le \bar{F}(x)$ and $Pr(Y > y|X = x) \le \bar{G}(y)$. Hence, we have

$$\begin{aligned}\frac{\partial^2 F(x,y)}{\partial x \partial y} &\ge f_0[1 - F(x)\bar{G}(y) - G(y)\bar{F}(x) + \bar{H}(x,y)] + H_0 h_{xy}\\ &= f_0[1 - F(x)\bar{G}(y) - G(y)\bar{F}(x) + \bar{H}(x,y)] + H_0 h_{xy}\\ &= f_0[\bar{H}_0(x,y) + H_0(x,y) + \bar{H}(x,y)] + H_0 h_{xy}\\ &\ge 0.\end{aligned}$$

(ii) According to [4], Additionally to the properties (P1)-(P3), to determine bivariate distribution uniquely by its marginals bivariate distribution must lie upper and lower Fréchet bounds. Therefore, this idea explains why we need negative dependence for the construction of distribution given in (ii).

(P1)

$$\lim_{x\to\infty} F(x,y) = \lim_{x\to\infty} F(x)G(y)[1-\bar{H}(x,y)] = G(y)$$
$$\lim_{y\to\infty} F(x,y) = \lim_{y\to\infty} F(x)G(y)[1-\bar{H}(x,y)] = F(x)$$
$$\lim_{x\wedge y\to\infty} F(x,y) = \lim_{x\wedge y\to\infty} F(x)G(y)[1-\bar{H}(x,y)] = 1.$$

(P2) $\frac{\partial F(x,y)}{\partial x} \ge 0$ and $\frac{\partial F(x,y)}{\partial y} \ge 0$.

$$\begin{aligned}\frac{\partial F(x,y)}{\partial x} &= f_x G(y)[1-\bar{H}(x,y)] - F(x)G(y)\frac{\partial \bar{H}(x,y)}{\partial x}\\ &\ge 0.\end{aligned}$$

Obviously, $\frac{\partial F(x,y)}{\partial y} \geq 0$.

(P3) $\frac{\partial^2 F(x,y)}{\partial x \partial y} \geq 0$.

$$\begin{aligned}\frac{\partial^2 F(x,y)}{\partial x \partial y} &= f_0[1-\bar{H}(x,y)] - f_x G(y)\frac{\partial \bar{H}(x,y)}{\partial y} - g_y F(x)\frac{\partial \bar{H}(x,y)}{\partial x} - H_0 h_{xy} \\ &= f_0[F(x)+G(y)-H(x,y)] - f_x G(y)\frac{\partial \bar{H}(x,y)}{\partial y} - g_y F(x)\frac{\partial \bar{H}(x,y)}{\partial x} - H_0 h_{xy}.\end{aligned}$$

Negatively dependence implies $h_{xy} \leq f_0$. Hence, we have

$$\begin{aligned}\frac{\partial^2 F(x,y)}{\partial x \partial y} &\geq f_0[F(x)-H_0+G(y)-H(x,y)] - f_x G(y)\frac{\partial \bar{H}(x,y)}{\partial y} - g_y F(x)\frac{\partial \bar{H}(x,y)}{\partial x} \\ &\geq 0.\end{aligned}$$

However, if X and Y are positively dependent, $F(x,y)$ can not bigger than Fréchet lower bound. Therefore, the assumption of negative dependence is needed. To show this situation, we assume $F(x)+G(y)>1$. Then we have

$$\begin{aligned}F(x,y)-\max\{F(x)+F(y)-1,0\} &= (F_0-H(x,y))(1-\bar{H}(x,y)) + (1-H(x,y))\bar{H}(x,y) \\ &\geq \big(F_0-H(x,y)\big)(F(x)+G(y)-1).\end{aligned}$$

As it can be seen that positivity of the above statement depends on $F_0 \geq H(x,y)$.

According to Theorem 1, first mixture component distribution can be positively, negatively dependent or independent. But the second component distribution must be negatively dependent or independent. Thus, for the base distribution $H(x,y)$, in order to be same structure for both mixture components, the random variables X and Y must be negatively dependent or independent. After this motivation, we can now propose the mixing of these two distributions as follows:

Let T and V be negatively dependent (or independent) continuous random variables. Then their joint distribution function denoted as $H(t,v)$ belongs to the distribution family $\mathcal{F}(F,G)$ where F and G denote respectively marginals of T and V. Let $H_1(x,y)$ and $H_2(t_2,v_2)$ respectively denote the distribution functions of (T_1,V_1) and (T_2,V_2) having the same marginals as H. The distribution functions of (T_1,V_1) and (T_2,V_2) are respectively defined by

$$H_1(t_1,v_1) = Pr(T_1 \leq t_1, V_1 \leq v_1) = F(t_1)G(v_1)[1+\bar{H}(t_1,v_1)] \tag{1}$$

and

$$H_2(t_2,v_2) = Pr(T_2 \leq t_2, V_2 \leq v_2) = F(t_2)G(v_2)[1-\bar{H}(t_2,v_2)] \tag{2}$$

where $\bar{H}(t,v)$ denotes survival function of (T,V) i.e., $Pr(T>t, V>v)$. As can be seen immediately from equations (1) and (2), T_1 and V_1 are positively dependent random pairs, and T_2 and V_2 are negatively dependent random pairs.

According to Therorem1, we can define a new pairs of random variables X and Y as below:

$$(X,Y) = \begin{cases}(T_1,V_1), \text{with probability } \alpha \\ (T_2,V_2), \text{with probability } 1-\alpha.\end{cases}$$

Hence, the distribution of (X,Y) obtained by mixing (1) and (2) which is given by

$$\begin{aligned}F(x,y) = Pr(X \leq x, Y \leq y) &= \alpha H_1(x,y) + (1-\alpha)H_2(x,y) \\ &= F(x)G(y) + (2\alpha-1)F(x)G(y)\bar{H}(x,y).\end{aligned} \tag{3}$$

By letting $\lambda = 2\alpha - 1$, where $\lambda \in [-1,1]$, eq. (3) can be rewritten as

$$\begin{aligned}F(x,y) &= H_0(x,y)[1+\lambda\bar{H}(x,y)] \\ &= (1+\lambda)H_0(x,y) - \lambda H_0(x,y)[1-\bar{H}(x,y)].\end{aligned} \tag{4}$$

$\lambda = 0$ indicates $F = H_0 = F_0 = FG$ i.e., independence of X and Y, $\lambda = -1$ indicates that X and Y negatively dependent, and $\lambda = 1$ indicates positive dependence between X and Y. Note that X and Y are independent from each other, F indicates well-known bivariate distribution which is Farlie-Gumbel-Morgenstern distribution (see, [3] and [5]).

We need the survival and probability density function for subsequent discussions. These functions are respectively given by

$$\begin{aligned}\bar{F}(x,y) &= \bar{H}_0(x,y) + \lambda H_0(x,y)\bar{H}(x,y) \\ &= \bar{H}_0(x,y)\big(1+\lambda H_0(x,y)\big) + \lambda H_0(x,y)[H(x,y)-H_0(x,y)],\end{aligned}$$

and

$$f(x,y) = h_0(x,y)[1+\lambda k(x,y)] + \lambda H_0(x,y)h(x,y)$$

where $k(x,y) = \bar{H}(x,y) - G(y)Pr(X>x|Y=y) - F(x)Pr(Y>y|X=x)$.

2. Reversed Hazard Rate of the New Family of Bivariate Distribution

The bivariate reversed hazard is defined by [1] as $r(x,y) = f(x,y)/F(x,y)$. Analogously to the hazard gradient by [7], [11] defined the bivariate reversed hazard rate as follows: $\boldsymbol{r}_{1,2}(x,y) = (r_1(x,y), r_2(x,y))$, where

$$r_1(x,y) = \frac{\partial log F(x,y)}{\partial x}, r_2(x,y) = \frac{\partial log F(x,y)}{\partial y}.$$

Reversed hazard rate gradients of $F(x,y)$ given by eq. (4) are as follows:

$$r_1(x,y) = r_1(x,\infty) - \lambda \frac{F(x) r_1(x,\infty) - H(x,y) r_{1_H}(x,y)}{1 + \lambda \bar{H}(x,y)}$$

$$r_2(x,y) = r_2(\infty,y) - \lambda \frac{G(y) r_2(\infty,y) - H(x,y) r_{2_H}(x,y)}{1 + \lambda \bar{H}(x,y)}.$$

Accordingly, after some simplifications, bivariate reversed hazard rate can be given by $r(x,y) = \frac{\lambda H(x,y)}{1+\lambda \bar{H}(x,y)} \left[r_1(x,\infty) r_{1_H}(x,y) + r_2(\infty,y) r_{2_H}(x,y) \right] + r_1(x,\infty) r_2(\infty,y) \left[\frac{1-\lambda+2\lambda \bar{H}(x,y)}{1+\lambda \bar{H}(x,y)} \right]$.

3. Lower and Upper Bounds on Spearman's Rho Measure for the New Family of Bivariate Distribution

This section deals with obtaining bounds for the bivariate distribution family given by the eq. (4). According to [6] and [4], for any bivariate distribution belonging to $\mathcal{F}(F,G)$ contains Fréchet a lower bound and an upper bound. These bounds are respectively defined as

$$F^-(x,y) = \max\{F(x) + G(y) - 1, 0\} \tag{5}$$

$$F^+(x,y) = \min\{F(x), G(y)\}. \tag{6}$$

For $F \in \mathcal{F}(F,G)$, Spearman's rho can be expressed as

$$\rho_s(X,Y) = 12 \int_{\mathbb{R}} \int_{\mathbb{R}} \{F(x,y) - F(x)G(y)\} dG(y) dF(x) \tag{7}$$

(see, [12]). The coefficient of Spearman's rho for the new family can be obtained by

$$\begin{aligned} \rho_s &= 12\lambda \int_{\mathbb{R}} \int_{\mathbb{R}} F(x)G(y)\bar{H}(x,y) dG(y) dF(x) \\ &= 12\lambda \int_0^1 \int_0^1 uv\bar{H}\left(F^{-1}(u), G^{-1}(v)\right) dv du \\ &= 12\lambda \int_0^1 \int_0^1 uv\left[1 - u - v + H\left(F^{-1}(u), G^{-1}(v)\right)\right] dv du. \end{aligned} \tag{8}$$

Hence, by using the eq. (5) for $\lambda > 0$, we have the lower bound as

$$\rho_s \geq -\lambda + 12\lambda \iint_{u+v-1>0} uv[u + v - 1] dv du = \frac{-\lambda}{12}.$$

To obtain the upper bound, we use the eq. (6), then

$$\rho_s \leq -\lambda + 12\lambda \iint_{v>u} u^2 v dv du + \iint_{u>v} uv^2 dv du = \frac{3\lambda}{5}.$$

According to sign of λ, we achieve the bounds as below:

$$\rho_s \in \begin{cases} \left[\frac{3\lambda}{5}, \frac{-\lambda}{12}\right], for\ \lambda < 0 \\ \left[\frac{-\lambda}{12}, \frac{3\lambda}{5}\right], for\ \lambda \geq 0. \end{cases}$$

We have two example to illustrate this family.

Example 1. The Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions are given by $H(x,y) = F(x)G(y)\left[1+\theta\,\bar{F}(x)\bar{G}(y)\right]$, for $\theta \in [-1,1]$. By taking $\theta \geq -1$, the distribution $F(x,y)$ is given by $F(x,y) = F(x)G(y)\left[1+\lambda\left(\bar{F}(x)\bar{G}(y)\left[1+\theta\,F(x)G(y)\right]\right)\right]$, where $\lambda \in [-1,1]$ and $\theta \in [-1,0]$. Hence, $\rho_s = \frac{\lambda}{12}(\theta+4)$. Since $\theta \in [-1,0]$, $\frac{\lambda}{4} \leq \rho_s \leq \frac{\lambda}{3}$. One can conclude that this family model weak dependence as FGM does.

Example 2. The bivariate Gumbel- Exponential (BGE) distribution is given by $H(x,y) = 1-e^{-x}-e^{-y}+e^{-x-y-\theta xy}$, for $\theta \in [0,1]$. The distribution $F(x,y)$ is given by $F(x,y) = 1-e^{-x}-e^{-y}+e^{-x-y}+\lambda(e^{-x-y-\theta xy}-e^{-2x-y-\theta xy}-e^{-x-2y-\theta xy}+e^{-2x-2y-\theta xy})$, where $\lambda \in [-1,1]$ and $\theta \in [0,1]$. According to [9], the Spearman's rho coefficient of BGE distribution is $\rho_s^{BGE} = 12\left[-\frac{e^{\frac{4}{\theta}}}{\theta}Ei\left(-\frac{4}{\theta}\right)-\frac{1}{4}\right]$, where $Ei(\cdot)$ is the exponential integral function. After some algebraic manipulation, ρ_s can be obtained as

$$\rho_s = 12\lambda\frac{e^{\frac{4}{\theta}}}{\theta}\left[-Ei\left(-\frac{4}{\theta}\right)+2e^{\frac{2}{\theta}}Ei\left(-\frac{6}{\theta}\right)-e^{\frac{5}{\theta}}Ei\left(-\frac{9}{\theta}\right)\right].$$

We calculate approximates values of ρ_s^{BGE} and ρ_s by using Maple with respect to some values of θ. Tabulated values are given as Table 1 below:

Table 1. Approximate Values of Spearman's Rho for BGE and F

	θ			
	$2/10$	$4/10$	$6/10$	$8/10$
ρ_s^{BGE}	-0.1369	-0.2531	-0.3542	-0.4437
ρ_s	$(0.2933)\lambda$	$(0.2624)\lambda$	$(0.2377)\lambda$	$(0.2173)\lambda$

For the proposed bivariate distribution families, it is generally expected to model a stronger correlation structure between two random variables. The distribution derived from the base of FGM can only reach the correlation structure that FGM can model. But, unlike in Example 1, as can be seen from the Table 1, it can be said that F can model negative dependence slightly better than BGE in small theta values. Then, we can choose this distribution as an alternative to BGE to provide a correlation structure between random variables.

4. Conclusions

In this study, we proposed a new bivariate distribution using a base distribution from the negative dependency class which is in $\mathcal{F}(F,G)$. Thus, this new distribution can reveal both negative dependence, positive dependence and independence between the random variables X and Y. The upper and lower bounds show that the values of the correlation coefficient for this family lies in the interval $[-.80,.80]$. Besides, as a result of illustrative examples, it can be said that distributions can be derived for pairs of random variables with higher correlations considering some base distributions. For further discussion, focusing on the negative dependence condition on $H(x,y)$, a new distribution family can be derived with any distribution function from $\mathcal{F}(F,G)$.

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