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Unified Cumulative Sum Control Chart for Moniroring Shifts in the Parameters of the Pareto Distribution

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Abstract In this study, we propose a unified Cumulative Sum (CUSUM) control chart for monitoring simultaneous shifts in the parameters of the Pareto distribution. The V-mask method of constructing a CUSUM control chart was used. The characteristics of the V-mask were investigated and it was observed that the lead distance, the mask angle and the Average Run Length (ARL) of the CUSUM control chart changed considerably for a small shift in the parameters of the Pareto distribution.

Keywords Unified CUSUM, Run length, Lead distance, V-Mask and simultaneous

1. Introduction

The quality of a product cannot be underestimated by both producers and consumers. Industry players therefore need a robust procedure to meet this desired quality and one of such procedures include the CUSUM control chart. The CUSUM control chart was proposed first by Page (1954) as a built up of the work done by Shewhart (1926). The Shewhart control charts are not able to detect a small to medium size shift in the process parameter. The CUSUM control chart has the tendency to detect this small to moderate size shift in the process parameter. Many researchers have studied enormously into the arena of CUSUM control charts. Notably among them include: Hawkins and Olwell (1998) stated that the CUSUM control charts are most sensitive Statistical Process Contol (SPC) to signal a persistent small step change in a process parameter. Also, Luguterah (2015) developed unified CUSUM control chart for monitoring simultaneous shifts in the parameters of the Elang-Truncated Exponential Distribution and therefore derived the parameters of the CUSUM chart proposed. Again, Naber and Bilgi (1994) developed a CUSUM control chart for the Gaussian distribution. Again, Kantam and Rao (2004) investigated the CUSUM control chart for the Log-Logistic Distribution and concluded that it was able to detect shifts on the average than the Shewhart control charts. Sasikumar and Bangusha (2014) explained the common uses of CUSUM control charts for monitoring performance overtime when the outcome is related to health science. The article provided a logical way to accumulate evidence over many patients, while adjusting for a changing mix of patients' characteristics that significantly affected risk. Bakhodir (2004) employed CUSUM control chart in economic and finance turning point in stock price indices.

Recently, Nasiru (2016) developed a one-sided CUSUM control chart for monitoring the shape parameter of the Pareto distribution.

In this study, we extended the study of Nasiru (2016) and therefore develop a two-sided CUSUM control chart for monitoring shifts in the shape parameter of the Pareto distribution.

2. Pareto Distribution

The Pareto distribution is lop-sided and a heavy-tailed distribution which was developed by the Italian economist and sociologist, Vilfredo Pareto (1848 – 1923). He worked in the field of national economy and sociology. The Pareto distribution is applied in modeling problems involving distributions of incomes or wealth and also many situations in which there will always be a shift in the equilibrium of the system.

If the random variable X has a Pareto distribution, then the density function is given by;

$$f_X(x;\ \gamma, c) = \frac{\gamma c^{\gamma}}{x^{\gamma+1}} \tag{1}$$

where $x \geq c$ and the parameters $\gamma > 0$ and $c > 0$ are the shape and scale parameters respectively. The corresponding cumulative distribution function is given by;

$$F_X(x;\ \gamma, c) = 1 - \left(\frac{c}{x}\right)^{\gamma} \tag{2}$$

The mean and variance of the Pareto distribution are given by;

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$$\mu = \frac{\gamma c}{\gamma - 1}, \gamma > 1 \quad (3)$$

and

$$\sigma^2 = \frac{\gamma c^2}{(\gamma - 2)(\gamma - 1)^2}, \gamma > 2 \quad (4)$$

respectively.

3. Sequential Probability Ratio Test

The Sequential Probability Ratio Test (SPRT) plays a significant role in the development of an acceptance sampling plan. The Wald's (1947) SPRT is a joint, subject by subject, Likelihood Ratio Test (LRT). In this approach each subject constitutes a stage. According to Johnson (1961), the CUSUM control charts are roughly equivalent to the SPRT. The SPRT have been used extensively in the development of an acceptance sampling plan and this acceptance sampling plan is used in determining the in and out-of-control limits in CUSUM procedures.

Suppose that we take a sample of m values $x_1, x_2, \ldots, x_m$ successively, from a population $f(x, \theta)$. Consider two hypotheses about θ, $H_0: \theta = \theta_0$ and $H_1: \theta = \theta_1$. The ratio of the probabilities of the sample is;

$$L_m = \frac{\prod_{i=1}^{m} f(x_i, \theta_1)}{\prod_{i=1}^{m} f(x_i, \theta_1)} \quad (5)$$

We select two numbers A and B, which are related to the desired α and β errors. The sequential test is set up as follows;

i. As long as $B < L_m < A$ we continue sampling.
ii. At the first i that $L_m \geq A$ we accept H_1.
iii. At the first i that $L_m \leq B$ we accept H_0.

An equivalent way for computation is to use the logarithm of L_m. Then, the inequality becomes;

$$\ln B < \sum_{i=1}^{m} \ln f(x_i, \theta_1) - \sum_{i=1}^{m} \ln f(x_i, \theta_0) < \ln A \quad (6)$$

The family of test is referred to as SPRT. If

$$Z_i = \ln \left\{ \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} \right\} \quad (7)$$

then the sampling terminates when

$$\sum Z_i \geq \ln A \quad (8)$$

or

$$\sum Z_i \leq \ln B \quad (9)$$

The Z_i's are independent random variables with variance, say, $\sigma^2 > 0$. Obviously $\sum_{i=1}^{m} Z_i$ has a variance $m\sigma^2$. As m increases the dispersion becomes greater and the probability that a value of $\sum Z_i$ will remain within the limits $\ln B$ and $\ln A$ tends to zero. The mean $\bar{Z}$ tends to a normal distribution with variance $\frac{\sigma^2}{m}$ and therefore the probability that it will fall between $\frac{\ln B}{m}$ and $\frac{\ln A}{m}$ tends to zero.

If we take a sample for which L_m lies between A and B for the first $n - 1$ trial and then $L_m \geq A$ at the n^{th} trial, so we accept H_1 (and reject H_0). By definition, the probability of getting such a sample is at least A times as large under H_1 as against H_0. The probability that we fail to reject H_0 when H_1 is true is β and the probability that we reject H_0 when H_1 is true is $1 - \beta$. Therefore

$$A \leq \frac{1-\beta}{\alpha} \quad (10)$$

Similarly, if we accept H_0

$$B \geq \frac{\beta}{1-\alpha} \quad (11)$$

Wald (1947) showed that for all practical purposes (10) and (11) holds as equalities. Thus

$$A = \frac{1-\beta}{\alpha} \quad (12)$$

and

$$B = \frac{\beta}{1-\alpha} \quad (13)$$

Suppose that $a = \frac{1-\beta}{\alpha}$ and $b = \frac{\beta}{1-\alpha}$ and that the true errors of first and second kind for the limits a and b are α' and β' respectively. Then, from (10)

$$\frac{\alpha'}{1-\beta'} \leq \frac{1}{a} = \frac{\alpha}{1-\beta} \quad (14)$$

and from (11)

$$\frac{\beta'}{1-\alpha'} \leq b = \frac{\beta}{1-\alpha} \quad (15)$$

Therefore

$$\alpha' \leq \frac{\alpha(1-\beta')}{1-\beta} \leq \frac{\alpha}{1-\beta} \quad (16)$$

and

$$\beta' \leq \frac{\beta(1-\alpha')}{1-\alpha} \leq \frac{\beta}{1-\alpha} \quad (17)$$

Furthermore

$$\alpha'(1-\beta) + \beta'(1-\alpha) \leq \alpha(1-\beta') + \beta(1-\alpha') \quad (18)$$

In practice α and β are small. From (16) and (17), the amount that α' can exceed α or β' exceed β is negligible. In addition, from relation (18) we see that either $\alpha' \leq \alpha$ or $\beta' \leq \beta$. Therefore the use of A and B can only increase one of the errors and only by a very small amount

4. CUSUM Control Chart for Monitoring the Shape (γ) and Scale Parameters (c)

In this section a unified CUSUM chart for monitoring both the scale and shape parameters of the Pareto distribution was constructed. The likelihood ratio test of the null hypothesis which says that there is no shift in the parameters of the Pareto distribution against the alternative that there is a shift in the parameters of the Pareto distribution is given as follows:

$$\frac{L_{1m}}{L_{0m}} = \frac{\prod_{i=1}^{m} f(x_i, \gamma_1, c_1)}{\prod_{i=1}^{m} f(x_i, \gamma_0, c_0)} \quad (19)$$

$$\frac{L_{1m}}{L_{0m}} = \frac{\prod_{i=1}^{m} \gamma_1 c_1^{\gamma_1}}{\prod_{i=1}^{m} x^{\gamma_1+1}} \times \frac{\prod_{i=1}^{m} x^{\gamma_0+1}}{\prod_{i=1}^{m} \gamma_0 c_0^{\gamma_0}} \quad (20)$$

The continuation region of the SPRT that discriminates between the two hypotheses is given by

$$\frac{\beta}{1-\alpha} < \frac{\prod_{i=1}^{m} \gamma_1 c_1^{\gamma_1}}{\prod_{i=1}^{m} x^{\gamma_1+1}} \times \frac{\prod_{i=1}^{m} x^{\gamma_0+1}}{\prod_{i=1}^{m} \gamma_0 c_0^{\gamma_0}} < \frac{1-\beta}{\alpha} \quad (21)$$

which becomes

$$\frac{\beta}{1-\alpha} < \left(\frac{\gamma_1}{\gamma_0}\right)^m \left(\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}\right)^m \prod_{i=1}^{m} x^{\gamma_0-\gamma_1} < \frac{1-\beta}{\alpha} \quad (22)$$

taking natural logarithm of both sides of (23)

$$\ln\left(\frac{\beta}{1-\alpha}\right) < m\ln\frac{\gamma_1}{\gamma_0} + m\ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}} + (\gamma_0-\gamma_1)\sum_{i=1}^{m}\ln x < \ln\frac{1-\beta}{\alpha} \quad (23)$$

Bearing in mind that the mean and variance of the Pareto distribution, any shift in the parameters affects these two moments. Let c_0 and γ_0 be the target values, let also $c_1\,(c_1 > c_0)$ and $\gamma_1\,(\gamma_1 > \gamma_0)$ be the changed values owing to the shift in the parameters. The SPRT will be stopped by rejecting or accepting the null hypothesis or continue to sample, as $\frac{L_{1m}}{L_{0m}}$ is outside or between $\ln A$ and $\ln B$. The process stops by rejecting H_0 if $\ln\frac{L_{1m}}{L_{0m}} \geq \ln A$; this gives a rejection line $\gamma_1 > \gamma_0$ and $c_1 > c_0$. Similarly, if we make use of SPRT with the same strength to the cases $c_1 < c_0$ and $\gamma_1 < \gamma_0$, where $\ln\frac{L_{1m}}{L_{0m}} \leq \ln A$, then another rejection line is determined. These two rejection lines indicates a symmetrical nature of masking. The observations in the sample enter the mask in a sequential way. Thus, the mask for the CUSUM chart for $\ln\frac{L_{1m}}{L_{0m}} \geq \ln A$ is

$$\sum_{i=1}^{m}\ln x_i \geq \frac{m\left[\ln\frac{\gamma_1}{\gamma_0} + \ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}\right] + \ln\alpha}{\gamma_1-\gamma_0} \quad (24)$$

This implies

$$\sum_{i=1}^{m}\ln x_i \geq mp + q \quad (25)$$

where

$$p = \frac{\ln\frac{\gamma_1}{\gamma_0} + \ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}}{\gamma_1-\gamma_0}$$

and

$$q = \frac{\ln\alpha}{\gamma_1-\gamma_0}$$

Similarly, the rejection line, when $\ln\frac{P_{1m}}{P_{0m}} \leq \ln A$, is given by

$$\sum_{i=1}^{m}\ln x_i \leq \frac{m\left[\ln\frac{\gamma_1}{\gamma_0} + \ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}\right] + \ln\alpha}{\gamma_1-\gamma_0} \quad (26)$$

which can be expressed as

$$\sum_{i=1}^{m}\ln x_i \leq mp^* + q^* \quad (27)$$

where

$$p^* = \frac{\ln\frac{\gamma_1}{\gamma_0} + \ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}}{\gamma_1-\gamma_0}$$

and

$$q^* = \frac{-\ln\alpha}{\gamma_1-\gamma_0}$$

Equations (26) and (28) form the regions above and below the plane $(m, \sum_{i=1}^{m}\ln x_i)$. If m is allowed sequentially, at some stage, $\sum_{i=1}^{m}\ln x_i$ satisfies either (26) or (28). Until this is achieved the process continues.

Using the slopes of the two lines (equations (26) and (28)), the parameters of the CUSUM chart, known as the angle of the mask and the lead distance, are obtained. From Figure 1, $tan\theta_1 =$ slope of the line S_1T_1 and $tan\theta_{-1} =$ slope of the line $S_{-1}T_{-1}$, hence

$$\theta_1 = tan^{-1}\left(\frac{\ln\frac{\gamma_1}{\gamma_0} + \ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}}{\gamma_1-\gamma_0}\right)$$

where $\gamma_1 > \gamma_0$ and $c_1 > c_0$.

And

$$\theta_{-1} = tan^{-1}\left(\frac{\ln\frac{\gamma_1}{\gamma_0} + \ln\frac{c_1^{\gamma_1}}{c_0^{\gamma_0}}}{\gamma_1-\gamma_0}\right)$$

where $\gamma_1 < \gamma_0$ and $c_1 < c_0$.

Various values of the angle are shown in Table 1. It was obvious from the results that as the values of $(\gamma_1-\gamma_0)$ and (c_1-c_0) increase the value of the angle increases. Also increasing values of $\frac{\gamma_1}{\gamma_0}$ and $\frac{c_1}{c_0}$ also increase the angle of the mask.

Table 1. Values of the angle θ for controlling both γ and c

γ_0	γ_1	c_0	c_1	θ
2.5	3.0	1.5	2.0	68.17
2.5	3.5	1.5	2.5	68.43
2.5	4.0	1.5	3.0	68.72
2.5	4.5	1.5	3.5	69.01
2.5	5.0	1.5	4.0	69.29
2.5	5.5	1.5	4.5	69.56
2.5	6.0	1.5	5.0	69.81
2.5	6.5	1.5	5.5	70.06
2.5	7.0	1.5	6.0	70.29
2.5	7.5	1.5	6.5	70.51

γ_0	γ_1	c_0	c_1	θ
5.5	4.5	7.0	6.0	70.6
5.5	4.0	7.0	5.0	71.9
5.5	3.5	7.0	4.0	72.4
5.5	3.0	7.0	3.0	72.7
5.5	2.5	7.0	2.0	72.9
5.5	2.0	7.0	1.0	73.4

When there is a negative shift in both parameters, that is where $\gamma_1 < \gamma_0$ and $c_1 < c_0$, it can be established from the second table of Table 1 that the mask angle (θ) increases as the values of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$ decreases.

The lead distance OS_1 is given by

$$d_1 = \frac{-\ln \alpha}{\ln \frac{\gamma_1}{\gamma_0} + \ln \frac{{c_1}^{\gamma_1}}{{c_0}^{\gamma_0}}}$$

where $\gamma_1 > \gamma_0$ and $c_1 > c_0$ and the lead distance OS_{-1} is given by

$$d_{-1} = \frac{-\ln \alpha}{\ln \frac{\gamma_1}{\gamma_0} + \ln \frac{{c_1}^{\gamma_1}}{{c_0}^{\gamma_0}}}$$

where $\gamma_1 < \gamma_0$ and $c_1 < c_0$.

Let $X_1, X_2, \ldots, X_m$ be a sample from Pareto distribution, if the points $(m, \sum_{i=1}^{m} \ln x_i)$ are plotted with a suitable scale, then the ordinates of the points represent the cumulative sum of the data. Equations (26) and (28) are the effects of a shift in the population parameters c and γ. Figure 1 shows a sizable shift in the parameters if $\sum_{i=1}^{m} \ln x_i$ falls outside the lines S_1T_1 and $S_{-1}T_{-1}$. The chart is interpreted by placing the mask over the last plotted point as shown in Figure 1. If any of the points lies below $S_{-1}T_{-1}$, then it indicates a decrease in c and γ and if any of the points falls above S_1T_1, then it shows an increase in c and γ.

Using some hypothetical values for $\gamma_{0,}\, \gamma_{1,}\, c$ and α, it can be determined that increasing the values of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$, the lead distance decreases given a fixed value of α. Again, the value of also increases with decreasing values of α and a given fixed value of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$. The details of the values of the lead distance are shown in Table 2.

Table 2. Values the lead distance d for controlling γ and c when $\gamma_1 > \gamma_0$ and $c_1 > c_0$

$\boldsymbol{\alpha = 0.005}$				
γ_0	γ_1	c_0	c_1	d
2.5	3.0	1.5	2.0	4.24
2.5	3.5	1.5	2.5	2.09
2.5	4.0	1.5	3.0	1.38
2.5	4.5	1.5	3.5	1.02
2.5	5.0	1.5	4.0	0.80
2.5	5.5	1.5	4.5	0.66
2.5	6.0	1.5	5.0	0.56
2.5	6.5	1.5	5.5	0.48
2.5	7.0	1.5	6.0	0.42
2.5	7.5	1.5	6.5	0.38
$\boldsymbol{\alpha = 0.025}$				
γ_0	γ_1	c_0	c_1	d
2.5	3.0	1.5	2.0	2.96
2.5	3.5	1.5	2.5	1.46
2.5	4.0	1.5	3.0	0.96
2.5	4.5	1.5	3.5	0.71
2.5	5.0	1.5	4.0	0.56
2.5	5.5	1.5	4.5	0.46
2.5	6.0	1.5	5.0	0.39
2.5	6.5	1.5	5.5	0.34
2.5	7.0	1.5	6.0	0.29
2.5	7.5	1.5	6.5	0.26
$\boldsymbol{\alpha = 0.05}$				
γ_0	γ_1	c_0	c_1	d
2.5	3.0	1.5	2.0	0.56
2.5	3.5	1.5	2.5	0.27
2.5	4.0	1.5	3.0	0.18
2.5	4.5	1.5	3.5	0.13
2.5	5.0	1.5	4.0	0.11
2.5	5.5	1.5	4.5	0.09
2.5	6.0	1.5	5.0	0.73
2.5	6.5	1.5	5.5	0.06
2.5	7.0	1.5	6.0	0.06
2.5	7.5	1.5	6.5	0.05

On the other hand, when there is a negative shift of the parameters, that is $\gamma_1 < \gamma_0$ and $c_1 < c_0$, it can be established that the lead distance (d) decreases as the value of alpha (α) increases and when the value of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$ decreases for a fixed value of α, the value of the lead distance increases. The details are shown in Table 3. The values of the lead distance are negative in this case because there is a negative shift in the parameters of the distribution.

Table 3. Values the lead distance d for controlling γ and c when $\gamma_1 < \gamma_0$ and $c_1 < c_0$

$\alpha = 0.01$				
γ_0	γ_1	c_0	c_1	d
5.5	4.5	8.0	6.0	-1.29
5.5	4.0	8.0	5.0	-0.87
5.5	3.5	8.0	4.0	-0.65
5.5	3.0	8.0	3.0	-0.53
5.5	2.5	8.0	2.0	-0.44
5.5	2.0	8.0	1.0	-0.37

$\alpha = 0.025$				
γ_0	γ_1	c_0	c_1	d
5.5	4.5	7	6	-1.03
5.0	4.0	6	5	-0.69
4.5	3.5	5	4	-0.52
4.0	3.0	4	3	-0.42
3.5	2.5	3	2	-0.35
3.0	2.0	2	1	-0.30

$\alpha = 0.05$				
γ_0	γ_1	c_0	c_1	d
5.5	4.5	7	6	-0.84
5.0	4.0	6	5	-0.56
4.5	3.5	5	4	-0.43
4.0	3.0	4	3	-0.34
3.5	2.5	3	2	-0.29
3.0	2.0	2	1	-0.24

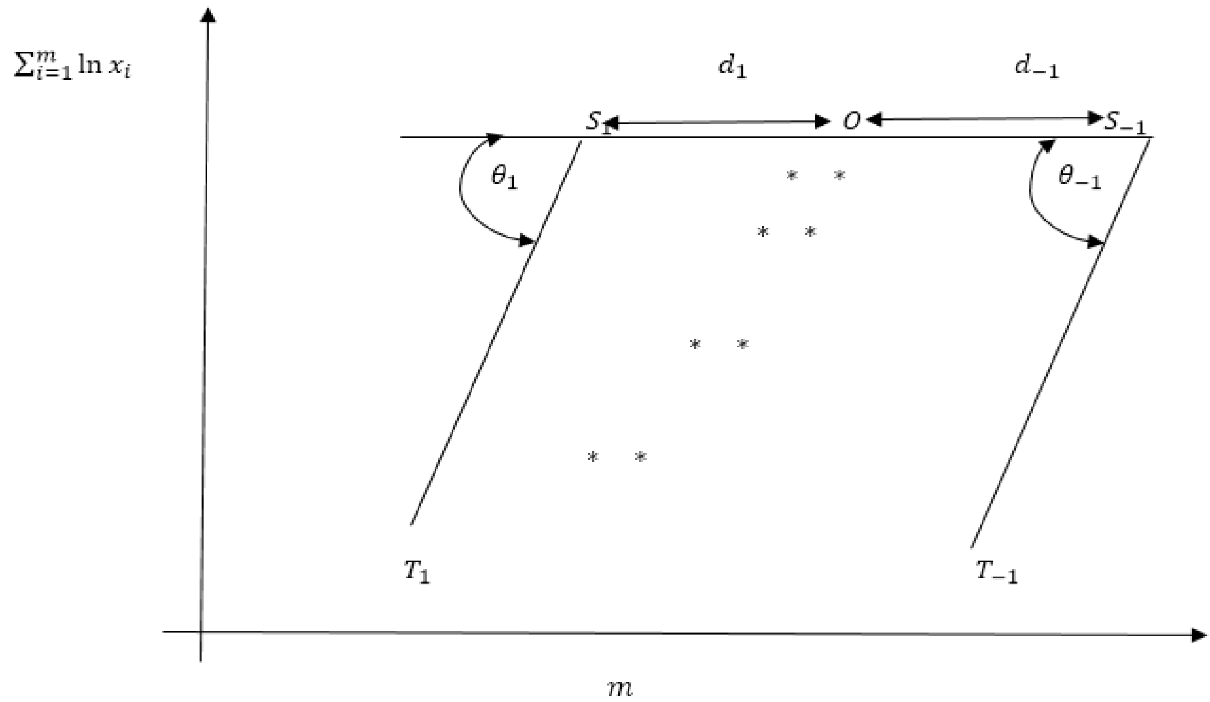

Figure 1. V-mask for unified CUSUM control chart

5. Average Run Length of the Unified CUSUM Control Chart

The Average Run Length of the unified CUSUM chart is the same as that of the two-sided CUSUM chart and is given by

$$ARL = \frac{-\ln \alpha}{\ln\left(\frac{\gamma_1}{\gamma_0}\right) + \frac{\gamma_0 - \gamma_1}{\gamma_1}} \tag{28}$$

Proof:

By definition

$$ARL = \frac{-\ln \alpha}{E(\ln Z)_{c=c_1, \gamma=\gamma_1}} \tag{29}$$

where $Z = \frac{f(x,\ \gamma_1,\ c_1)}{f(x,\ \gamma_0,\ c_0)}$.

using equation (1), Z can be written as

$$Z = \frac{\gamma_1 {c_1}^{\gamma_1}}{x^{\gamma_1 + 1}} \times \frac{x^{\gamma_0 + 1}}{\gamma_0 {c_0}^{\gamma_0}} \tag{30}$$

Taking natural logarithm of (30)

$$\ln Z = \ln\left(\frac{\gamma_1}{\gamma_0}\right) + \ln\left(\frac{{c_1}^{\gamma_1}}{{c_0}^{\gamma_0}}\right) + \ln\left[x^{(\gamma_0 - \gamma_1)}\right] \tag{31}$$

By simplifying (31), it reduces to

$$\ln Z = \ln\left(\frac{\gamma_1}{\gamma_0}\right) + \ln\left(\frac{{c_1}^{\gamma_1}}{{c_0}^{\gamma_0}}\right) + (\gamma_0 - \gamma_1)\ln x \tag{32}$$

Taking expectation of (32)

$$E[\ln Z] = \ln\left(\frac{\gamma_1}{\gamma_0}\right) + \ln\left(\frac{{c_1}^{\gamma_1}}{{c_0}^{\gamma_0}}\right) + (\gamma_0 - \gamma_1)E[\ln x] \tag{33}$$

The expectation, $E[\ln x]$ is obtained as follows

$$E[\ln x] = \int_c^\infty \ln x\ f(x,\ \gamma_1,\ c)\, dx \tag{34}$$

By substituting the density function for $f(x,\ \gamma_1,\ c)$ into (34), we obtained

$$E[\ln x] = \int_c^\infty \frac{\gamma_1 c^{\gamma_1}}{x^{\gamma_1 + 1}} \ln x\ dx \tag{35}$$

Simplifying (35), we obtained

$$E[\ln x] = \gamma_1 c^{\gamma_1}\left[0 - \left(\frac{-c}{\gamma_1^2} - \frac{c \ln c}{\gamma_1}\right) e^{-(\gamma_1 + 1)\ln c}\right] \tag{36}$$

and finally,

$$E[\ln x] = \left[\frac{\gamma_1 + \gamma_1^2 \ln c}{\gamma_1^2}\right] \tag{37}$$

Substituting the value of $E[\ln x]$ in (37) into (33) we obtain

$$E[\ln Z] = \ln\left(\frac{\gamma_1}{\gamma_0}\right) + \ln\left(\frac{{c_1}^{\gamma_1}}{{c_0}^{\gamma_0}}\right) + (\gamma_0 - \gamma_1)\left(\frac{\gamma_1 + \gamma_1 \ln c_1}{\gamma_1^2}\right) \tag{38}$$

Simplifying (38), we obtain

$$E[\ln Z] = \ln\left(\frac{\gamma_1}{\gamma_0}\right) + \frac{\gamma_0 - \gamma_1}{\gamma_1} \tag{39}$$

Substituting (39) into (29), the ARL formular is derived as required.

$$ARL = \frac{-\ln \alpha}{\ln\left(\frac{\gamma_1}{\gamma_0}\right) + \frac{\gamma_0 - \gamma_1}{\gamma_1}}$$

Using some hypothetical values of γ_0, γ_1, and α, it can be established that as $(\gamma_1 - \gamma_0)$ increases in value, the ARL tends to decrease with any given value of α. Also as the value of α decreases, the ARL also increases for fixed value of $(\gamma_1 - \gamma_0)$ as displayed in Table 4.

Also it can be established in the second table of Table 4 that when there is a negative shift $(\gamma_1 < \gamma_0)$, the ARL decreases as α increases and when $(\gamma_0 - \gamma_1)$ decreases, the ARL also increases for any given value of α. The negative values of the ARL obtained only indicates the negative shift in both parameters of the Pareto distribution.

Table 4. Average Run Length for the parameters of the Pareto distribution

		α						
γ_0	γ_1	0.5	0.005	0.005	0.025	0.1	0.01	0.001
0.5	1.60	1.48	6.30	11.14	7.76	4.84	9.68	14.52
0.5	1.65	1.39	6.03	10.66	7.42	4.63	9.27	13.90
0.5	1.70	1.34	5.79	10.23	7.12	4.45	8.89	13.34
0.5	1.75	1.29	5.56	9.84	6.85	4.45	8.55	12.83
0.5	1.80	1.24	5.36	9.48	6.60	4.12	8.24	12.36
0.5	1.85	1.20	5.18	9.16	6.38	3.98	7.96	11.94
0.5	1.90	1.16	5.01	8.86	6.17	3.85	7.69	11.55
0.5	1.95	1.12	4.85	8.58	5.98	3.73	7.46	11.19
0.5	2.00	1.09	4.71	8.33	5.80	3.62	7.24	10.86
0.5	2.05	1.06	4.57	8.09	5.63	3.52	7.03	10.55

γ_0	γ_1	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
5.5	4.5	-20.72	-16.60	-13.48
5.5	4.0	-12.28	-9.84	-7.99
5.5	3.5	-8.06	-6.46	-5.24
5.5	3.0	-5.52	-4.42	-3.59
5.5	2.2	-3.84	-3.07	-2.50
5.5	2.0	-2.63	-2.11	-1.71

6. Practical Demonstration of the Unified CUSUM Control Chart

In this section, the application of the proposed CUSUM chart has been illustrated using a hypothetical data simulated from the Pareto distribution. The first ten observations were simulated with $\gamma_0 = 2.5$ and $c_0 = 1.5$. The last five observations were simulated with $\gamma_1 = 5$ and $c_1 = 3$ for a simultaneous shift in both parameters. Table 5 displays the simulated data points and the corresponding cumulative sum. The parameters of the V-mask were calculated using $\gamma_0 = 2.5,\ c_0 = 1.5,\ \gamma_1 = 5,\ c_1 = 3$ and $\alpha = 0.01$. The lead distance and the mask angle were obtained as 0.9 and 64° respectively. The sample number (m) was plotted against the cumulative sum of the data. The V-mask was then placed at the last plotted point to monitor whether the process is in control or out of control as shown in Figure 2. It can be established from Figure 2, the process was out of control as observations 1 to 13 fell above line S_1T_1 indicating an increase in γ and c. On the other hand if the plotted points fall below the line $S_{-1}T_{-1}$, means there is a negative shift in the parameters. Anytime any of the above scenarios are experience then an action should be taken in order to bring the process back to control.

In Table 6, the first ten observations were simulated with $\gamma_0 = 5.5$ and $c_0 = 8$. The last five observations were simulated with $\gamma_1 = 4.5$ and $c_1 = 6$ for a simultaneous shift in both parameters. The parameters of the V-mask were calculated using $\gamma_0 = 5.5,\ c_0 = 8,\ \gamma_1 = 4.5,\ c_1 = 6$ and $\alpha = 0.05$. The lead distance and the mask angle were obtained as - 0.8 and 74° respectively. Since distance cannot be negative, an absolute value of the distance was then used in the placing of the V-mask. The sample number was then plotted against the cumulative sum (CUSUM) of the data. The V-mask was then placed at the last plotted point to monitor whether the process is in control or out of control as shown in Figure 3. It was established from Figure 3 that the process was out of control as observations 1 to 13 fell above line S_1T_1 indicating a shift in γ and c. This calls for a corrective measure to be taken in order to bring the process back to control.

Table 5. Simulated hypothetical data for the unified CUSUM

Sample	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Data(x)	1.8	1.5	3.8	1.5	1.2	5.2	2.0	14.7	1.9	11.8	4.0	1.8	1.4	2.6	2.4
$\ln x$	0.6	0.5	1.3	0.5	0.2	1.7	0.7	2.7	0.6	2.5	1.4	0.6	0.3	1.0	0.9
CUSUM	0.6	1.1	2.4	2.8	3.0	4.7	5.4	8.1	8.7	11.2	12.6	13.2	13.5	14.5	15.4

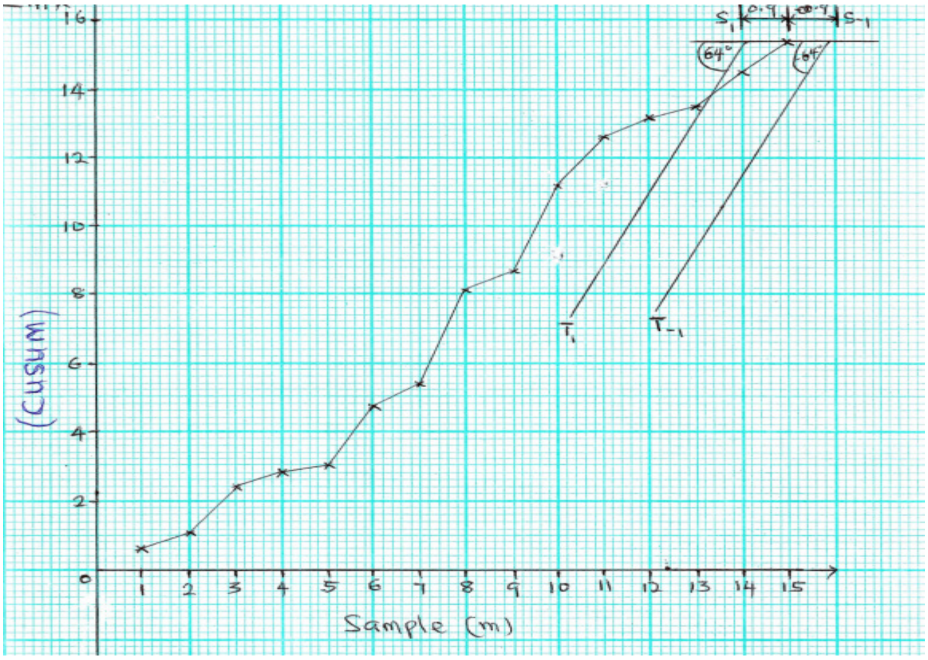

Figure 2. Unified CUSUM plot for the simulated hypothetical data

Table 6. Simulated hypothetical data for the unified CUSUM

Sample	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Data	0.1	0.6	1.2	0.1	2.0	0.2	0.3	0.6	0.5	0.2	0.4	0.2	0.9	0.4	0.4
CUSUM	0.1	0.7	1.9	2.0	4.0	4.2	4.5	5.1	5.6	5.8	6.2	6.4	7.3	7.7	8.1

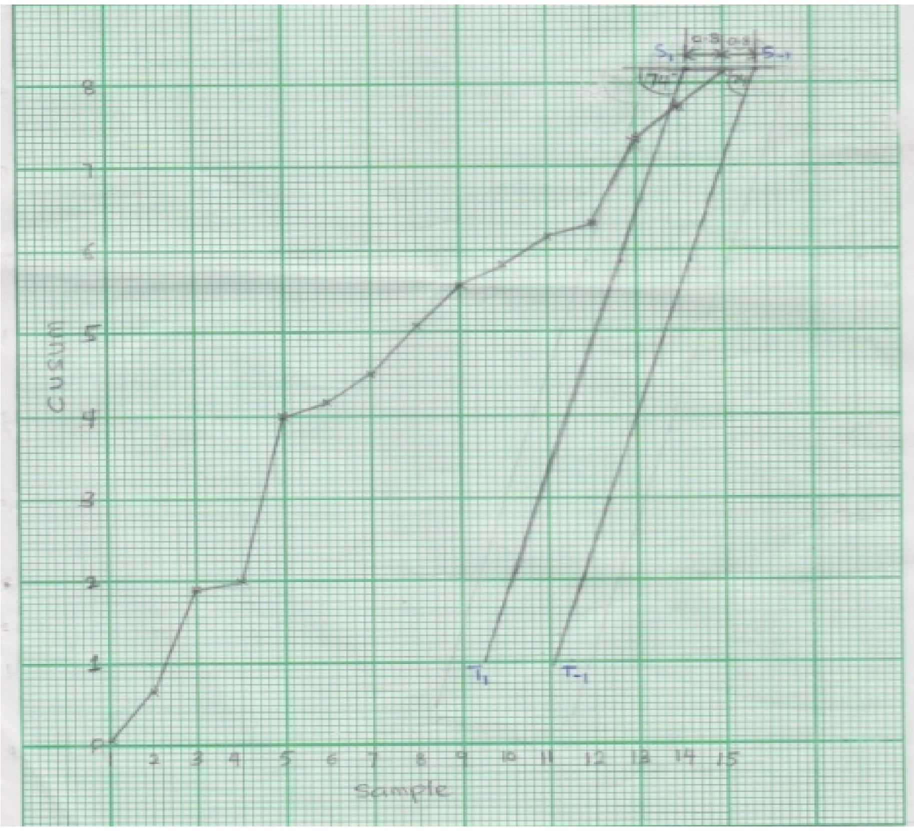

Figure 3. Unified CUSUM plot for the simulated hypothetical data

7. Conclusions

The results of the study showed that as the values of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$ increase the size of the mask angle (θ) increases. Also, increasing values of $\frac{\gamma_1}{\gamma_0}$ and $\frac{c_1}{c_0}$ increases the size of the mask angle (θ). Furthermore, when there is a negative shift in both parameters, that is where $\gamma_1 < \gamma_0$ and $c_1 < c_0$, it can be established from the second table of Table 1 that the mask angle (θ) increases as the values of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$ decreases.

With regard to the lead distance it can be determined that increasing the values of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$, the lead distance decreases given a fixed value of α. Again, the value of the lead distance increases with decreasing values of α and a fixed value of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$. However, When there is a negative shift in the parameters of the distribution, that is $\gamma_1 < \gamma_0$ and $c_1 < c_0$, it can be established that the lead distance decreases as the value of alpha (α) increases and when the value of $(\gamma_1 - \gamma_0)$ and $(c_1 - c_0)$ decreases for a fixed value of α, the value of lead distance increases.

On the ARL, it was established that as $(\gamma_1 - \gamma_0)$ increases in value, the ARL tends to decrease with any given value of α. Also, as the value of α decreases, the ARL also increases for a fixed value of $(\gamma_1 - \gamma_0)$. Again when there is a negative shift $(\gamma_1 < \gamma_0)$, the ARL decreases as α increases and when $(\gamma_0 - \gamma_1)$ decreases, the ARL also increases for any given value of α.

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