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A Zero-Truncated Poisson-Shanker Distribution and Its Applications

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Abstract In this paper, a zero-truncated Poisson-Shanker distribution (ZTPSD) has been introduced by taking the zero-truncated version of Poisson-Shanker distribution (PSD) of Shanker (2016). The expression for the *r*th factorial moment of the ZTPSD has been obtained and hence the first four moments about origin and central moments have been obtained. The expressions for coefficient of variation, skewness, kurtosis and index of dispersion have been presented. Method of maximum likelihood estimation and the method of moments have been discussed for estimationg the parameter of ZTPSD. Three examples of observed data sets have been presented to test the goodness of fit of ZTPSD over zero-truncated Poisson distribution (ZTPD) and Zero truncated Poisson-Lindley distribution (ZTPLD).

Keywords Zero-truncation, Poisson-Shanker distribution, Moments, Parameter estimation, Goodness of fit

1. Introduction

Suppose $P_0\left(x;\theta\right)$ is the original distribution. Then the zero-truncated version of $P_0\left(x;\theta\right)$ is defined as

$$P_1\left(x;\theta\right)=\frac{P_0\left(x;\theta\right)}{1-P_0\left(0;\theta\right)}\quad ;x=1,2,3,... \tag{1.1}$$

In probability theory, zero-truncated distribution is a certain class of discrete distribution whose support is the set of positive integers. When the data to be modeled originate from a mechanism which generates data that structurally excludes zero counts, zero-truncated distribution is the appropriate choice.

The probability mass function (pmf) of Poisson-Lindley distribution (PLD) given by

$$P_2\left(x;\theta\right)=\frac{\theta^2\left(x+\theta+2\right)}{\left(\theta+1\right)^{x+3}}\;;x=0,1,2,....,\theta>0 \tag{1.2}$$

has been introduced by Sankaran (1970) to model count data. It is a Poisson mixture of Lindley (1958) distribution having probability density function (pdf)

$$f\left(x,\theta\right)=\frac{\theta^2}{\theta+1}\left(1+x\right)e^{-\theta x}\;;x>0,\theta>0 \tag{1.3}$$

Ghitany *et al* (2008 a) has detailed study on various properties, estimation of parameter and application of Lindley distribution. Ghitany and Al-Mutairi (2009) have discussed the estimation methods of PLD along with simulation study and application. Shanker *et al* (2015) has detailed discussion on the applications of exponential and Lindley distributions for modeling lifetimes data from different fields of knowledge. Shanker and Hagos (2015) have discussed the applications of Poisson-Lindley distribution in biological sciences.

Using (1.1) and (1.2), Ghitany *et al* (2008 b) obtained zero-truncated Poisson -Lindley distribution (ZTPLD) defined by its pmf

$$P_3\left(x;\theta\right)=\frac{\theta^2}{\theta^2+3\theta+1}\frac{x+\theta+2}{\left(\theta+1\right)^x}\quad ;x=1,2,3,....,\theta>0 \tag{1.4}$$

Shanker *et al* (2015) has done comparative study on applications of ZTPLD and zero-truncated Poisson distribution (ZTPD) on different real data sets from different fields of knowledge and showed that ZTPLD gives better fit than ZTPD in almost all data sets relating to demography, biological sciences and social sciences.

The zero-truncated Poisson distribution (ZTPD) is defined by its pmf

$$P_4\left(x;\theta\right)=\frac{e^{-\theta}\theta^x}{\left(1-e^{-\theta}\right)x!}\quad ;x=1,2,3,....,\theta>0 \tag{1.5}$$

In this paper, a zero-truncated Poisson-Shanker distribution (ZTPSD) has been introduced by taking the zero-truncated version of Poisson-Shanker distribution (PSD)

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suggested by Shanker (2016). The first four moments about origin and the moments about mean of ZTPSD have been obtained and thus expressions for coefficient of variation, skewness, kurtosis, and index of dispersion have been given. The estimation of its parameter has been discussed using maximum likelihood estimation and method of moments. Finally, applications of ZTPSD to three observed real data sets have been given to test its goodness of fit over zero-truncated Poisson distribution (ZTPD) and Zero-truncated Poisson-Lindley distribution (ZTPLD).

2. Zero-Truncated Poisson-Shanker Distribution (ZTPSD)

The Poisson-Shanker distribution defined by its pmf

$$P_0(x;\theta)=\frac{\theta^2}{\theta^2+1}\frac{x+\left(\theta^2+\theta+1\right)}{(\theta+1)^{x+2}}\quad ;x=0,1,2,3,...,\ \theta>0 \tag{2.1}$$

has been introduced by Shanker (2016) to model count data. The distribution arises from the Poisson distribution when its parameter λ follows Shanker distribution introduced by Shanker (2015) with pdf

$$f(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\quad x>0\ ,\theta>0 \tag{2.2}$$

Shanker (2015) showed that (2.2) is a better model than both exponential and Lindley (1958) distributions for modeling lifetime data from biomedical science and engineering.

Using (1.1) and (2.1), the pmf of zero-truncated Poisson-Shanker distribution (ZTPSD) can be obtained as

$$P_5(x;\theta)=\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\frac{x+\left(\theta^2+\theta+1\right)}{(\theta+1)^{x}}\quad ;x=1,2,3,...,\ \theta>0 \tag{2.3}$$

The ZTPSD can also be obtained from the size-biased Poisson distribution (SBPD) with p.m.f.

$$g(x\,|\,\lambda)=\frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)}\quad ;x=1,2,3,...\ ,\lambda>0 \tag{2.4}$$

when its parameter λ follows a distribution having p.d.f.

$$h(\lambda;\theta)=\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\left[(\theta+1)\lambda+\left(\theta^2+\theta+1\right)\right]e^{-\theta x}\quad ;\lambda>0,\theta>0 \tag{2.5}$$

Thus the p.m.f. of ZTPSD can be obtained as

$$P(x;\theta)=\int_0^\infty g(x\,|\,\lambda)\cdot h(\lambda;\theta)\,d\lambda$$

$$=\int_0^\infty\frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)}\cdot\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\left[(\theta+1)\lambda+\left(\theta^2+\theta+1\right)\right]e^{-\theta\lambda}d\lambda\quad ;\lambda>0,\theta>0$$

$$=\frac{\theta^2}{\left(\theta^3+\theta^2+2\theta+1\right)\Gamma(x)}\int_0^\infty e^{-(\theta+1)\lambda}\cdot\left[(\theta+1)\lambda^x+\left(\theta^2+\theta+1\right)\lambda^{x-1}\right]d\lambda$$

$$=\frac{\theta^2}{\left(\theta^3+\theta^2+2\theta+1\right)\Gamma(x)}\left[\frac{(\theta+1)\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{\left(\theta^2+\theta+1\right)\Gamma(x)}{(\theta+1)^{x}}\right]$$

$$= \frac{\theta^2}{\left(\theta^3+\theta^2+2\theta+1\right)}\left[\frac{x}{(\theta+1)^x}+\frac{\left(\theta^2+\theta+1\right)}{(\theta+1)^x}\right]$$

$$= \frac{\theta^2}{\left(\theta^3+\theta^2+2\theta+1\right)}\cdot\frac{x+\left(\theta^2+\theta+1\right)}{(\theta+1)^x}\;;x=1,2,3,...,\;\theta>0 \tag{2.6}$$

which is the p.m.f. of ZTPSD, as obtained earlier in (2.3). The main motivation of considering a continuous distribution in (2.5) is that moments of ZTPSD (2.3) can easily be obtained from the size-biased Poisson mixture of the continuous distribution using (2.6).

To have a comparative study on the nature and behavior of ZTPSD and ZTPLD, several graphs of their pmf's have been drawn for varying values of their parameter θ and presented in figure 1.

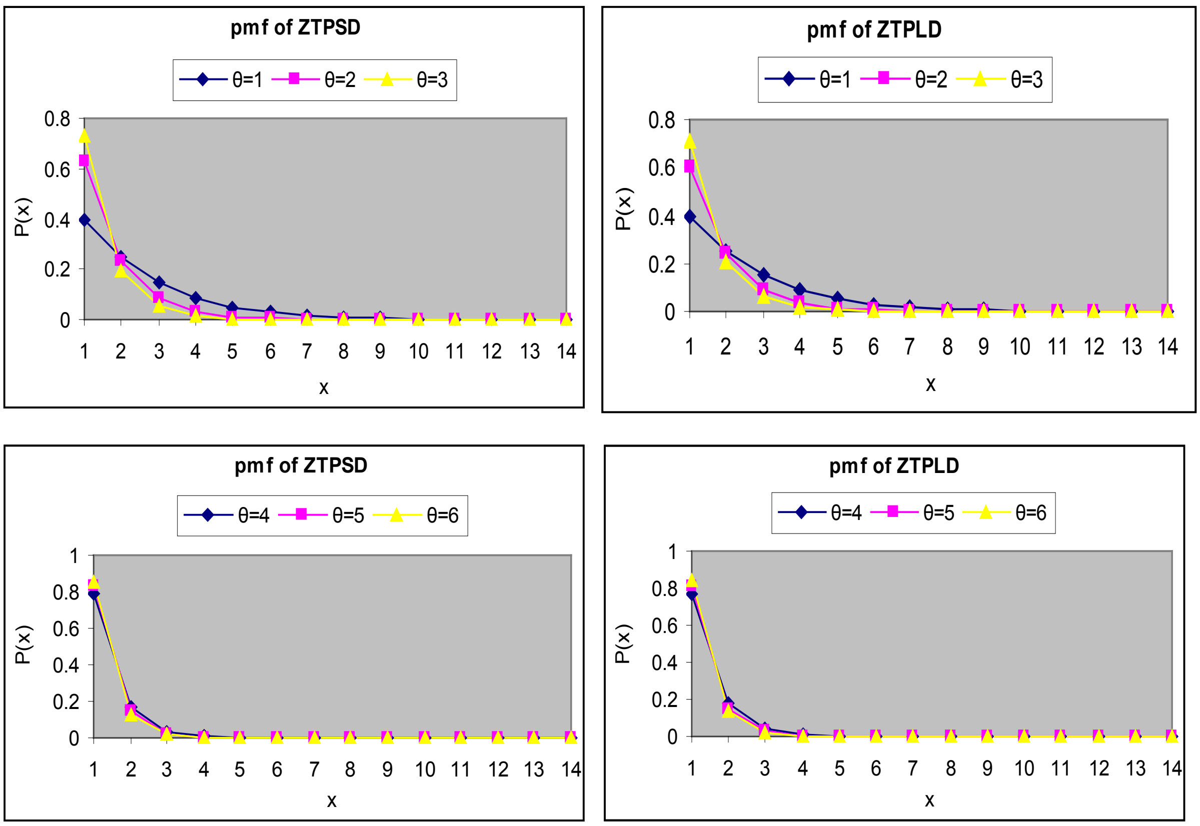

Figure 1. Graphs of ZTPSD and ZTPLD for varying values of the parameter θ

A separate graph of ZTPSD for varying values of parameter θ has been shown in figure 2.

Since $\frac{P_5(x+1;\theta)}{P_5(x;\theta)}=\left(\frac{1}{\theta+1}\right)\left(\frac{\theta}{\theta+1}\right)^x\left(1+\frac{1}{x}\right)\left[1+\frac{1}{x+\left(\theta^2+\theta+1\right)}\right]$ is a decreasing function of x, $P_5(x;\theta)$ is log-concave. Therefore, ZTPSD is unimodal, has increasing failure rate (IFR), and hence increasing failure rate average (IFRA). It is new better than used (NBU), new better than used in expectation (NBUE), and has decreasing mean residual life (DMRL). Detailed discussions about the definitions of these aging concepts are available in Barlow and Proschan (1981).

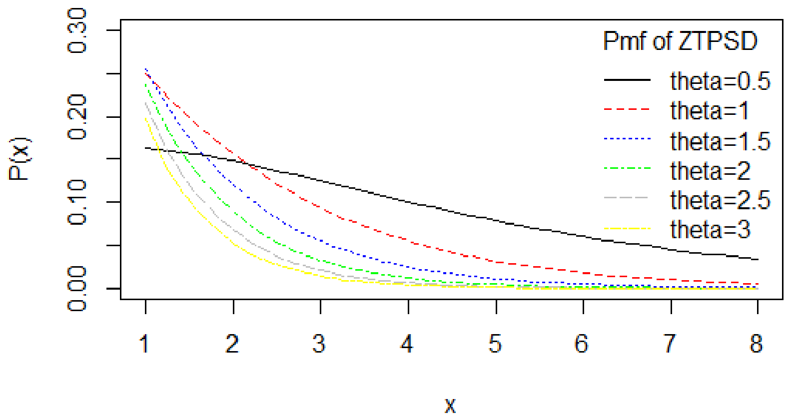

Figure 2. Graphs of ZTPSD for varying values of the parameter θ

3. Moments and Related Measures

From (2.6), the rth factorial moment about origin of ZTPSD (2.3) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right], \text{ where } \quad X^{(r)} = X\left(X-1\right)\left(X-2\right)...\left(X-r+1\right)$$

$$= \int_0^\infty \left[\sum_{x=1}^\infty x^{(r)} \frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)}\right] \cdot \frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\left[\left(\theta+1\right)\lambda+\left(\theta^2+\theta+1\right)\right]e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\int_0^\infty \lambda^{r-1}\left[\sum_{x=1}^\infty x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right] \cdot \left[\left(\theta+1\right)\lambda+\left(\theta^2+\theta+1\right)\right]e^{-\theta\lambda}d\lambda$$

Taking $x+r$ in place of x, we get

$$\mu_{(r)}' = \frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\int_0^\infty \lambda^{r-1}\left[\sum_{x=1}^\infty (x+r)\frac{e^{-\lambda}\lambda^{x}}{x!}\right] \cdot \left[\left(\theta+1\right)\lambda+\left(\theta^2+\theta+1\right)\right]e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\int_0^\infty \lambda^{r-1}(\lambda+r) \cdot \left[\left(\theta+1\right)\lambda+\left(\theta^2+\theta+1\right)\right]e^{-\theta\lambda}d\lambda$$

$$= \frac{r!(\theta+1)\left\{\theta^3+\theta^2+(r+1)\theta+(r+1)\right\}}{\theta^r\left(\theta^3+\theta^2+2\theta+1\right)}; \quad r=1,2,3,... \tag{3.1}$$

Taking r = 1,2,3 and 4 in (3.1), first four factorial moments about origin can be obtained and then using the relationship between factorial moments and moments about origin, the first four moments about origin of ZTPSD can be obtained as

$$\mu_1' = \frac{\left(\theta+1\right)\left(\theta^3+\theta^2+2\theta+2\right)}{\theta\left(\theta^3+\theta^2+2\theta+1\right)}$$

$$\mu_2' = \frac{\left(\theta+1\right)\left(\theta^4+3\theta^3+4\theta^2+8\theta+6\right)}{\theta^2\left(\theta^3+\theta^2+2\theta+1\right)}$$

$$\mu_3' = \frac{\left(\theta+1\right)\left(\theta^5+7\theta^4+14\theta^3+26\theta^2+42\theta+24\right)}{\theta^3\left(\theta^3+\theta^2+2\theta+1\right)}$$

$$\mu_4' = \frac{(\theta+1)\left(\theta^6+15\theta^5+50\theta^4+104\theta^3+210\theta^2+264\theta+120\right)}{\theta^4\left(\theta^3+\theta^2+2\theta+1\right)}$$

Using the relationship between moments about origin and moments about mean, the moments about mean of ZTPSD are thus obtained as

$$\mu_2 = \sigma^2 = \frac{(\theta+1)\left(\theta^6+2\theta^5+6\theta^4+9\theta^3+10\theta^2+8\theta+2\right)}{\theta^2\left(\theta^3+\theta^2+2\theta+1\right)^2}$$

$$\mu_3 = \frac{(\theta+1)\left(\theta^{10}+5\theta^9+16\theta^8+42\theta^7+79\theta^6+120\theta^5+137\theta^4+116\theta^3+72\theta^2+26\theta+4\right)}{\theta^3\left(\theta^3+\theta^2+2\theta+1\right)^3}$$

$$\mu_4 = \frac{(\theta+1)\begin{pmatrix}\theta^{14}+11\theta^{13}+54\theta^{12}+199\theta^{11}+566\theta^{10}+1271\theta^9+2333\theta^8+3471\theta^7\\+4231\theta^6+4177\theta^5+3268\theta^4+1972\theta^3+842\theta^2+216\theta+24\end{pmatrix}}{\theta^4\left(\theta^3+\theta^2+2\theta+1\right)^4}$$

The coefficient of variation (C.V), coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis $\left(\beta_2\right)$ and index of dispersion $\left(\gamma\right)$ of ZTPSD are thus obtained as

$$C.V. = \frac{\sigma}{\mu} = \sqrt{\frac{\theta^6+2\theta^5+6\theta^4+9\theta^3+10\theta^2+8\theta+2}{(\theta+1)\left(\theta^3+\theta^2+2\theta+2\right)^2}}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\begin{pmatrix}\theta^{10}+5\theta^9+16\theta^8+42\theta^7+79\theta^6+120\theta^5+137\theta^4\\+116\theta^3+72\theta^2+26\theta+4\end{pmatrix}}{\sqrt{(\theta+1)\left(\theta^6+2\theta^5+6\theta^4+9\theta^3+10\theta^2+8\theta+2\right)^3}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{\begin{pmatrix}\theta^{14}+11\theta^{13}+54\theta^{12}+199\theta^{11}+566\theta^{10}+1271\theta^9+2333\theta^8+3471\theta^7+4231\theta^6\\+4177\theta^5+3268\theta^4+1972\theta^3+842\theta^2+216\theta+24\end{pmatrix}}{(\theta+1)\left(\theta^6+2\theta^5+6\theta^4+9\theta^3+10\theta^2+8\theta+2\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu} = \frac{\theta^6+2\theta^5+6\theta^4+9\theta^3+10\theta^2+8\theta+2}{\theta\left(\theta^6+2\theta^5+5\theta^4+7\theta^3+7\theta^2+6\theta+2\right)}$$

It can be easily verified that the ZTPSD is over dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under dispersed $\left(\mu>\sigma^2\right)$ for $\theta<(=)>\theta^*=1.24166$ respectively. Note that the ZTPLD is over dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under dispersed $\left(\mu>\sigma^2\right)$ for $\theta<(=)>\theta^*=1.258627$ respectively.

The nature of coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion of ZTPSD for varying values of the parameter θ are shown in figure 3.

To study the comparative nature of mean, variance, coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion of ZTPSD and ZTPLD for varying values of the parameter θ, a table for their values have been prepared and presented in table 1.

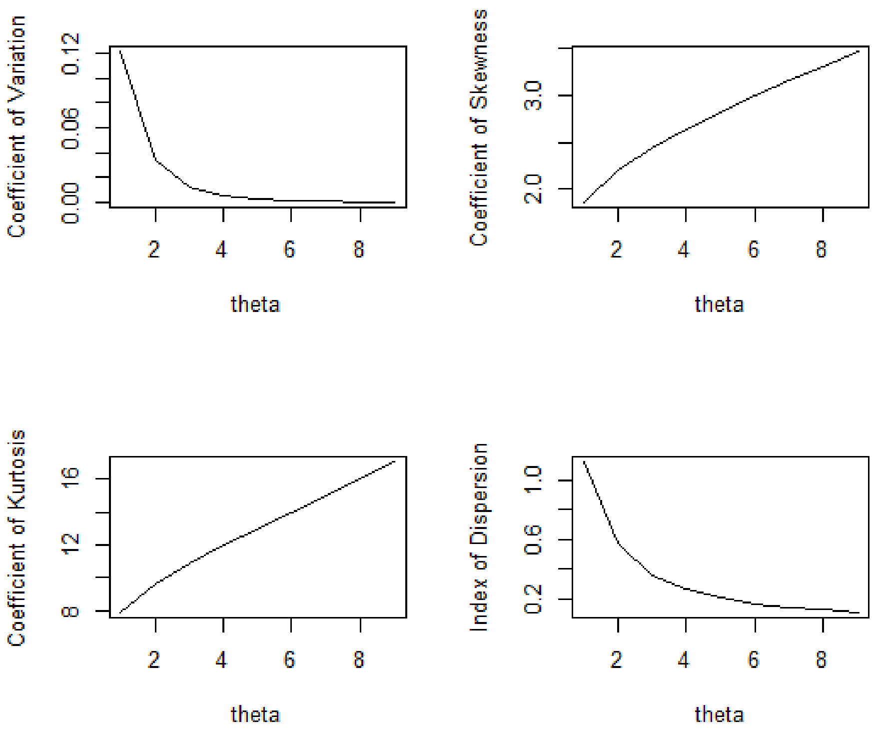

Figure 3. Graphs of coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion of ZTPSD for varying values of the parameter θ

Table 1. Numerical values of μ_1', μ_2, CV, $\sqrt{\beta_1}$, β_2 and γ of ZTPSD and ZTPLD

	Values of θ for ZTPSD					
	1	**2**	**3**	**4**	**5**	**6**
μ_1'	2.4	1.588235	1.364341	1.264045	1.207453	1.171069
μ_2	3.04	0.918685	0.495163	0.33337	0.250379	0.200295
CV	0.726483	0.603488	0.515764	0.456773	0.414409	0.382166
$\sqrt{\beta_1}$	1.865512	2.203138	2.438585	2.640492	2.8252	2.997752
β_2	7.924515	9.683935	10.89364	11.95264	12.97332	13.9824
γ	1.266667	0.578431	0.362932	0.263733	0.207361	0.171036

	Values of θ for ZTPLD					
	1	**2**	**3**	**4**	**5**	**6**
μ_1'	2.4	1.636364	1.403509	1.293103	1.229268	1.187879
μ_2	3.04	1.004132	0.556479	0.375297	0.280119	0.222277
CV	0.726483	0.612372	0.531507	0.473756	0.430551	0.396895
$\sqrt{\beta_1}$	1.865512	2.130283	2.346599	2.542147	2.724109	2.895526
β_2	7.924515	9.227252	10.31062	11.32858	12.32376	13.31073
γ	1.266667	0.613636	0.396491	0.29023	0.227875	0.187121

4. Estimation of Parameter

4.1. Maximum Likelihood Estimate (MLE) of Parameter: Let $x_1, x_2, ..., x_n$ be a random sample of size n from the ZTPSD (2.3) and let f_x be the observed frequency in the sample corresponding to $X = x(x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the ZTPSD (2.3) is given by

$$L = \left(\frac{\theta^2}{\theta^3 + \theta^2 + 2\theta + 1} \right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}} \prod_{x=1}^{k} \left[x + \left(\theta^2 + \theta + 1 \right) \right]^{f_x}$$

The log likelihood function is given by

$$\log L = n \log \left(\frac{\theta^2}{\theta^3 + \theta^2 + 2\theta + 1} \right) - \sum_{x=1}^{k} x\, f_x \log(\theta+1) + \sum_{x=1}^{k} f_x \log \left[x + \left(\theta^2 + \theta + 1 \right) \right]$$

and the log likelihood equation is thus obtained as

$$\frac{d \log L}{d\theta} = \frac{2n}{\theta} - \frac{n\left(3\theta^2 + 2\theta + 2\right)}{\theta^3 + \theta^2 + 2\theta + 1} - \frac{n\bar{x}}{\theta+1} + \sum_{x=1}^{k} \frac{(2\theta+1) f_x}{x + \left(\theta^2 + \theta + 1\right)}$$

The maximum likelihood estimate $\hat{\theta}$ of θ is the solution of the equation $\frac{d \log L}{d\theta} = 0$ and is given by the solution of the following non-linear equation

$$\sum_{x=1}^{k} \frac{(2\theta+1) f_x}{x + \left(\theta^2 + \theta + 1\right)} - \frac{n\bar{x}}{\theta+1} - \frac{n\left(\theta^3 - 2\theta - 2\right)}{\theta\left(\theta^3 + \theta^2 + 2\theta + 1\right)} = 0$$

where $\bar{x}$ is the sample mean. This non-linear equation can be solved by any numerical iteration methods such as Newton-Raphson method, Bisection method, Regula –Falsi method etc. In the following theorem, the consistency and asymptotic normality of maximum likelihood estimator of ZTPSD has been established.

Theorem: The ML estimator $\hat{\theta}$ of θ of the ZTPSD is consistent and asymptotically normal. That is $\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} N\left[0,\ I^{-1}(\theta)\right]$, where

$I(\theta) = \frac{-4\theta^6 - 3\theta^5 + 9\theta^3 + 6\theta^2 + 6\theta + 2}{\theta^2\left(\theta^3 + \theta^2 + 2\theta + 1\right)^2} + \frac{\theta^2 (2\theta+1)^2}{\theta^3 + \theta^2 + 2\theta + 1} \int_0^1 \frac{t^{\theta^2+\theta+1}}{\theta + 1 - t} dt$ is the Fisher's information about θ.

Proof: The ZTPSD satisfies the regularity conditions under which the ML estimator $\hat{\theta}$ of θ is consistent and asymptotically normal [see Hogg *et al* (2005), chapter 6]. We have

$$I(\theta) = E\left[-\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta) \right]$$

$$= E\left[\frac{2}{\theta^2} - \frac{3\theta^4 + 4\theta^3 + 2\theta^2 - 2\theta + 2}{\left(\theta^3 + \theta^2 + 2\theta + 1\right)^2} - \frac{X}{(\theta+1)^2} - \frac{2}{X + \left(\theta^2 + \theta + 1\right)} + \frac{(2\theta+1)^2}{\left\{X + \left(\theta^2 + \theta + 1\right)\right\}^2} \right]$$

$$= E\left[\frac{-\theta^6+8\theta^4+14\theta^3+10\theta^2+8\theta+2}{\theta^2\left(\theta^3+\theta^2+2\theta+1\right)^2}-\frac{X}{(\theta+1)^2}-\frac{2}{X+\left(\theta^2+\theta+1\right)}+\frac{(2\theta+1)^2}{\left\{X+\left(\theta^2+\theta+1\right)\right\}^2}\right]$$

$$=\frac{-\theta^6+8\theta^4+14\theta^3+10\theta^2+8\theta+2}{\theta^2\left(\theta^3+\theta^2+2\theta+1\right)^2}-\frac{\mu}{(\theta+1)^2}-2E\left[\frac{1}{X+\left(\theta^2+\theta+1\right)}\right]$$

$$+(2\theta+1)^2 E\left[\frac{1}{\left\{X+\left(\theta^2+\theta+1\right)\right\}^2}\right] \quad (4.1.3)$$

where

$$\mu=\frac{(\theta+1)\left(\theta^3+\theta^2+2\theta+2\right)}{\theta\left(\theta^3+\theta^2+2\theta+1\right)} \quad (4.1.4)$$

$$E\left[\frac{1}{X+\left(\theta^2+\theta+1\right)}\right]=\sum_{x=1}^{\infty}\frac{1}{x+\left(\theta^2+\theta+1\right)}\cdot\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\cdot\frac{x+\left(\theta^2+\theta+1\right)}{(\theta+1)^x}$$

$$=\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\sum_{x=1}^{\infty}\frac{1}{(\theta+1)^x}=\frac{\theta}{\theta^3+\theta^2+2\theta+1} \quad (4.1.5)$$

and

$$E\left[\frac{1}{\left\{X+\left(\theta^2+\theta+1\right)\right\}^2}\right]=\sum_{x=1}^{\infty}\frac{1}{\left\{x+\left(\theta^2+\theta+1\right)\right\}^2}\cdot\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\cdot\frac{x+\left(\theta^2+\theta+1\right)}{(\theta+1)^x}$$

$$=\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\sum_{x=1}^{\infty}\frac{1}{\left\{x+\left(\theta^2+\theta+1\right)\right\}}\frac{1}{(\theta+1)^x}$$

$$=\frac{\theta^2}{\theta^3+\theta^2+2\theta+1}\int_0^1\frac{t^{\theta^2+\theta+1}}{\theta+1-t}dt \quad (4.1.6)$$

Using equations (4.1.4), (4.1.5), and (4.1.6) in (4.1.3), we get

$$I(\theta)=\frac{-4\theta^6-3\theta^5+9\theta^3+6\theta^2+6\theta+2}{\theta^2\left(\theta^3+\theta^2+2\theta+1\right)^2}+\frac{\theta^2(2\theta+1)^2}{\theta^3+\theta^2+2\theta+1}\int_0^1\frac{t^{\theta^2+\theta+1}}{\theta+1-t}dt\,.$$

4.2. Method of Moment Estimate (MOME) of Parameter: Equating the population mean to the corresponding sample mean, MOME $\tilde{\theta}$ of θ of ZTPSD is the solution of the following non-linear equation

$$(1-\bar{x})\theta^4+(2-\bar{x})\theta^3+(3-2\bar{x})\theta^2+(4-\bar{x})\theta+2=0$$

where $\bar{x}$ is the sample mean.

5. Applications to Real Data Sets

The ZTPSD has been fitted to a number of data - sets to test its goodness of fit over ZTPD and ZTPLD. The maximum likelihood estimate (MLE) has been used to fit the ZTPSD, ZTPLD and ZTPD. Three examples of observed data-sets, for which the ZTPD, ZTPLD and ZTPSD has been fitted, are presented. The first data-set is the number of flower heads as per the number of fly eggs reported by Finney and Varley (1955), the second data- set is the number of yeast cell counts observed per mm square reported by Student (1907) and the third data-set is animal abundance data of Keith and Meslow (1968) regarding the distribution of snowshoe hares captured over 7 days.

Table 2. The numbers of counts of flower heads as per the number of fly eggs reported by Finney and Varley (1955)

Number of fly eggs	Observed Frequency	Expected Frequency		
		ZTPD	ZTPLD	ZTPSD
1	22	15.3	26.8	26.3
2	18	21.9	19.8	19.8
3	18	20.8	13.9	14.1
4	11	14.9	9.5	9.7
5	9	8.5	6.4	6.5
6	6	4.1 }	4.2 }	4.2 }
7	3	1.7 }	2.7 }	2.7 }
8	0	0.6 }	1.7 }	1.7 }
9	1	0.3 }	3.0 }	2.9 }
Total	88	88.0	88.0	88.0
ML estimate		$\hat{\theta} = 2.860402$	$\hat{\theta} = 0.718559$	$\hat{\theta} = 0.740658$
χ^2		6.677	3.743	3.281
d.f.		4	4	4
p-value		0.1540	0.4419	0.5119

Table 3. Number of yeast cell counts observed per mm square reported by Student (1907)

Number of yeast cell counts per mm square	Observed Frequency	Expected Frequency		
		ZTPD	ZTPLD	ZTPSD
1	128	121.3	127.6	128.0
2	37	49.2	40.9	40.5
3	18	13.3 }	12.8 }	12.8 }
4	3	2.7 }	4.0 }	4.0 }
5	1	0.4 }	1.2 }	1.2 }
6	0	0.1 }	0.5 }	0.5 }
Total	187	187.0	187.0	187.0
ML estimate		$\hat{\theta} = 0.811276$	$\hat{\theta} = 2.667323$	$\hat{\theta} = 2.453055$
χ^2		5.228	1.034	0.960
d.f.		1	1	1
p-value		0.0222	0.3092	0.3272

Table 4. Number of Snowshoe hares counts captured over 7 days reported by Keith and Meslow (1968)

Number of times hares caught	Observed Frequency	Expected Frequency		
		ZTPD	ZTPLD	ZTPSD
1	184	176.6	182.6	183.0
2	55	66.0	55.3	54.8
3	14	16.6 }	16.6	16.3
4	4	3.1 }	4.8 }	4.8 }
5	4	0.7 }	1.9 }	2.1 }
Total	261	261.0	261.0	261.0
ML estimate		$\hat{\theta} = 0.756171$	$\hat{\theta} = 2.863957$	$\hat{\theta} = 2.625645$
χ^2		2.45	0.61	0.49
d.f.		1	2	2
p-value		0.1175	0.7371	0.7827

6. Conclusions

A zero-truncated Poisson-Shanker distribution (ZTPSD) has been introduced. The first four moments about origin and the moments about mean have been obtained. The expressions for coefficient of variation, skewness, kurtosis and index of dispersion of ZTPSD have been given. The method of maximum likelihood and the method of moments have also been discussed for estimating its parameter. Three examples of observed real data- sets have been given to test its goodness of fit over ZTPD and ZTPLD. The goodness of fit of ZTPSD gives quite satisfactory fit over both ZTPD and ZTPLD and thus ZTPSD can be considered an important distribution for modeling zero-truncated count data.

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