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# The Variance Weighted Gradient Projection (VWGP): An Alternative Optimization Approach of Response Surfaces

**Otaru O. A. P., Iwundu M. P.***

Department of Mathematics and Statistics, University of Port Harcourt, Port Harcourt, Nigeria

**Abstract** A Variance Weighted Gradient Projection (VWGP) method that uses experimental design principles based on variance, and simultaneously optimizes several response surfaces is introduced as an alternative to popularly used one-at-a-time optimization methods. The method relies on the general line search equation whose components are the starting (initial) point of search, the direction of search, the step-length and the point arrived at the $j^{th}$ iteration. At the end of an iteration, the optimizer(s) reached are successively added to the previous immediate design measure(s). The use of projection operator scheme that allows the projection of design points from one design space to another is employed. Unlike most existing optimization methods which use guess initial point of search, a weighted average of design points selected from the design region and sufficiently spread over the entire region, is proposed as the initial point of search. By the choice of the proposed initial point of search, two limitations of the guess point methods are overcome, namely cycling and possible lack of convergence. Results obtained using the VWGP simultaneous optimization method have been compared with the BFGS Quasi-Newton algorithm and the VWGP method is seen comparatively efficient in locating the optimizers of several response surfaces.

**Keywords** Simultaneous Optimization, Response Surfaces, Variance Weighted Gradients, Projection Operator, Quasi-Newton's Algorithm

## 1. Introduction

The optimization of response surfaces plays a vital role in locating the best set of factor levels to achieve some goals. Scientists and researchers have explored Response Surface Methodology (RSM) in various areas such as chemical, manufacturing and processing industries, managerial studies, engineering and science disciplines, etc. RSM is essentially a sequential procedure and entails response surface designs, modeling and optimization. It usually explores the relationships between several explanatory variables and one or more response variables. The response model used is only an approximation to the true unknown model. In many practical situations, the need for a solution of a set of approximating polynomial response functions arises. Attempts have been made by several researchers to address the optimization of response surfaces. In fact, various gradient and non-gradient based optimization methods have been formulated to address the need. Some of the methods include Newton's Method (NM), Quasi-Newton's Method (QNM), Genetic Algorithm (GA), Particle Swarm Algorithm (PSA), Mesh Adaptive Direct Search Method, etc. Comparatively, the gradient methods seem to be faster than the non-gradient methods in terms of the required iterative steps to convergence. The Newton's Method, which was developed by [7] optimizes polynomial functions when the functions are differentiable. Due to the computational rigour involved in the Newton's method, [10] gave an improvement to the Newton's Method thus overcoming the tedious computational challenges. For historical development of the Newton-Raphson method, see [11]. Quasi-Newton Methods are used as alternatives to the Newton's method and are frequently used in Non-Linear Programming to improve the computational speed of the Newton's method. The first Quasi Newton Method, though rarely used today, was developed by [2] and was later given attention by other researchers like [3]. Although the Quasi Newton Method has faster computational time than the Newton's Method, the convergence rate is however still slow. The most commonly used Quasi Newton algorithms are the SR1 formular, the BHHH method and the BFGS. The BFGS Quasi Newton Method, suggested independently by Broyden, Fletcher, Goldfarb and Shanno (See [1]), is the method used in the mathematical software, MATLAB. Although having their drawbacks, Newton's method as well as Quasi Newton Methods have been extensively and successfully used in single objective optimization problems as seen in [5] and [9]. As opposed to single-objective optimization problems, interests are moving deeply into multiobjective optimization

* Corresponding author:
mary.iwundu@uniport.edu.ng (Iwundu M. P.)
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problems involving two or more objective functions to be optimized simultaneously. For both single- and multiple-objective optimization problems, the use of gradient-based algorithms seem successfully utilized. [4] presented an extension of Newton's method for solving unconstrained multiobjective optimization problems. One very recent paper on the subject of multiobjective optimization is due to [12]. The paper presented a quasi-Newton's method for solving unconstrained multiobjective optimization problems when the objective functions are strongly convex. For the successes in using gradient-based algorithms, we present in this paper the Variance Weighted Gradient Projection (VWGP) method for simultaneously optimizing several response functions defined over different constraints and having disjoint feasible regions. The method is gradient-based, relying on weighted gradients of the response functions resulting from the variances of the design points. The method uses the properties of line equations, such as could be seen in [8] to obtain the optimum responses. The consideration for the Variance Weighted Gradient Projection algorithm stems from the successful use of the Variance Weighted Gradient algorithm in optimizing response surfaces defined over same feasible region and having same constraints as in [6]. In handling the problem involving different regions and different constraints, a projection scheme that allows the projection of design points from one design region to another is proposed. The projection scheme enhances fast convergence of the algorithm to the desired optima as measured by the number of iterative moves made. It is possible to have one or more response functions converge before others, however, all functions are certain to converge to the required optima.

## 2. Methodology

The proposed method relies on the properties of line equation, having a starting point of search, a direction of search and a step length. The three parameters of the line search are optimally chosen.

### 2.1. The Starting Point of Search

The starting point of search is obtained as the weighted mean of the selected design points from the design region, where the weighting factor is a function of the variance of the design points.

For an N-point design measure, $\xi_N$, comprising of the design points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ we define a weighted mean vector

$$\overline{\underline{x}} = \sum_{i=1}^{N} w_i \underline{x}_i = X'\underline{w}$$

where,

$$X = \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_N \end{pmatrix}$$

is an $N \times n$ design matrix and

$$\underline{w} = (w_1, w_2, \cdots, w_N)'; w_i \geq 0, \sum_{i=1}^{N} w_i = 1$$

is the vector of weights associated with the design points.

The optimal starting point $\overline{\underline{x}}^*$ is obtained by minimizing the norm

$$\overline{\underline{x}}'\overline{\underline{x}} = \sum_{i=1}^{N-1} w_i^2 \underline{x}_i' \underline{x}_i + (1 - \sum_{i=1}^{N-1} w_i)^2 \underline{x}_N' \underline{x}_N$$

To minimize $\overline{\underline{x}}'\overline{\underline{x}}$ with respect to $w_i$, we solve

$$\partial \overline{\underline{x}}'\overline{\underline{x}} \big/ \partial w_i = 0$$

The optimal starting point of search is

$$\overline{\underline{x}}^* = \sum_{i=1}^{N} w_i^* \underline{x}_i$$

### 2.2. The Direction of Search

The direction of search is in the direction of minimum variance and is computed as

$$\underline{d} = \sum_{i=1}^{N} \theta_i g_i ; \theta_i \in (0,1)$$

where $\underline{g}_i$ is the gradient vector and $\theta_i$ weighting factor vector.

The direction variance is given as

$$\mathrm{Var}(\underline{d}) = \sum_{i=1}^{N} \theta_i^2 V_i$$

where $V_i$ is the variance of the function at the $i^{th}$ support point.

The weighting vector $\{\theta_i\}_{i=(1,N)}$ is obtained from the partial derivatives

$$\partial V(\underline{d}) / \partial \theta_i = 0 ; \ \theta_i = A^{-1}B$$

where

$$A = \begin{bmatrix} V_1 + V_N & V_N & \cdots & V_N \\ V_N & V_2 + V_N & \cdots & V_N \\ \vdots & \vdots & \cdots & \vdots \\ V_N & V_N & \cdots & V_{N-1} + V_N \end{bmatrix}_{N-1}$$

$$B = \begin{bmatrix} V_N \\ V_N \\ \vdots \\ V_N \end{bmatrix}_{N-1}$$

and

$$\theta_N = 1 - \sum_{i=1}^{N-1} \theta_i$$

The normalized weighting factor vector $\theta_i^*$ is given as

$$\theta_i^* = \theta_i \left( \sum_{i=2}^{N} \theta_i^2 \right)^{-\frac{1}{2}} ; \sum_{i=1}^{N} \theta_i^{*2} = 1$$

### 2.3. The Step Length of Search

Given that the response function is constrained and the optimizer lies on the boundary of the feasible region governed by

$$\underline{c}_s \underline{x} = b_s ; s = 1, 2, \cdots, S$$

The optimal step-length $\rho^*$ of the function $f(.)$ is obtained as the minimum distance covered in an average step for *s* number of constraint and is defined as:

$$\rho^* = \min_s \left\{ \frac{\underline{c}_s \underline{x}^* - b_s}{\underline{c}_s \underline{d}^*} \right\}$$

Where $\underline{c}_s$ and $\underline{x}$ are vectors and $b_s$ is a scalar for s number of constraints, while $\underline{x}^*$ is the optimal starting point and $\underline{d}^*$ is the optimal direction vector.

### 2.4. The Variance Weighted Gradient Projection (VWGP) Approach

Let

$$f_r(x) = \underline{a}' \underline{x} + e$$

be the n-variate, p-parameter polynomial of degree m, defined on the r$^{th}$ feasible regions $\tilde{X}_r$ supported by s constraints. Such that

$$\underline{x} \in \tilde{X}_r = \{ \underline{c}_{sr}' \underline{x} \leq, =, \geq b_{sr} \} ; r = 1, 2, \cdots, R ; s = 1, 2, \cdots, S$$

where $\underline{a}$ is a *p*-component vector of known coefficients, independent random variable $\underline{x} \in \tilde{X}_r$ and *e* is the random error component assumed normally and independently distributed with zero mean and constant variance. While $\underline{c}_s$ is a component vector of know coefficients and $b_s$ is a scalar for *s* number of constraints in the r$^{th}$ region.

The Variance Weighted Gradient (VWG) method is given by the following sequential steps:

i) From $\tilde{X}_r$ obtain the design measures $\xi_{rN_r}^j$ which are made up of support points from respective regions such that

$$\xi_{rN_r+j}^{(j)} = \begin{pmatrix} \frac{\underline{x}_{r1}}{N_r} \\ \frac{\underline{x}_{r2}}{N_r} \\ \vdots \\ \frac{\underline{x}_{rN_r}}{N_r} \end{pmatrix}$$

where $N_r$ support points are spread evenly in $\tilde{X}_r$

ii) From the support points that make up the design measure compute *R* starting points as, the arithmetic mean vectors.

$$\underline{x}_r^* = (\underline{x}_{r1}, \underline{x}_{r2}, \cdots, \underline{x}_{rn})' ; \underline{x}_{ri} = \frac{\sum_{l=1}^{N_r} x_{rli}}{N_r} ; \begin{pmatrix} i = (1, n) \\ l = (1, N_r) \\ r = (1, R) \end{pmatrix}$$

iii) Obtain the n-component gradient function for the r$^{th}$ region.

$$\underline{g}_r = \left\{ \frac{\partial f_r(x)}{\partial x_i} \right\} = \begin{pmatrix} g_{r1}(x) \\ g_{r2}(x) \\ \vdots \\ g_{rn}(x) \end{pmatrix}$$

where

$g_{ri}(x) = \underline{q}' \underline{x} + e$ is an $(m-1)$ degree polynomial; $i = (1, n)$

$\underline{q}$ is a t-component vector of known coefficients

iv) Compute the corresponding r gradient vectors, by substituting each design point defined on the r$^{th}$ region to the gradient function $\underline{g}_r$ as

$$\underline{g}_{rl} = \begin{pmatrix} \underline{g}_{r1} \\ \underline{g}_{r2} \\ \vdots \\ \underline{g}_{rN_r} \end{pmatrix}$$

v) Using the gradient function and design measures obtain the corresponding design matrices $X_r$.

$$X_r(\xi^{(j)}_{rN_r+j}) = \begin{pmatrix} x_{r11} & x_{r12} \cdots & x_{r1t} \\ x_{r21} & x_{r22} \cdots & x_{r2t} \\ \vdots & & \\ x_{rN_r1} & x_{rN_r2} \cdots & x_{rN_rt} \end{pmatrix} = \begin{pmatrix} \underline{\underline{x}}_{r1}' \\ \underline{\underline{x}}_{r2}' \\ \vdots \\ \underline{\underline{x}}_{rN_r}' \end{pmatrix}$$

In order to form the design matrix $X_r$, a single polynomial that combines the respective gradient function $\underline{g}_{ri}(x)$ associated with each response function $f_r(x)$ is $\underline{g}_r(x)$.

vi) Compute the variances of each *l* design point $\underline{x}_{rl}$ defined on the r$^{th}$ region as

$$V_{rl} = \underline{\underline{x}}_{rl}' M_r^{-1} \underline{\underline{x}}_{rl} \ ; M_r = X_r(\xi^{(j)}_{rN_r+j})' X_r(\xi^{(j)}_{rN_r+j})$$

vii) Obtain the direction vector in the r$^{th}$ region as

$$\underline{d}_r = \sum_{l=1}^{N_r} \theta_{rl}\, \underline{g}_{rl} \ ; \theta_{rl} \in (0,1)$$

and the normalize direction vector $\underline{d}_r^* = \sum_{l=1}^{N_r} \theta_{rl}^* \underline{g}_{rl}$ such that $\underline{d}_r^{*\prime} \underline{d}_r = 1$.

viii) Compute the step-length $\rho_r^*$ as

$$\rho_r^* = \min_s \left\{ \frac{\underline{c}_{sr} \bar{\underline{x}}_r^* - b_{sr}}{\underline{c}_{sr} \underline{d}_r^*} \right\}$$

with $\bar{\underline{x}}_r^*$, $\rho_r^*$ and $\underline{d}_r^*$ make a move to

$$\underline{x}_{r,j}^* = \bar{\underline{x}}_r^* - \rho_r^* \underline{d}_r^*$$

using $\underline{x}_{r,j}^*$ evaluate the projection operator $P_r$ as

$$P_r = \underline{x}_r^* (\underline{x}_r^{*\prime} \underline{x}_r^*) \underline{x}_r^{*\prime}$$

and obtain the projector optimizers for each region $\underline{\underline{x}}_r^*$ as

$$\underline{\underline{x}}_1^* = P_1 \underline{x}_R^*$$
$$\underline{\underline{x}}_2^* = P_2 \underline{x}_{R-1}^*$$
$$\vdots$$
$$\underline{\underline{x}}_{R-1}^* = P_{R-1} \underline{x}_2^*$$
$$\underline{\underline{x}}_R^* = P_R \underline{x}_1^*$$

ix) To make a next move set $j = j+1$ and define the design measure as

$$\xi^{(j)}_{rN_r+j} = \begin{pmatrix} \xi_{rN_r}^{(0)} \\ \underline{\underline{x}}_r^* \end{pmatrix}$$

and repeat the process from step (ii) then obtain

$$\underline{x}_{r,j+1}^* = \bar{\underline{x}}_r^* - \rho_r^* \underline{d}_r^*$$

x) If $f_r(\underline{x}_{r,j}^*) \le f_r(\underline{x}_{r,j+1}^*)$.

where $f_r(\underline{x}_{r,j}^*)$ is the r$^{th}$ feasible region at the j$^{th}$ step and $f_r(\underline{x}_{r,j+1,})$ is the r$^{th}$ feasible region at the next j+1$^{th}$ step.
then set the optimizers as

$$\underline{x}_{r,j+1}^* = \underline{x}_r^*$$

and STOP. Else, set $j = j+1$ and repeat the process from (ii).

**Theorem**

Let the direction vector $\underline{d} = \sum_{i=1}^{N} \theta_i g_i$, the weighting factor vector $\theta_i$ is inversely proportional to the variance vector of the function $V(f(.))$; where $V(f(.)) = \underline{x}' M^{-1}(\xi_N)\underline{x}$.

**Proof: (By induction)**

$$Var(\underline{d}) = \sum_{i=1}^{N} \theta_i^2 V_i \ ; \underline{d} = \sum_{i=1}^{N} \theta_i\, \underline{g}_i$$

$$Var(\underline{d}) = \theta_1^2 V_1 + \theta_2^2 V_2 + \theta_3^2 V_3 + \ldots + \theta_{N-1}^2 V_{N-1} (1 - \theta_1 - \theta_2 - \theta_3 - \ldots - \theta_{N-1})^2 V_N$$

$$\text{Min } V(\underline{d}) \text{ w.r.t } \theta \Rightarrow \partial V(\underline{d}) \big/ \partial \theta_i \ ; i = 1,2,3,\cdots,N$$

$$\Rightarrow \partial V(\underline{d}) \big/ \partial \theta = \left\{ \begin{array}{l} 2\theta_1 V_1 - 2(1-\theta_1-\theta_2-\theta_3-\ldots-\theta_{N-1})V_N \\ 2\theta_2 V_2 - 2(1-\theta_1-\theta_2-\theta_3-\ldots-\theta_{N-1})V_N \\ 2\theta_3 V_3 - 2(1-\theta_1-\theta_2-\theta_3-\ldots-\theta_{N-1})V_N \\ \vdots\vdots \quad - \ \vdots \qquad \cdots\cdots\cdots\cdots \qquad \vdots \\ 2\theta_{N-1} V_{N-1} - 2(1-\theta_1-\theta_2-\theta_3-\ldots-\theta_{N-1})V_N \end{array} \right\} = 0$$

$$\Rightarrow \begin{pmatrix} V_1+V_N & V_N & V_N & \cdots & V_N \\ V_N & V_2+V_N & V_N & \cdots & V_N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ V_N & V_N & V_N & \cdots & V_{N-1}+V_N \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-1} \end{pmatrix} = \begin{pmatrix} V_N \\ V_N \\ \vdots \\ V_N \end{pmatrix}$$

$$\Rightarrow (D+JV_N)\theta = \underline{1}V_N; D = \text{diag}\{V_1, V_2, V_3, \cdots, V_N\}, \underline{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ and } J = \underline{11}'$$

$$\Rightarrow \theta = (D+\underline{11}'V_N)^{-I}\underline{1}V_N ; (D+\underline{11}'V_N)^{-I} = D^{-I} - D^{-I}\underline{1}V_N(1+\underline{1}'D^{-I}\underline{1}V_N)^{-I}\underline{1}'D^{-I}$$

$$\Rightarrow \theta = \left[D^{-I} - D^{-I}\underline{1}V_N(1+\underline{1}'D^{-I}\underline{1}V_N)^{-I}\underline{1}'D^{-I}\right]\underline{1}\,V_N$$

$$\Rightarrow \theta = D^{-I}\underline{1}'V_N - D^{-I}\underline{1}V_N(1+\underline{1}'D^{-I}\underline{1}V_N)^{-I}\underline{1}'D^{-I}\underline{1}V_N$$

$$\Rightarrow \theta = D^{-I}\underline{1}V_N - D^{-I}\underline{1}V_N(1+a)^{-I}a \; ; \; \underline{1}'D^{-I}\underline{1}V_N = a$$

$$\theta = D^{-I}\underline{1}V_N\left(1-\frac{a}{1+a}\right)$$

$$\theta = D^{-I}\underline{1}V_N\left(\frac{1}{1+a}\right)$$

$$\Rightarrow \theta = D^{-I}\underline{1}\left(\frac{V_N}{1+\underline{1}'D^{-I}\underline{1}V_N}\right)$$

$$\Rightarrow \theta = D^{-I}\underline{1}k; \; D^{-I} = \text{diag}\{V_1^{-I}, V_2^{-I}, V_3^{-I}, \cdots, V_N^{-I}\}, \; k = \frac{V_N}{1+\underline{1}'D^{-I}\underline{1}V_N}$$

$$\therefore \theta_i \propto \frac{1}{V_i}$$

## 3. Results

We present the working of the algorithm using a numerical illustration involving two response functions. The problem is;

$$\text{Minimize } f_1(x_1,x_2) = x_1^3 - 2x_1x_2 - 3x_1^2 - 4 \text{ subject to } 2x_1 + x_2 \le 3 \; ; x_1, x_2 \le 0$$

and

$$\text{Minimize } f_2(x_1,x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6 \text{ subject to } x_1 + x_2 \le 2 \; ; x_1, x_2 \le 0$$

The design regions are defined by

$$\tilde{X}_1 = \{2x_1 + x_2 \le 3 \; ; x_1, x_2 \le 0\}$$

$$\tilde{X}_2 = \{x_1 + x_2 \le 2 \; ; x_1, x_2 \le 0\}$$

To solve this problem using the VWGP technique, we select design points from $\tilde{X}_1$ and $\tilde{X}_2$ to make up the respective initial design measures as

$$\xi_{1,10}{}^{(0)} = \begin{pmatrix} x_1 & x_2 \\ 1.5 & 0 \\ 1 & 1 \\ 1 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 1 \\ 1 & 0 \\ 0 & 0.5 \\ 0 & 1.5 \\ 0.5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \xi_{2,15}{}^{(0)} = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0 \\ 2 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 1.5 & 0.5 \\ 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \\ 0 & 1.5 \\ 0.5 & 1.5 \\ 0 & 2 \end{pmatrix}$$

Taking the weighted averages, the initial starting points for the two functions are, respectively,

$$\underline{\bar{x}}_1^* = \begin{pmatrix} 0.6 \\ 0.6 \end{pmatrix} \quad \text{and} \quad \underline{\bar{x}}_2^* = \begin{pmatrix} 0.66 \\ 0.66 \end{pmatrix}$$

The gradient vectors are obtained as

$$\underline{g}_1 = \begin{Bmatrix} g_{11} = 3x_1^2 - 6x_1 - 2x_2 \\ g_{12} = -2x_1 \end{Bmatrix} \qquad \underline{g}_2 = \begin{Bmatrix} g_{21} = 4x_1 - 2x_2 - 6 \\ g_{22} = -2x_1 + 4x_2 \end{Bmatrix}$$

Hence the gradient functions are, respectively,

$$\underline{g}_1(x_1, x_2) = b_0 x_1^2 - b_1 x_1 - b_2 x_2 \quad \text{and} \quad \underline{g}_2(x_1, x_2) = b_0 + b_1 x_1 + b_2 x_2$$

Using the gradient functions and the design measures, we form the corresponding design matrices as

$$X_1 = \begin{pmatrix} 1.5 & 0 & 2.25 \\ 1 & 1 & 1 \\ 1 & 0.5 & 1 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 1 & 0.25 \\ 1 & 0 & 1 \\ 0 & 0.5 & 0 \\ 0 & 1.5 & 0 \\ 0.5 & 0 & 0.25 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 1 & 0 \\ 1 & 1.5 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0.5 \\ 1 & 0.5 & 0.5 \\ 1 & 1 & 0.5 \\ 1 & 1.5 & 0.5 \\ 1 & 0 & 1 \\ 1 & 0.5 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1.5 \\ 1 & 0.5 & 1.5 \\ 1 & 0 & 2 \end{pmatrix}$$

The variances associated with the design points of each design measure are computed as illustrated below for the design point

$$\underline{x}_{11} = (1.5 \quad 0 \quad 2.25)^T.$$

The variance of $\underline{x}_{11}$ denoted $V_{11}$ is computed as

$$V_{11} = (1.5 \quad 0 \quad 2.25)\left(\begin{pmatrix} 1.5 & 0 & 2.25 \\ 1 & 1 & 1 \\ 1 & 0.5 & 1 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 1 & 0.25 \\ 1 & 0 & 1 \\ 0 & 0.5 & 0 \\ 0 & 1.5 & 0 \\ 0.5 & 0 & 0.25 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1.5 & 0 & 2.25 \\ 1 & 1 & 1 \\ 1 & 0.5 & 1 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 1 & 0.25 \\ 1 & 0 & 1 \\ 0 & 0.5 & 0 \\ 0 & 1.5 & 0 \\ 0.5 & 0 & 0.25 \\ 0 & 1 & 0 \end{pmatrix}\right)^{-1} (1.5 \quad 0 \quad 2.25)^T$$

$$= 0.8571$$

The process continues similarly in obtaining the variance of each design point from the two regions. Thus the vectors of variances are respectively,

$$V_{1i} = \begin{pmatrix} 0.8571 \\ 0.2460 \\ 0.1905 \\ 0.1905 \\ 0.2460 \\ 0.2460 \\ 0.0556 \\ 0.5000 \\ 0.2460 \\ 0.2222 \end{pmatrix} \quad \text{and} \quad V_{2l} = \begin{pmatrix} 0.3702 \\ 0.2080 \\ 0.1545 \\ 0.2097 \\ 0.3737 \\ 0.1832 \\ 0.0857 \\ 0.0857 \\ 0.2009 \\ 0.1492 \\ 0.1063 \\ 0.1726 \\ 0.2707 \\ 0.2875 \\ 0.5470 \end{pmatrix}$$

The weighting factors are obtained as

$$\theta_{rN-1} = A_r^{-1} B_r$$

where

$$A_r = \begin{pmatrix} V_{r1} + V_{rN} & V_{rN} & \cdots & V_{rN} \\ V_{rN} & V_{r2} + V_{rN} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ V_{rN} & V_{rN} & V_{rN} & V_{rN-1} + V_{rN} \end{pmatrix} \text{ and } B_r = \begin{pmatrix} V_{rN} \\ V_{rN} \\ \vdots \\ V_{rN} \end{pmatrix}$$

Thus

$$\theta_{11} = \begin{pmatrix} 1.0793 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 \\ 0.2222 & 0.4682 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.4127 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 & 0.4127 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.4682 & 0.2222 & 0.2222 & 0.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.4682 & 0.2222 & 0.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2778 & 0.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.7222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.2222 & 0.4682 \end{pmatrix}^{-1} \begin{pmatrix} 0.2222 \\ 0.2222 \\ 0.2222 \\ 0.2222 \\ 0.2222 \\ 0.2222 \\ 0.2222 \\ 0.2222 \\ 0.2222 \end{pmatrix} = \begin{pmatrix} 0.0222 \\ 0.0775 \\ 0.1001 \\ 0.1001 \\ 0.0775 \\ 0.0775 \\ 0.3431 \\ 0.0381 \\ 0.0775 \end{pmatrix}$$

and

$$\theta_{1N_1} = 1 - \sum_{l=1}^{N_1 - 1} \theta_{1l} = 0.0858$$

$\theta_{1l}$ is normalized as

$$\theta_{1l}^{*} = \begin{pmatrix} 0.0537 \\ 0.1874 \\ 0.2420 \\ 0.2420 \\ 0.1874 \\ 0.1874 \\ 0.8293 \\ 0.0922 \\ 0.1874 \\ 0.2075 \end{pmatrix}$$

$$\theta_{21} = \begin{pmatrix} 0.9172 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.7550 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.7015 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.7567 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.9207 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.7302 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.6327 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.6327 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.7479 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.6962 & 0.5470 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.6533 & 0.5470 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.7196 & 0.5470 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.8177 & 0.5470 \\ 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.5470 & 0.8345 \end{pmatrix}^{-1} \begin{pmatrix} 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \\ 0.5470 \end{pmatrix}$$

Thus

$$\theta_{21} = \begin{bmatrix} 0.0313 & 0.0558 & 0.0751 & 0.0553 & 0.0310 & 0.0633 & 0.1355 & 0.1355 & 0.0578 & 0.0778 & 0.1092 & 0.0672 & 0.0429 & 0.0403 \end{bmatrix}'$$

and

$$\theta_{2N_2} = 1 - \sum_{l=1}^{N_2-1} \theta_{21} = 0.0220 .$$

$\theta_{21}$ is normalized as

$$\theta_{21}^* = \left[0.1078\ \ 0.1922\ \ 0.2587\ \ 0.1905\ \ 0.1067\ \ 0.2180\ \ 0.4667\ \ 0.4667\ \ 0.1991\ \ 0.2680\ \ 0.3761\ \ 0.2315\ \ 0.1477\ \ 0.1388\ \ 0.0757\right]'$$

The computed statistics are summarized in Table 1.

**Table 1.** Summary statistics for the first and second response functions at respective design points in the first iteration

| Runs | Design Points $N_1 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_1 = (g_{1i}, g_{2i})$ | Variances $(V_{11})$ | Weighting Factor $(\theta_{11}^*)$ | Design Points $N_2 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_2 = (g_{1i}, g_{2i})$ | Variances $(V_{21})$ | Weighting Factor $(\theta_{21}^*)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.5, 0 | -2.25, -3 | 0.8571 | 0.0537 | 0, 0 | -6, 0 | 0.3702 | 0.1078 |
| 2 | 1, 1 | -5, -2 | 0.2460 | 0.1874 | 0.5, 0 | -4, -1 | 0.2080 | 0.1922 |
| 3 | 1, 0.5 | -4, -2 | 0.1905 | 0.2420 | 1, 0 | -2, -2 | 0.1545 | 0.2587 |
| 4 | 0.5, 0.5 | -3.25, -1 | 0.1905 | 0.2420 | 1.5, 0 | 0, -3 | 0.2097 | 0.1905 |
| 5 | 0.5, 1 | -4.25, -1 | 0.2460 | 0.1874 | 2, 0 | 2, -4 | 0.3737 | 0.1067 |
| 6 | 1, 0 | -3, -2 | 0.2460 | 0.1874 | 0, 0.5 | -6.5, 2 | 0.1832 | 0.2180 |
| 7 | 0, 0.5 | -1, 0 | 0.0556 | 0.8293 | 0.5, 0.5 | -4.5, 1 | 0.0857 | 0.4667 |
| 8 | 0, 1.5 | -3, 0 | 0.5000 | 0.0922 | 1, 0.5 | -2.5, 0 | 0.0857 | 0.4667 |
| 9 | 0.5, 0 | -2.25, -1 | 0.2460 | 0.1874 | 1.5, 0.5 | -0.5 -1 | 0.2009 | 0.1991 |
| 10 | 0, 1 | -2, 0 | 0.2222 | 0.2075 | 0, 1 | -7, 4 | 0.1492 | 0.2680 |
| 11 | | | | | 0.5, 1 | -5, 3 | 0.1063 | 0.3761 |
| 12 | | | | | 1, 1 | -5, 2 | 0.1726 | 0.2315 |
| 13 | | | | | 0, 1.5 | -7.5, 6 | 0.2707 | 0.1477 |
| 14 | | | | | 0.5, 1.5 | -5.5, 5 | 0.2875 | 0.1388 |
| 15 | | | | | 0, 2 | -8, 8 | 0.5470 | 0.0757 |

The direction vectors are respectively

$$\underline{d}_1 = \begin{pmatrix} -6.1119 \\ -2.0094 \end{pmatrix}$$

and

$$\underline{d}_2 = \begin{pmatrix} -13.4341 \\ 3.8462 \end{pmatrix}$$

The normalized direction vectors are respectively

$$\underline{d}_1^* = \begin{pmatrix} -0.9499 \\ -0.3123 \end{pmatrix}$$

and

$$\underline{d}_2^* = \begin{pmatrix} -0.9613 \\ -0.2752 \end{pmatrix}$$

The associated step-lengths are computed as

$$\rho_1{}^* = \left\{ \frac{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 0.60 \\ 0.60 \end{pmatrix} - 3}{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} -0.9499 \\ -0.3123 \end{pmatrix}} \right\} = \frac{-1.2}{-2.2121} = 0.5424$$

$$\rho_2{}^* = \left\{ \frac{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0.66 \\ 0.66 \end{pmatrix} - 2}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -0.9613 \\ 0.2752 \end{pmatrix}} \right\} = \frac{-0.68}{-0.6861} = 0.9911$$

With the starting points of search, the directions of search and the step-lengths we make a first move for each of the functions to

$$\underline{x}_1^* = \begin{pmatrix} 0.6 \\ 0.6 \end{pmatrix} - \left(0.5424\right)\begin{pmatrix} -0.9499 \\ -0.3123 \end{pmatrix} = \begin{pmatrix} 1.12 \\ 0.77 \end{pmatrix}$$

$$\underline{x}_2^* = \begin{pmatrix} 0.66 \\ 0.66 \end{pmatrix} - \left(0.9911\right)\begin{pmatrix} -0.9613 \\ 0.2752 \end{pmatrix} = \begin{pmatrix} 1.61 \\ 0.39 \end{pmatrix}$$

**Table 2.** Summary statistics for the two response functions at respective design points in the second iteration

| Runs | Design Points $N_1 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_1 = (g_{1i}, g_{2i})$ | Variances $(V_{11})$ | Weighting Factor $(\theta_{11}^*)$ | Design Points $N_2 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_2 = (g_{1i}, g_{2i})$ | Variances $(V_{21})$ | Weighting Factor $(\theta_{21}^*)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.5, 0 | -2.25, -3 | 0.7203 | 0.0614 | 0, 0 | -6, 0 | 0.3651 | 0.0929 |
| 2 | 1, 1 | -5, -2 | 0.2216 | 0.1996 | 0.5, 0 | -4, -1 | 0.1997 | 0.1700 |
| 3 | 1, 0.5 | -4, -2 | 0.1650 | 0.2679 | 1, 0 | -2, -2 | 0.1406 | 0.2414 |
| 4 | 0.5, 0.5 | -3.25, -1 | 0.1892 | 0.2339 | 1.5, 0 | 0, -3 | 0.1875 | 0.1810 |
| 5 | 0.5, 1 | -4.25, -1 | 0.2449 | 0.1806 | 2, 0 | 2, -4 | 0.3406 | 0.0996 |
| 6 | 1, 0 | -3, -2 | 0.2195 | 0.2015 | 0, 0.5 | -7, 2 | 0.1986 | 0.1709 |
| 7 | 0, 0.5 | -1, 0 | 0.0555 | 0.7969 | 0.5, 0.5 | -5, 1 | 0.0850 | 0.3994 |
| 8 | 0, 1.5 | -3, 0 | 0.4999 | 0.0883 | 1, 0.5 | -3, 0 | 0.0775 | 0.4380 |
| 9 | 0.5, 0 | -2.25, -1 | 0.2445 | 0.1808 | 1.5, 0.5 | -1 -1 | 0.1762 | 0.1926 |
| 10 | 0, 1 | -2, 0 | 0.2222 | 0.1989 | 0, 1 | -8, 4 | 0.1428 | 0.2377 |
| 11 | 1.2, 0.3 | -3.44, -2.46 | 0.2171 | 0.2038 | 0.5, 1 | -6, 3 | 0.0809 | 0.4196 |
| 12 | | | | | 1, 1 | -4, 2 | 0.1251 | 0.2713 |
| 13 | | | | | 0, 1.5 | -9, 6 | 0.1977 | 0.1717 |
| 14 | | | | | 0.5, 1.5 | -7, 5 | 0.1875 | 0.1810 |
| 15 | | | | | 0, 2 | -10, 8 | 0.3632 | 0.0934 |
| 16 | | | | | 1.2, 0.8 | -2.8, 0.8 | 0.1319 | 0.2574 |

The resulting values of response functions are, respectively, $f_1 = -8.083$ and $f_2 = 0.572$.

In order to make a second move, a projector operator $P_r$ is formulated by projecting design points from one design space or feasible regions to another feasible region. The resulting projector operator $P_r$ is used to compute the projector optimizers $\underline{\underline{x}}_1^*$ and $\underline{\underline{x}}_2^*$ which are added to the design measure of respective response function and the process of search continues.

The computations are as follows:

$$P_1 = \begin{pmatrix} 1.12 \\ 0.77 \end{pmatrix} \left[ \begin{pmatrix} 1.12 \\ 0.77 \end{pmatrix}' \begin{pmatrix} 1.12 \\ 0.77 \end{pmatrix} \right]^{-I} \begin{pmatrix} 1.12 \\ 0.77 \end{pmatrix}' = \begin{pmatrix} 0.6790 & 0.4668 \\ 0.4668 & 0.3210 \end{pmatrix}$$

$$\underline{\underline{x}}_2^* = P_1 \underline{x}_2^* = \begin{pmatrix} 0.6790 & 0.4668 \\ 0.4668 & 0.3210 \end{pmatrix} \begin{pmatrix} 1.61 \\ 0.39 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 0.8 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1.61 \\ 0.39 \end{pmatrix} \left[ \begin{pmatrix} 1.61 \\ 0.39 \end{pmatrix}' \begin{pmatrix} 1.61 \\ 0.39 \end{pmatrix} \right]^{-I} \begin{pmatrix} 1.61 \\ 0.39 \end{pmatrix}' = \begin{pmatrix} 0.9446 & 0.2288 \\ 0.2288 & 0.0554 \end{pmatrix}$$

$$\underline{\underline{x}}_1^* = P_2 \underline{x}_1^* = \begin{pmatrix} 0.9446 & 0.2288 \\ 0.2288 & 0.0554 \end{pmatrix} \begin{pmatrix} 1.12 \\ 0.77 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 0.3 \end{pmatrix}$$

Adding the projector optimizers to the initial design measures result in the following augmented design measures

$$\xi_{1,11}^{(1)} = \begin{pmatrix} 1.5 & 0 \\ 1 & 1 \\ 1 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 1 \\ 1 & 0 \\ 0 & 0.5 \\ 0 & 1.5 \\ 0.5 & 0 \\ 1.2 & 0.3 \end{pmatrix} \text{ and } \xi_{2,16}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0 \\ 2 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 1.5 & 0.5 \\ 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \\ 0 & 1.5 \\ 0.5 & 1.5 \\ 0 & 2 \\ 1.2 & 0.8 \end{pmatrix}$$

With the new design measures, the computed statistics are summarized in Table 2.

The starting points of search at the second iteration are respectively

$$\overline{\underline{x}}_1^* = \begin{pmatrix} 0.657 \\ 0.582 \end{pmatrix}$$

and

$$\underline{\bar{x}}_2^* = \begin{pmatrix} 0.700 \\ 0.675 \end{pmatrix}$$

As previously described, the directions of search and the step-lengths of search at the second iteration are computed similarly as

$$\underline{d}_1 = \begin{pmatrix} -6.9081 \\ -2.6191 \end{pmatrix}$$

$$\underline{d}_2 = \begin{pmatrix} -16.1968 \\ 4.5961 \end{pmatrix}$$

and normalized as

$$\underline{d}_1^* = \begin{pmatrix} -0.9350 \\ -0.3545 \end{pmatrix}$$

$$\underline{d}_2^* = \begin{pmatrix} -0.9620 \\ 0.2729 \end{pmatrix}$$

Compute the step-lengths of search are respectively

$$\rho_1^* = \left\{ \frac{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 0.6570 \\ 0.572 \end{pmatrix} - 3}{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} -0.9350 \\ -0.3545 \end{pmatrix}} \right\} = \frac{-1.114}{-2.2245} = 0.5007$$

$$\rho_2^* = \left\{ \frac{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0.70 \\ 0.67 \end{pmatrix} - 2}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -0.9620 \\ 0.2729 \end{pmatrix}} \right\} = \frac{-0.63}{-0.6891} = 0.9142$$

The points reached at the second iteration are respectively

$$\underline{x}_1^* = \begin{pmatrix} 0.657 \\ 0.572 \end{pmatrix} - (0.5007)\begin{pmatrix} -0.9350 \\ -0.3545 \end{pmatrix} = \begin{pmatrix} 1.12 \\ 0.75 \end{pmatrix}$$

$$\underline{x}_2^* = \begin{pmatrix} 0.700 \\ 0.675 \end{pmatrix} - (0.9142)\begin{pmatrix} -0.9620 \\ 0.2729 \end{pmatrix} = \begin{pmatrix} 1.58 \\ 0.43 \end{pmatrix}$$

The corresponding values of objective functions are respectively $f_1 = -8.038$ and $f_2 = 0.529$. Since for the first function, $f_1(\underline{x}_{12}^*) = -8.038 > f_1(\underline{x}_{11}^*) = -8.083$, convergence is established for the first function. However, since for the second function $f_2(\underline{x}_{22}^*) = 0.529 < f_2(\underline{x}_{21}^*) = 0.572$, a next iteration is required. Using a projector operator, a new design point $(1.57 \quad 0.43)^T$ is obtained and thus the design measure is augmented as

$$\xi_{2,17}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0 \\ 2 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 1.5 & 0.5 \\ 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \\ 0 & 1.5 \\ 0.5 & 1.5 \\ 0 & 2 \\ 1.2 & 0.8 \\ 1.57 & 0.43 \end{pmatrix}$$

Continuing the process, the computed statistics using the new design measure are summarized in Table 3.

**Table 3.** Summary statistics for the second response function at respective design points in the third iteration

| Runs | Design Points $N_2 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_2 = (g_{1i}, g_{2i})$ | Variances $(V_{21})$ | Weighting Factor $(\theta_{21}^*)$ |
|---|---|---|---|---|
| 1 | 0, 0 | -6, 0 | 0.3606 | 0.0863 |
| 2 | 0.5, 0 | -4, -1 | 0.1997 | 0.1562 |
| 3 | 1, 0 | -2, -2 | 0.1343 | 0.2320 |
| 4 | 1.5, 0 | 0, -3 | 0.1644 | 0.1898 |
| 5 | 2, 0 | 2, -4 | 0.2900 | 0.1076 |
| 6 | 0, 0.5 | -7, 2 | 0.1960 | 0.1592 |
| 7 | 0.5, 0.5 | -5, 1 | 0.0845 | 0.3692 |
| 8 | 1, 0.5 | -3, 0 | 0.0684 | 0.4559 |
| 9 | 1.5, 0.5 | -1 -1 | 0.1479 | 0.2107 |
| 10 | 0, 1 | -8, 4 | 0.1417 | 0.2201 |
| 11 | 0.5, 1 | -6, 3 | 0.0794 | 0.3927 |
| 12 | 1, 1 | -4, 2 | 0.1127 | 0.2769 |
| 13 | 0, 1.5 | -9, 6 | 0.1974 | 0.1580 |
| 14 | 0.5, 1.5 | -7, 5 | 0.1845 | 0.1689 |
| 15 | 0, 2 | -10, 8 | 0.3632 | 0.0859 |
| 16 | 1.2, 0.8 | -2.8, 0.8 | 0.1139 | 0.2739 |
| 17 | 1.57 0.43 | -0.58 1.42 | 0.1613 | 0.1932 |

The starting point of search at the third iteration is

$$\underline{\bar{x}}_2^* = \begin{pmatrix} 0.7511 \\ 0.6605 \end{pmatrix}$$

The direction of search, normalized direction of search and step-length of search at the third iteration are respectively

$$\underline{d}_2 = \begin{pmatrix} -15.499 \\ 3.894 \end{pmatrix}$$

$$\underline{d}_2^* = \begin{pmatrix} -0.9698 \\ 0.2436 \end{pmatrix}$$

and

$$\rho_2{}^* = \left\{ \frac{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0.7511 \\ 0.6605 \end{pmatrix} - 2}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -0.9698 \\ 0.2436 \end{pmatrix}} \right\} = \frac{-0.5884}{-0.7262} = 0.8102$$

The point reached at the third iteration is

$$\underline{x}_2^* = \begin{pmatrix} 0.7511 \\ 0.6605 \end{pmatrix} - \begin{pmatrix} 0.8102 \end{pmatrix}\begin{pmatrix} -0.9698 \\ 0.2436 \end{pmatrix} = \begin{pmatrix} 1.54 \\ 0.46 \end{pmatrix}$$

The corresponding value of objective function is $f_2 = 0.509$

Since $f_2(\underline{x}_{23}^*) = 0.509 < f_2(\underline{x}_{22}^*) = 0.529$, a next iteration is required.

Using a projector operator, a new design point $(1.54 \quad 0.46)^T$ is obtained and thus the design measure is augmented as

$$\xi_{2,18}{}^{(3)} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0 \\ 2 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 1.5 & 0.5 \\ 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \\ 0 & 1.5 \\ 0.5 & 1.5 \\ 0 & 2 \\ 1.2 & 0.8 \\ 1.57 & 0.43 \\ 1.54 & 0.46 \end{pmatrix}$$

The computed statistics using the new design measure are summarized in Table 4.

**Table 4.** Summary statistics for the second response function at respective design points in the fourth iteration

| Runs | Design Points $N_2 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_2 = (g_{1i}, g_{2i})$ | Variances $(V_{21})$ | Weighting Factor $(\theta_{21}^*)$ |
|---|---|---|---|---|
| 1 | 0, 0 | -6, 0 | 0.3573 | 0.0801 |
| 2 | 0.5, 0 | -4, -1 | 0.1997 | 0.1433 |
| 3 | 1, 0 | -2, -2 | 0.1302 | 0.2201 |
| 4 | 1.5, 0 | 0, -3 | 0.1488 | 0.1927 |
| 5 | 2, 0 | 2, -4 | 0.2554 | 0.1121 |
| 6 | 0, 0.5 | -7, 2 | 0.1943 | 0.1576 |
| 7 | 0.5, 0.5 | -5, 1 | 0.0841 | 0.3407 |
| 8 | 1, 0.5 | -3, 0 | 0.0620 | 0.4625 |
| 9 | 1.5, 0.5 | -1 -1 | 0.1280 | 0.2239 |
| 10 | 0, 1 | -8, 4 | 0.1410 | 0.2031 |
| 11 | 0.5, 1 | -6, 3 | 0.0782 | 0.3665 |
| 12 | 1, 1 | -4, 2 | 0.1035 | 0.2771 |
| 13 | 0, 1.5 | -9, 6 | 0.1973 | 0.1453 |
| 14 | 0.5, 1.5 | -7, 5 | 0.1819 | 0.1576 |
| 15 | 0, 2 | -10, 8 | 0.3632 | 0.0790 |
| 16 | 1.2, 0.8 | -2.8, 0.8 | 0.1010 | 0.2837 |
| 17 | 1.57 0.43 | -0.58 -1.42 | 0.1397 | 0.2050 |
| 18 | 1.54 0.46 | -0.76 -1.24 | 0.1344 | 0.2135 |

The starting point of search at the fourth iteration is

$$\underline{\bar{x}}_2^* = \begin{pmatrix} 0.7950 \\ 0.6494 \end{pmatrix}$$

The direction of search, normalized direction of search and step-length of search at the fourth iteration are respectively

$$\underline{d}_2 = \begin{pmatrix} -14.8298 \\ 3.2315 \end{pmatrix}$$

$$\underline{d}_2^* = \begin{pmatrix} -0.9770 \\ 0.2129 \end{pmatrix}$$

and

$$\rho_2{}^* = \left\{ \frac{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0.7950 \\ 0.6494 \end{pmatrix} - 2}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -0.9770 \\ 0.2129 \end{pmatrix}} \right\} = \frac{-0.5556}{-0.7641} = 0.7271$$

The point reached at the fourth iteration is

$$\underline{x}_2^* = \begin{pmatrix} 0.7950 \\ 0.6494 \end{pmatrix} - \begin{pmatrix} 0.7271 \end{pmatrix}\begin{pmatrix} -0.9770 \\ 0.2129 \end{pmatrix} = \begin{pmatrix} 1.51 \\ 0.49 \end{pmatrix}$$

The corresponding value of objective function is $f_2 = 0.500$

Since $f_2(\underline{x}_{24}^*) = 0.500 < f_2(\underline{x}_{23}^*) = 0.509$, a next iteration is required.

Using a projector operator, a new design point $(1.51 \quad 0.49)^T$ is obtained and thus the design measure is augmented as

$$\xi_{2,19}^{(4)} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0 \\ 2 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 1.5 & 0.5 \\ 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \\ 0 & 1.5 \\ 0.5 & 1.5 \\ 0 & 2 \\ 1.2 & 0.8 \\ 1.57 & 0.43 \\ 1.54 & 0.46 \\ 1.51 & 0.49 \end{pmatrix}$$

The computed statistics using the new design measure are summarized in Table 5.

The starting point of search at the fifth iteration is

$$\underline{\bar{x}}_2^* = \begin{pmatrix} 0.8326 \\ 0.6410 \end{pmatrix}$$

The direction of search, normalized direction of search and step-length of search at the fifth iteration are respectively

$$\underline{d}_2 = \begin{pmatrix} -14.2093 \\ 2.6246 \end{pmatrix}$$

$$\underline{d}_2^* = \begin{pmatrix} -0.9833 \\ 0.1816 \end{pmatrix}$$

and

$$\rho_2^* = \left\{ \frac{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0.8326 \\ 0.6410 \end{pmatrix} - 2}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -0.9833 \\ 0.1816 \end{pmatrix}} \right\} = \frac{-0.5264}{-0.8017} = 0.6566$$

The point reached at the fifth iteration is

$$\underline{x}_2^* = \begin{pmatrix} 0.8326 \\ 0.6410 \end{pmatrix} - \begin{pmatrix} 0.6566 \end{pmatrix}\begin{pmatrix} -0.9833 \\ 0.1819 \end{pmatrix} = \begin{pmatrix} 1.478 \\ 0.521 \end{pmatrix}$$

The corresponding value of objective function is $f_2 = 0.503$

Since $f_2(\underline{x}_{25}^*) = 0.503 > f_2(\underline{x}_{24}^*) = 0.500$, convergence is established and hence no indication for a further search.

**Table 5.** Summary statistics for the second response function at respective design points in the fifth iteration

| Runs | Design Points $N_2 = (x_{1i}, x_{2i})$ | Gradient Vector $\underline{g}_2 = (g_{1i}, g_{2i})$ | Variances $(V_{21})$ | Weighting Factor $(\theta_{21}^*)$ |
|---|---|---|---|---|
| 1 | 0, 0 | -6, 0 | 0.3547 | 0.0742 |
| 2 | 0.5, 0 | -4, -1 | 0.1997 | 0.1318 |
| 3 | 1, 0 | -2, -2 | 0.1273 | 0.2068 |
| 4 | 1.5, 0 | 0, -3 | 0.1376 | 0.1913 |
| 5 | 2, 0 | 2, -4 | 0.2305 | 0.1143 |
| 6 | 0, 0.5 | -7, 2 | 0.1931 | 0.1365 |
| 7 | 0.5, 0.5 | -5, 1 | 0.0839 | 0.3140 |
| 8 | 1, 0.5 | -3, 0 | 0.0573 | 0.4597 |
| 9 | 1.5, 0.5 | -1 -1 | 0.1133 | 0.2326 |
| 10 | 0, 1 | -8, 4 | 0.1406 | 0.1873 |
| 11 | 0.5, 1 | -6, 3 | 0.0772 | 0.3414 |
| 12 | 1, 1 | -4, 2 | 0.0964 | 0.2731 |
| 13 | 0, 1.5 | -9, 6 | 0.1973 | 0.1333 |
| 14 | 0.5, 1.5 | -7, 5 | 0.1796 | 0.1468 |
| 15 | 0, 2 | -10, 8 | 0.3631 | 0.0726 |
| 16 | 1.2, 0.8 | -2.8, 0.8 | 0.0911 | 0.2894 |
| 17 | 1.57 0.43 | -0.58 -1.42 | 0.1237 | 0.2131 |
| 18 | 1.54 0.46 | -0.76 -1.24 | 0.1190 | 0.2215 |
| 19 | 1.51 0.49 | -0.94 -1.06 | 0.1147 | 0.2294 |

# 4. Discussion

Variance is a principle commonly used in statistical theories. It has been used in design constructions as well as in optimization problems. Variances of direction vectors have been studied and it is well established that a good direction of search should satisfy the minimum variance property. In optimization, moving in the direction of minimum variance has been shown to optimize performance. This has been clearly stated and proved in literature. Available literatures also show advantage of moving in the direction of minimum variance. The use of variance principle in the VWGP algorithm employed in solving multi-objective problems has assisted in getting optimal solutions with possible minimum iterative moves. Also, optimal choices on the starting point of search and the step-length of search greatly enhanced performance.

As mentioned in the literature review, various gradient and non-gradient based optimization methods have been formulated to obtain optimal solutions for constrained polynomial response functions defined over continuous variables. We have compared the solutions obtained using the new algorithm with those obtained using the Quasi-Newton method. With the Quasi-Newton method, results show that the function $f_1(x_1, x_2) = x_1^3 - 2x_1x_2 - 3x_1^2 - 4 + e$ converged after four iterations with optimal values as $x_1 = 1.12, x_2 = 0.76$ and a corresponding value of response function, $f_1 = -8.06$. Also using the Quasi-Newton method, the function $f_2(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6 + e$ converged after four iterations with optimal values as $x_1 = 1.5, x_2 = 0.5$ and a corresponding value of response function, $f_2 = 0.5$. We observed that the guess initial point requirement of the Quasi-Newton method poses limitation in the use of the Quasi-Newton method. It is sometimes difficult to get a good guess point and thus results in slow convergence of the algorithm or even non-convergence of the algorithm. For the illustration presented in this paper, the optimal solution for the first function was found at the first iterative move and the optimal solution for the second function was found at the fourth iterative move. The printout from MATLAB showing the results are presented in Appendices A and B.

The use of the Variance Weighted Gradient Projection (VWGP) method is a reliable optimization method for optimizing polynomial response surfaces defined by constraints on different feasible regions. The algorithm is reliable and converges to the desired optima with few iterative steps. The performance of the new algorithm agrees with the suggestion of [7] that a good algorithm should have less computation and a quick convergence. The present study may be extended to searching for optimal solutions to multi-objective functions defined on distinct regions but where some or all of the regions may have two or more objective functions.

## 5. Limitation of the Method

The VWGP algorithm is not yet implemented in a Mathematical or Statistical software and hence the computation time and memory resources are not addressed.

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## Appendix 1

**The Quasi-Newton Optimization with the aid of MATLAB**

**Response Function:** minimize $f_1(x_1, x_2) = x_1^3 - 2x_1x_2 - 3x_1^2 - 4 + e$

$$f_2(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6 + e$$

subject to

$$\tilde{X}_1 = \{x_1, x_2 : 2x_1 + x_2 \leq 3\}$$

**QN**

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
Diagnostic Information
Number of variables: 2
Functions
Objective: Objfun1
Gradient: finite-differencing
Hessian: finite-differencing (or Quasi-Newton)
Constraints
Nonlinear constraints: do not exist
Number of linear inequality constraints: 1
Number of linear equality constraints: 0
Number of lower bound constraints: 0
Number of upper bound constraints: 0
Algorithm selected
medium-scale
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
End diagnostic information

| Iter | F-count | f(x) | Max constraint | Line search steplength | Directional derivative | First-order optimality | Procedure |
|---|---|---|---|---|---|---|---|
| 0 | 3 | -4 | -2 | | | | |
| 1 | 6 | -8.032 | 0 | 1 | -3.94 | 3.28 | |
| 2 | 9 | -8.04036 | 0 | 1 | -0.00763 | 0.38 | Hessian modified |
| 3 | 12 | -8.06066 | 0 | 1 | -0.00103 | 0.0156 | |
| 4 | 15 | -8.06067 | 0 | 1 | -6.17e-007 | 0.000354 | |

Optimization terminated: magnitude of directional derivative in search direction less than 2*options.TolFun and maximum constraint violation is less than options.TolCon.

Active inequalities (to within options.TolCon = 1e-006):

lower upper ineqlin ineqnonlin

x =

1.12 0.761

fval =

-8.06

# Appendix 2

**The Quasi-Newton Optimization with the aid of MATLAB**

**Response Function:** minimize $f_2(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6 + e$

subject to

$$\tilde{X}_2 = \{x_1, x_2 : 2x_1 + x_2 \le 3\}$$

**QN**

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Diagnostic Information

Number of variables: 2

Functions

Objective: Objfun2

Gradient: finite-differencing

Hessian: finite-differencing (or Quasi-Newton)

Constraints

Nonlinear constraints: do not exist

Number of linear inequality constraints: 1

Number of linear equality constraints: 0

Number of lower bound constraints: 0

Number of upper bound constraints: 0

Algorithm selected

medium-scale

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%

End diagnostic information

| Iter | F-count | f(x) | Max constraint | Line search steplength | Directional derivative | First-order optimality | Procedure |
|---|---|---|---|---|---|---|---|
| 0 | 3 | 6 | -2 | | | | |
| 1 | 8 | 3.5 | -1.5 | 0.25 | 4 | 3 | |
| 2 | 11 | 0.514139 | -2.22e-016 | 1 | -1.33 | 1.17 | |
| 3 | 14 | 0.500197 | 0 | 1 | 0.00373 | 0.0481 | |
| 4 | 17 | 0.5 | 0 | 1 | -1.31e-012 | 1.36e-008 | |

Optimization terminated: first-order optimality measure less than options.TolFun and maximum constraint violation is less than options.TolCon.

Active inequalities (to within options.TolCon = 1e-006):

```
   lower      upper     ineqlin    ineqnonlin
                  1
 x =
     1.5    0.5
 fval =    0.5
```

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