

# A New Method of Construction of E-optimal Generalized Group Divisible Designs with Two Groups

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**Abstract** In this paper, we consider the construction of generalized group divisible designs with two groups (GGDD(2)) from balanced incomplete block designs (BIBD). We also verify the E-optimality of these designs.

**Keywords** Generalized Group Divisible Design (GGDD), Balanced Incomplete Block Design (BIBD), E-optimality

## 1. Introduction

Generalized Group Divisible Designs and their optimality have been studied by Jacroux (1980) [1], Srivastav and Morgan (1998) [2], Thannippara et al. (2009) [3], Ghosh et al. (2012) [4] and others. In this article we present a new method for constructing generalized group divisible designs with two groups from balanced incomplete block designs. We prove that the constructed generalized group divisible designs are also E-optimal.

*Balanced Incomplete Block Designs (BIBD):* An incomplete block design with  $v$  treatments allocated over  $b$  blocks, each of size  $k$  ( $k < v$ ), such that each treatment appears in  $r$  blocks, no treatment appears more than once in a block and each pair of treatments appears in exactly  $\lambda$  blocks, is called a balanced incomplete block design (BIBD). The numbers  $v, b, k, r$ , and  $\lambda$  are called the parameters of the design.

*Generalized Group Divisible Designs with two groups (GGDD(2)):* Let  $d(v, b, k)$  be any design having  $v$  treatments allocated in  $b$  blocks each of size  $k$ . Then the design  $d$  is a generalized group divisible design with two groups if the treatments can be divided into two mutually disjoint sets  $V_1$  and  $V_2$ , each with size  $v_1$  and  $v_2$  respectively ( $v_1 < v_2$ ), such that

- (i) for  $i = 1, 2$  and for all  $c \in V_i$ ,  $\lambda_{dcc} = r_{dc} = \text{constant}$

- (ii) for all  $i \in V_1$  and  $j \in V_1$

$$(i, j = 1, 2, \dots, v_1; i \neq j), \lambda_{dij} = \lambda_1 + \lambda_2 = \gamma_{11} \text{ (say),}$$

- (iii) for all  $i \in V_2$  and  $j \in V_2$ ,

$$(i, j = v_1 + 1, v_1 + 2, \dots, v_2; i \neq j), \lambda_{dij} = \lambda_2 = \gamma_{22} \text{ (say), and}$$

- (iv) for all  $i \in V_1$  and  $j \in V_2$

$$(i = 1, 2, \dots, v_1; j = v_1 + 1, v_1 + 2, \dots, v_2), \lambda_{dji} = \lambda_2 = \gamma_{12} \text{ (say)}$$

where  $\lambda_{dij}$  denotes the  $(i, j)$ th entry of the concurrence matrix  $N_d N'_d$ .

*E-Optimality:* Let  $D(v, b, k)$  be a set of designs each having  $v$  treatments allocated in  $b$  blocks of size  $k$  each. A design  $d \in D(v, b, k)$  is said to be E-Optimal if it maximizes  $z_{d_1} = \min \{z_{d_i}\}$  where  $0 < z_{d_1} \leq z_{d_2} \leq \dots \leq z_{d_{v-1}}$  are the non-zero eigenvalues of the C-matrix of the design.

## 2. Method of Construction

**Theorem 2.1.** The design obtained by merging two BIBDs with same block sizes is a generalized group divisible design with two groups.

**Proof.** Let  $d_1(v_1, b_1, k, r_1, \lambda_1)$  and  $d_2(v_2, b_2, k, r_2, \lambda_2)$  be two balanced incomplete block designs with treatments labelled  $1, 2, \dots, v_1$  and

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$1, 2, \dots, v_2$  ( $v_2 > v_1$ ), respectively. Suppose that we merge these two designs to obtain a design, say,  $d$ . Obviously, the new design has  $v_2$  treatments and  $b_1 + b_2$  blocks. The  $v_2$  treatments in the design  $d$  can be grouped into two, viz.,  $V_1 = \{1, 2, \dots, v_1\}$  and  $V_2 = \{v_1 + 1, v_1 + 2, \dots, v_2\}$  such that the treatments in  $V_1$  i.e.,  $1, 2, \dots, v_1$  will be replicated  $r_1 + r_2$  times and those in  $V_2$ , i.e.,  $v_1 + 1, v_1 + 2, \dots, v_2$  will be replicated  $r_2$  times. In addition,

for all  $i \in V_1$  and  $j \in V_1$ , ( $i, j = 1, 2, \dots, v_1; i \neq j$ ),  $\lambda_{ij} = \lambda_1 + \lambda_2 = \gamma_{11}$  (say),

for all  $i \in V_2$  and  $j \in V_2$ , ( $i, j = v_1 + 1, v_1 + 2, \dots, v_2; i \neq j$ ),  $\lambda_{ij} = \lambda_2 = \gamma_{22}$  (say), and

for all  $i \in V_1$  and  $j \in V_2$ , ( $i = 1, 2, \dots, v_1; j = v_1 + 1, v_1 + 2, \dots, v_2$ )

$\lambda_{ji} = \lambda_2 = \gamma_{12}$  (say).

Thus, the design  $d$  is a GGDD(2).

### 3. Example

Consider the design  $d_1$ :

1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

and the design  $d_2$ :

1	4	7	1	1	1	2	2	2	3	3	3
2	5	8	4	7	5	4	8	6	4	7	6
3	6	9	8	6	9	7	5	9	9	5	8

Note that  $d_1$  is a BIBD(7, 3, 3, 1) and  $d_2$  is a BIBD(9, 12, 3, 4, 1). Combining these two balanced incomplete block designs, we get a new design  $d$ :

1	1	1	2	2	3	3	1	4	7	1	1	1	2	2	2	3	3	3
2	4	6	4	5	4	5	2	5	8	4	7	5	4	8	6	4	7	6
3	5	7	6	7	7	6	3	6	9	8	6	9	7	5	9	9	5	8

Obviously, the design  $d$  is a GGDD(2) with  $V_1 = \{1, 2, 3, 4, 5, 6, 7\}$  and  $V_2 = \{8, 9\}$ . The parameters of  $d$  are  $v = 9 (= v_2)$ ,  $b = 19 (= b_1 + b_2)$ ,  $k = 3$ ,  $\gamma_{11} = 2 (= \lambda_1 + \lambda_2)$ ,  $\gamma_{22} = 1 (= \lambda_2)$ , and  $\gamma_{12} = 1 (= \lambda_2)$ . The number of replications for the treatments in  $V_1$  is 7 ( $= r_1 + r_2$ ) and that for the treatments in  $V_2$  is 4 ( $= r_2$ ).

### 4. E-Optimality

**Theorem 4.1.** The class of GGDD's constructed in section 2 is E-optimal.

**Proof.** We will prove the theorem using the Lemma 4.1 given below. It can be verified that for the class of GGDD's constructed as per Theorem 2.1, the smallest off-diagonal entry is  $\lambda_2$  and the smallest value of replication is  $r_2$ . Keeping this in mind, we shall proceed as follows:

Let  $C_d$  denote the  $C$ -matrix of the design  $d$ . Consider the matrix

$$T_{xd} = kC_d - x \left[ \frac{v_2}{(v_2 - 1)} I_{v_2} - \frac{1}{(v_2 - 1)} E_{v_2} \right],$$

where  $x$  is a real number,  $I_{v_2}$  is the  $v_2 \times v_2$  identity matrix, and  $E_{v_2}$  is the  $v_2 \times v_2$  matrix of ones. The matrix

$T_{xd}$  has the following form  $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ , where

$$A_{11} = \left[ (k-1)(r_1 + r_2) + (\lambda_1 + \lambda_2) - \frac{xv_2}{(v_2 - 1)} \right] I_{v_1} - \left[ (\lambda_1 + \lambda_2) - \frac{x}{(v_2 - 1)} \right] E_{v_1},$$

$$A_{12} = \left[ -\lambda_2 + \frac{x}{(v_2 - 1)} \right] E_{v_1 \times (v_2 - v_1)}, \quad A_{21} = \left[ -\lambda_2 + \frac{x}{(v_2 - 1)} \right] E_{(v_2 - v_1) \times v_1}, \text{ and}$$

$$A_{22} = \left[ (k-1)r_2 + \lambda_2 - \frac{xv_2}{(v_2 - 1)} \right] I_{(v_2 - v_1)} - \left[ \lambda_2 - \frac{x}{(v_2 - 1)} \right] E_{(v_2 - v_1)}.$$

The eigenvalues of  $T_{xd}$  are

$$0 < kz_{d_1} - \frac{xv_2}{v_2 - 1} \leq kz_{d_2} - \frac{xv_2}{v_2 - 1} \leq \dots \leq kz_{d_{v-1}} - \frac{xv_2}{v_2 - 1} \quad (1)$$

For a given value of  $x$ , let  $t_{xd_{ij}}$  denote the  $(i, j)$  th entry of  $T_{xd}$ . Note that

for all  $i \in V_1$ ,

$$t_{xd_{ii}} = (k-1)(r_1 + r_2) - x, \quad (2)$$

for all  $i \in V_1$  and  $j \in V_1$ , ( $i \neq j$ ),

$$t_{xd_{ij}} = -(\lambda_1 + \lambda_2) - \frac{x}{(v_2 - 1)}, \quad (3)$$

for all  $i \in V_2$ ,

$$t_{xd_{ii}} = (k-1)r_2 - x, \quad (4)$$

for all  $i \in V_2$  and  $j \in V_2$ , ( $i \neq j$ ),

$$t_{xd_{ij}} = -\lambda_2 + \frac{x}{(v_2 - 1)}, \quad (5)$$

and for all  $i \in V_1$  and  $j \in V_2$ , ( $i \neq j$ ),

$$t_{xd_{ij}} = -\lambda_2 + \frac{x}{(v_2 - 1)}. \quad (6)$$

First let us find an upper bound for  $z_{d_1}$ . Letting  $x = r_2(k-1)$  in (4), we get, for all  $i \in V_2$ ,  $t_{xd_{ii}} = 0$ . Similarly, for all  $i \in V_1$  and  $j \in V_2$ , ( $i \neq j$ ), from (6) we get

$$t_{xd_{ij}} = -\lambda_2 + \frac{r_2(k-1)}{(v_2 - 1)} = -\lambda_2 + \frac{\lambda_2(v_2 - 1)}{(v_2 - 1)} = 0.$$

and for all  $i \in V_1$  and  $j \in V_1$ , ( $i \neq j$ ), from (3),

$$\begin{aligned} t_{xd_{ij}} &= -(\lambda_1 + \lambda_2) + \frac{r_2(k-1)}{(v_2 - 1)} \\ &= -(\lambda_1 + \lambda_2) + \frac{\lambda_2(v_2 - 1)}{(v_2 - 1)} = -\lambda_1 < 0 \end{aligned}$$

Hence  $t_{xd_{ij}} < 0$  for all  $i \neq j$ . Thus  $T_{xd}$  must possess a negative eigenvalue or at least two zero eigenvalues. Now from (1), we get  $kz_{d_1} - r_2 \frac{(k-1)v_2}{(v_2 - 1)} \leq 0$  i.e.,

$$z_{d_1} \leq \frac{r_2(k-1)v_2}{k(v_2 - 1)}.$$

Note that  $r_2k - \lambda_2 \geq r_2k - r_2 = r_2(k-1) = \lambda_2(v_2 - 1) = m(v_2 - 1)$ . Consider the matrix  $T_{xd}$  with

$$x = m(v_2 - 1). \text{ From (1), we have } kz_{d_1} - x \frac{v_2}{(v_2 - 1)} \geq 0.$$

Now substituting  $x = m(v_2 - 1)$ , and rearranging terms we have  $z_{d_1} \geq \frac{mv_2}{k}$ . Combining the upper and lower bound obtained above, we have

$$\frac{mv_2}{k} \leq z_{d_1} \leq \frac{r_2(k-1)v_2}{k(v_2 - 1)}.$$

Further if  $\lambda_{d_{ij}} \geq \frac{r_2(k-1)}{(v_2 - 1)}$  for all  $i \neq j$ , then

$$m = \frac{r_2(k-1)}{(v_2 - 1)} \text{ so that}$$

$$\frac{r_2(k-1)v_2}{k(v_2 - 1)} \leq z_{d_1} \leq \frac{r_2(k-1)v_2}{k(v_2 - 1)}.$$

This gives  $z_{d_1} = \frac{r_2(k-1)v_2}{k(v_2 - 1)}$ . Hence, by the Lemma

4.1 given below, the class of GGDD's constructed using Theorem 2.1 is E-optimal.

We state the Lemma 4.1 which is given in Jacroux (1980) [1] and used in the proof of the above theorem.

**Lemma 4.1.** Suppose  $d \in D(r_1, r_2, \dots, r_v; b, k)$  has  $C$ -matrix  $C_d$  and  $m$  is the smallest off-diagonal element occurring in the matrix  $N_d N'_d = (\lambda_{dij})$ . Then

$$\frac{vm}{k} \leq z_{d_1} \leq \frac{r_p(k-1)v}{(v-1)k}.$$

Further, if  $\lambda_{d_{ij}} \geq \frac{r_p(k-1)}{v-1}$  for all  $i \neq j$ , then  $z_{d_1} = \frac{r_p(k-1)v}{(v-1)k}$  and hence  $d$  is

E-optimal in  $D(r_1, r_2, \dots, r_v; b, k)$ .

## 5. Illustrations

Consider the example of GGDD(2) constructed in Section 3. The incidence matrix  $N_d$  of this design is given by

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The concurrence matrix  $N_d N'_d$  of this design is given by

$$\begin{bmatrix} 7 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 7 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 7 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 7 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 7 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 7 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 7 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}.$$

The information matrix  $C_d$  is given by  $C_d = \text{diag}(r_1, r_2, \dots, r_v) - N_d K^{-1} N'_d$ . Since  $k_1 = k_2 = \dots = k_b = k$ , therefore  $C_d = \text{diag}(r_1, r_2, \dots, r_v) - \frac{N_d N'_d}{k}$ . In this example  $r_1 = r_2 = \dots = r_7 = 7$ ,  $r_8 = r_9 = 4$ , and  $k = 3$ , finally we get the information matrix  $C_d$  as

$$\begin{bmatrix} 14/3 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & -1/3 & -1/3 \\ -2/3 & 14/3 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & -1/3 & -1/3 \\ -2/3 & -2/3 & 14/3 & -2/3 & -2/3 & -2/3 & -2/3 & -1/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & 14/3 & -2/3 & -2/3 & -2/3 & -1/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & 14/3 & -2/3 & -2/3 & -1/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & 14/3 & -2/3 & -1/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & -2/3 & 14/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & 8/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & 8/3 \end{bmatrix}.$$

Here  $x = r_2(k-1) = (4)(3-1) = 8$ . On substituting the

values of  $C_d$ ,  $x$ , and other parameters, we obtain the

matrix  $T_{xd} = kC_d - x \left[ \frac{v_2}{(v_2-1)} I_{v_2} - \frac{1}{(v_2-1)} E_{v_2} \right]$  as

$$\begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 6 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 6 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 6 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 6 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero eigenvalues of  $C_d$  are 5.33 and 3. The minimum nonzero eigenvalue of  $C_d$  is 3 and satisfies the conditions of Lemma 4.1. Hence the design  $d$  constructed in Section 3 is E-optimal.

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