An Inequality between the Arithmetic Mean of Some Numbers and the Arithmetic Mean of Their Images through a Convex Function

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Abstract The purpose of this note is to present a relation between the arithmetic mean, of a finite number of real numbers, and the arithmetic mean of their images through a convex function. Some applications of this inequality are also included.

1. Introduction

Let $n \geq 1$ be a fixed natural number, and I an interval of real numbers. For every $a = (a_1, a_2, \ldots, a_n) \in I^n$, the *arithmetic mean* associated to a is defined as:

$$A_n[a] := \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Let $I \subset \mathbf{R}$ be an interval. If $f : I \to \mathbf{R}$ is a convex (concave) function, then the well known Jensen inequality (see [2]-[4], [10], [11] says that:

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \leq (\geq) \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n},$$

which can be written, using the above notation, as:

$$f\left(A_n[a]\right) \leq (\geq) A_n\left[f(a)\right]. \tag{1.1}$$

The inequality (1.1) represents a relation between the image of $A_n[a]$, through f, and the arithmetic mean of the images of the numbers $\{a_i\}_{1 \leq i \leq n}$, i.e., $A_n[f(a)]$.

We can imagine the numbers $a_1, a_2, \ldots, a_n$ as being some data $\mathcal{D}$ received by a statistician. For large values of n, it is hard for the statistician to look at each number, a_i, $1 \leq i \leq n$, in these data. So, the statistician decides to make a skeleton of the data, composed of the minimum m and maximum M values of these data, a center c of the data, and a number s measuring the spread of the data. While

m and M are easy to define as:

$$m = \min\{a_i | 1 \leq i \leq n\} \tag{1.2}$$

and

$$M = \max\{a_i | 1 \leq i \leq n\}, \tag{1.3}$$

there are many different ways to define the center c. One very popular way is to a apply first a strictly monotone function f to the data $\mathcal{D}$, obtaining the new data $f(\mathcal{D}) : f(a_1), f(a_2), \ldots, f(a_n)$. Then we compute the average value $A_n[f(\mathcal{D})]$, defined as:

$$A_n\left[f(\mathcal{D})\right] := A_n\left[f(a)\right] \tag{1.4}$$

$$= \frac{1}{n}\sum_{i=1}^{n} f(a_i). \tag{1.5}$$

Finally, we apply the inverse function f^{-1} to $A_n[f(\mathcal{D})]$, defining the center of the data to be:

$$c_f := f^{-1}\left(A_n\left[f(a)\right]\right). \tag{1.6}$$

A typical example of such functions, used in computing the center of data consisting of positive numbers, is given by the functions: $f_r : (0, \infty) \to \mathbb{R}$, $f_r(x) = x^r$, for $r \in \mathbb{R} \setminus \{0\}$. Following the above procedure, these functions give rise to the Hölder means of the positive numbers $a_1, a_2, \ldots, a_n$.

Because different functions f and g produce different centers c_f and c_g of the data, it is important to find inequalities relating $A_n[f(a)]$ and $A_n[g(a)]$ in order to understand the inequalities between c_f and c_g.

2. Main Result

We present now the result leading to the main result of this note.

Proposition 2.1 If
1. $f : I \subset \mathbf{R} \to \mathbf{R}$ is a convex (concave) function;
2. $a_i \in I \cap [m, M]$, for all $i \in \{1, 2, \ldots, n\}$, where m, $M \in \mathbf{R}$,

then the following inequality:

$$(M - m) \cdot A_n\left[f(a)\right] - \left[f(M) - f(m)\right] \cdot A_n[a]$$
$$\leq (\geq) M \cdot f(m) - m \cdot f(M) \tag{2.7}$$

holds.

Proof. Let $i \in \{1, 2, \ldots, n\}$ be fixed. Since $a_i \in [m, M]$, a_i can be written as the following convex combination of M and m:

$$a_i = \frac{a_i - m}{M - m} \cdot M + \frac{M - a_i}{M - m}.$$

If we denote $p_i := (a_i - m)/(M - m)$ and $q_i := (M - a_i)/(M - m)$, then since $p_i \geq 0$, $q_i \geq 0$, and $p_i + q_i = 1$, we conclude from the definition of convexity of f, that:

$$f(a_i) = f(p_i \cdot M + q_i \cdot m)$$
$$\leq p_i f(M) + q_i f(m)$$
$$= \frac{(a_i - m) f(M) + (M - a_i) f(m)}{M - m}. \tag{2.8}$$

Summing up in the last inequality from $i = 1$ to $i = n$, we obtain:

$$\sum_{i=1}^{n} f(a_i) \leq \frac{\left(\sum_{i=1}^{n} a_i - n \cdot m\right) \cdot f(M) + \left(n \cdot M - \sum_{i=1}^{n} a_i\right) \cdot f(m)}{M - m}$$
$$= \frac{n \cdot \left(\frac{1}{n}\sum_{i=1}^{n} a_i - m\right) \cdot f(M) + n \cdot \left(M - \frac{1}{n} \cdot \sum_{i=1}^{n} a_i\right) \cdot f(m)}{M - m}$$

After multiplying both sides of the last relation by $(M - m)/n$, we obtain:

$$(M - m) \cdot A_n\left[f(a)\right]$$
$$\leq (A_n[a] - m) \cdot f(M) + (M - A_n[a]) \cdot f(m),$$

which is equivalent to:

$$(M - m) \cdot A_n\left[f(a)\right] - \left[f(M) - f(m)\right] \cdot A_n[a]$$
$$\leq M \cdot f(m) - m \cdot f(M).$$

If f is concave all the inequalities from this proof are reversed.

Corollary 2.2. If we impose the additional condition that $f(m) < f(M)$ to the assumptions from Proposition 2.1, then:

$$\frac{A_n\left[f(a)\right]}{f(M) - f(m)} - \frac{A_n[a]}{M - m} \leq (\geq) \frac{M \cdot f(m) - m \cdot f(M)}{(M - m) \cdot \left(f(M) - f(m)\right)}. \tag{2.9}$$

Proof. Since $f(m) \neq f(M)$, we have $m \neq M$. Dividing both sides of the inequality (2.7) by the strictly positive number $(M - m)\left(f(M) - f(m)\right)$, we obtain (2.9).

Theorem 2.3 Let a_1, a_2, …, a_n be real numbers, $m := \min\{a_i | 1 \leq i \leq n\}$ and $M := \max\{a_i | 1 \leq i \leq n\}$. Let $f : [m, M] \to [r, s]$ and $g : [m, M] \to [u, v]$, be two strictly increasing and bijective functions, such that: $f \circ g^{-1}$ is convex on $[u, v]$. Then:

$$\left[g(M) - g(m)\right] A_n\left[f(a)\right] - \left[f(M) - f(m)\right] A_n\left[g(a)\right]$$
$$\leq g(M) f(m) - g(m) f(M). \tag{2.10}$$

Proof. We apply Proposition 2.1 to the numbers: $g(a_1)$, $g(a_2)$, …, $g(a_n)$ and the convex function $f \circ g^{-1}$ obtaining inequality (2.10).

Application 2.4 Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = e^x$. It is clear that f is convex and strictly increasing on $\mathbf{R}$. Thus for all $\{a_i\}_{i=1}^{n} \subset [m, M] \subset \mathbf{R}$, such that $m < M$, we conclude from (2.9), that:

$$\frac{e^{a_1} + e^{a_2} + \cdots + e^{a_n}}{n(e^M - e^n)} - \frac{a_1 + a_2 + \cdots + a_n}{n(M - m)}$$
$$\leq \frac{M e^m - m e^M}{(M - m)(e^M - e^m)}. \tag{2.11}$$

If $a_1 \leq a_2 \leq \ldots \leq a_n$, with at least one of these inequalities being strict, then setting: $m = a_1$ and $M = a_n$, we can rewrite (2.11) as:

$$\frac{e^{a_1} + e^{a_2} + \cdots + e^{a_n}}{n(e^{a_n} - e^{a_1})} - \frac{a_1 + a_2 + \cdots + a_n}{n(a_n - a_1)}$$
$$\leq \frac{a_n e^{a_1} - a_1 e^{a_n}}{(a_n - a_1)(e^{a_n} - e^{a_1})}. \tag{2.12}$$

Corollary 2.5 If $a_i \in [m, \infty)$, m, $M \in (0, \infty)$, for all $i \in \{1, 2, \ldots, n\}$, then:

$$G_n^{M-m}[a] \geq M^{A_n[a] - m} \cdot m^{M - A_n[a]},$$

where $G_n[a] := \sqrt[n]{a_1 \cdot a_2 \cdots a_n}$ is the geometric mean of the numbers $\{a_i\}_{i=1}^{n}$.

Proof. Since the function $f : (0, \infty) \to \mathbf{R}$, $f(x) = \ln x$ is concave, it follows from the inequality (2.7) that:

$$(M - m) \cdot A_n[\ln a] - (\ln M - \ln m) \cdot A_n[a] \geq M \ln m - m \ln M$$

$$\Leftrightarrow (M - m) \cdot \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n} - \ln\left(\frac{M}{m}\right) \cdot A_n[a]$$

$$\geq \ln\left(\frac{m^M}{M^m}\right)$$

$$\Leftrightarrow \ln G_n^{M-m}[a] - \ln\left[\left(\frac{M}{m}\right)^{A_n[a]}\right] \geq \ln\left(\frac{m^M}{M^m}\right)$$

$$\Leftrightarrow \ln\left(\frac{G_n^{M-m}[a]}{\left(\frac{M}{m}\right)^{A_n[a]}}\right) \geq \ln\left(\frac{m^M}{M^m}\right)$$

$$\Leftrightarrow G_n^{M-m}[a] \geq \left(\frac{M}{m}\right)^{A_n[a]} \cdot \frac{m^M}{M^m}$$

$$\Leftrightarrow G_n^{M-m}[a] \geq M^{A_n[a]-m} \cdot m^{M-A_n[a]}.$$

If $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then we can take $m = a_1$ and $M = a_n$, and obtain:

$$G_n^{a_n-a_1}[a] \geq a_n^{A_n[a]-a_1} \cdot a_1^{a_n-A_n[a]}. \tag{2.13}$$

This inequality holds even when $a_1 = a_2 = \cdots = a_n$.

Application 2.6 *Let* $a_i = i+1$*, for all* $i \in \{1, 2, \ldots, n\}$. *Then we can take* $m = a_1 = 2$ *and* $M = a_n = n+1$. *We have:*

$$A_n[a] = \frac{2 + 3 + \cdots + (n+1)}{n}$$

$$= \frac{n+3}{2}$$

$$= \frac{n}{2} + \frac{3}{2}.$$

It follows now from (2.13) that:

$$\left[\sqrt[n]{(n+1)!}\right]^{n-1} \geq 2^{\frac{n-1}{2}} \cdot (n+1)^{\frac{n-1}{2}}.$$

This is equivalent to:

$$\sqrt[n]{(n+1)!} \geq \sqrt{2(n+1)}.$$

Corollary 2.7 *If* $a_i \in [m, M]$*, for all* $i \in \{1, 2, \ldots, n\}$*, and* $[m, M] \subset (0, \infty)$*, then:*

$$A_n[a] \leq M + m - \frac{M \cdot m}{H_n[a]}, \tag{2.14}$$

where

$$H_n[a] := \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n}$$

is the harmonic mean of the numbers $\{a_i\}_{i=1}^n$.

Proof. The function $f : (0, \infty) \to \mathbb{R}$, $f(x) = 1/x$ is convex. Applying the inequality (2.7) to this function, we get:

$$(M-m) \cdot A_n[1/a] - \left(\frac{1}{M} - \frac{1}{m}\right) \cdot A_n[a] \leq M \cdot \frac{1}{m} - m \cdot \frac{1}{M}$$

$$\Leftrightarrow (M-m) \cdot \frac{1/a_1 + 1/a_2 + \cdots + 1/a_n}{n} - \frac{m-M}{M \cdot m} \cdot A_n[a]$$

$$\leq \frac{M}{m} - \frac{m}{M}$$

$$\Leftrightarrow (M-m) \cdot \frac{1}{H_n[a]} + \frac{M-m}{M \cdot m} \cdot A_n[a] \leq \frac{M^2 - m^2}{M \cdot m}$$

$$\Leftrightarrow \frac{1}{H_n[a]} + \frac{A_n[a]}{M \cdot m} \leq \frac{M+m}{M \cdot m}$$

$$\Leftrightarrow A_n[a] \leq M + m - \frac{M \cdot m}{H_n[a]}.$$

If $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then by taking $m := a_1$ and $M := a_n$ in (2.14), we obtain:

$$A_n[a] \leq a_1 + a_n - \frac{a_1 \cdot a_n}{H_n[a]}.$$

Alternative characterizations for means, obtained by different methods, can be found in [6]-[9].

Observation 2.8 *The means* $A_n[a]$ *and* $H_n[a]$ *make sense for negative numbers, too. If* $\{a_i\}_{i=1}^n \subset (-\infty, 0)$*, then the inequality (2.14) is reversed, which can be proved very simply by multiplying both sides of this inequality by* -1.

Application 2.9 *Let* $a_i = i$*, for all* $i \in \{1, 2, \ldots, n\}$. *Then we can take* $m = a_1 = 1$*,* $M = a_n = n$*, and we have* $A_n[a] = (n+1)/2$. *It follows now from (2.14) that:*

$$\frac{n+1}{2} \leq n + 1 - \frac{n \cdot 1}{H_n[a]}.$$

This is equivalent to

$$\frac{n}{H_n[a]} \leq \frac{n+1}{2}.$$

The last inequality means:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \frac{n+1}{2}.$$

Corollary 2.10 *Let* $r \geq 1$ *and* $\{a_i\}_{i=1}^n \subset [m, M] \subset (0, \infty)$. *Then the following inequality holds:*

$$(M-m) \cdot \left(M_n^r[a]\right)^r - (M^r - m^r) \cdot A_n[a]$$
$$\leq M \cdot m^r - m \cdot M^r, \tag{2.15}$$

where:

$$M_n^r[a] := \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{1/r}$$

is the power mean of order r (*called also the* Hölder mean

of order r) *of the numbers* a_1 , a_2 , $\dots$, a_n . *In particular if* $\min\{a_i | 1 \le i \le n\} = a_1$, $\max\{a_i | 1 \le i \le n\} = a_n$, *and* $a_1 < a_n$, *then:*

$$\frac{\left(M_n^r[a]\right)^r}{a_n^r - a_1^r} - \frac{A_n[a]}{a_n - a_1} \le \frac{a_n \cdot a_1^r - a_1 \cdot a_n^r}{(a_n - a_1) \cdot (a_n^r - a_1^r)}. \quad (2.16)$$

Proof. Everything follow by applying the inequality (2.7) to the convex and increasing function $f : [0, \infty) \to \mathbf{R}$, $f(x) = x^r$.

Application 2.11 *For all* $q > p > 0$ *and all positive numbers* a_1 , a_2 , $\dots$, a_n , *we have:*

$$\left\{H_\infty^p([a]) - H_{-\infty}^p([a])\right\} H_q^q[a] - \left\{H_\infty^q([a]) - H_{-\infty}^q([a])\right\} H_p^p[a]$$

$$\le H_\infty^p([a]) H_{-\infty}^q([a]) - H_{-\infty}^p([a]) H_\infty^q([a]),$$

where $H_r([a])$ *denotes the* r *-Hölder mean of* a_1 , a_2 , $\dots$, a_n , *for all* $r \in [-\infty, \infty]$.

Proof. We simply apply Theorem 2.3 to the numbers a_1 , a_2 , $\dots$, a_n , and the function $f(x) = x^q$ and $g(x) = x^p$. We have $(f \circ g^{-1})(x) = x^{q/p}$, which is a convex function since $q/p > 1$, and thus the result of our application follows easily.

Observation 2.12 *All the inequalities from this paper can be reformulated using weighted means, too.*

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