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Comellas' Deterministic Small-Worlds: A Review

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Abstract In this piece, the paper Deterministic Small-world Communication Networks, a paper from the year of 2000 by Comellas, Ozon, and Peters, is discussed. From the introduction, where some mismatch between sigmatoids, and intended senses is identified, to the conclusion, where the same mismatch appears in the shape of wonder, the findings are of surprising nature: major misunderstandings in the interpretation of the research or teaching or invention of others may lead to great new theories, which may lead to wonderful sets of new, and meaningful mathematical paradigms.

Keywords Small-world, Comellas, Network, Deterministic

1. Introduction

"No one set out to invent sticky notes. Instead, in 1968, Dr. Spencer Silver, a chemist at 3M Company, invented a unique, low-tack adhesive that would stick to things but also could be repositioned multiple times. He was trying to invent a super-strong adhesive, but he came up with a super-weak one instead"

(National Center for Families Learning 2018, para. 5)

Comellas et al. happened to create a new graph, which is a variation of the Circulant Graph, which was seen before in Algebra through matrices:

For $n \in \mathbb{N}$ and $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, look at the circulant matrix

$$\mathbf{C_x} := \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1 \\ x_1 & \cdots & x_{n-1} & x_0 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

(Lindner 2018, p. 1)

Definition 1.4. A **circulant graph** $X(n; S)$ is a Cayley graph on $\mathbb{Z}_n$. That is, it is a graph whose vertices are labelled $\{0, 1, \ldots, n-1\}$, with two vertices labelled i and j adjacent iff $i - j \pmod n \in S$, where $S \subset \mathbb{Z}_n$ has $S = -S$ and $0 \notin S$.

(Morris 2004, p. 2)

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The "circulant graph" $G(n, S)$ with symbol S (so named because its vertex-vertex adjacency matrix is a circulant) has as its vertices the elements of the cyclic group Z_n, and as its edges those unordered pairs $\{x, y\}$ such that $x - y \in S$, where $0 \notin S = -S \subseteq Z_n$ (that is, S is a subset of nonzero elements which is closed under taking additive inverses). Such graphs are familiar to geometers as "star-polygons" [6]. It is easy to see that $G(n, S)$ is connected if and only if S generates the group Z_n. Therefore the connected circulant graphs are precisely the Cayley graphs for cyclic groups, to which we restrict our discussion.

(Parsons 1980)

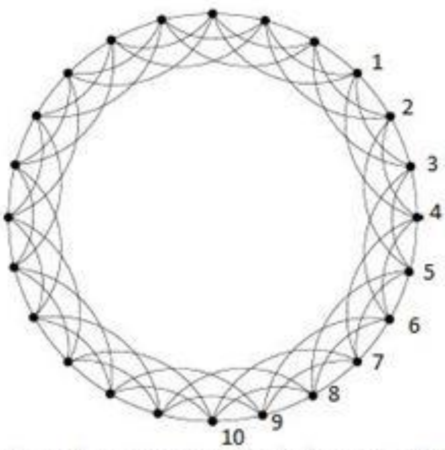


Fig. 1. $C_{n,\Delta}$ with $n = 24$ and $\Delta = 6$, a circulant graph with steps $\pm 1; \pm 2; \pm 3$.

$$\left\{\begin{matrix} 1 & 2 & 3 & 4 & X \\ 2 & 3 & 4 & 5 & X \\ 3 & 4 & 5 & 6 & X \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}\right\}$$

(1 2)(1 3)(1 4)(2 3)(2 4)(2 5)(3 4)(3 5)(3 6)(4 5)(4 6)(4 7)...

(Comellas, Ozon, & Peters 2000, p. 84)

The original circulant graph, which connects to the matrix exhibited above:

Example 0.3 We observe in Figure 2 that the circulant graph $circ(24 : 6, 9)$ has three components, which are mutually isomorphic.

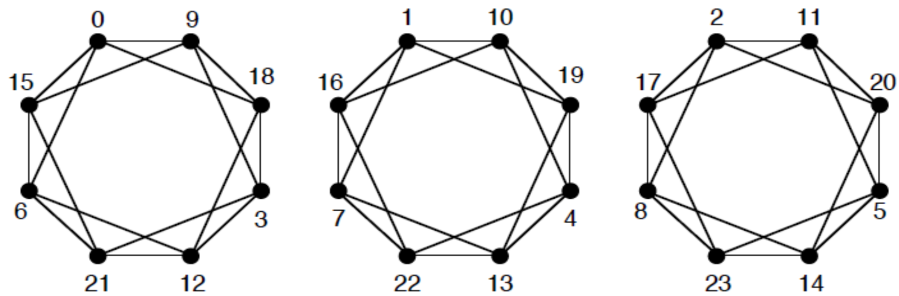


Figure 2: The circulant graph $circ(24 : 6, 9)$ has three components.

(Alspach 2010, para. 16)

Because the name above coincides with the name chosen by Comellas et al., Circulant Graph, we may think that Comellas et al. would have to have the same sort of symbol to represent this graph, so circ(x: y,w). In this case, there is hope that his symbol, $C_{m,n}$ will lead to another type of graph because the graph above is certainly not Comellas' Circulant (two reference numbers, not three).

Let's see:

Now, for $1 \leq m < n/2$, let $C_{m,n}$ denote the graph $K_m \vee (K_m^c + K_{n-2m})$, depicted in figure 4.9*a*; two specific examples, $C_{1,5}$ and $C_{2,5}$, are shown in figures 4.9*b* and 4.9*c*.

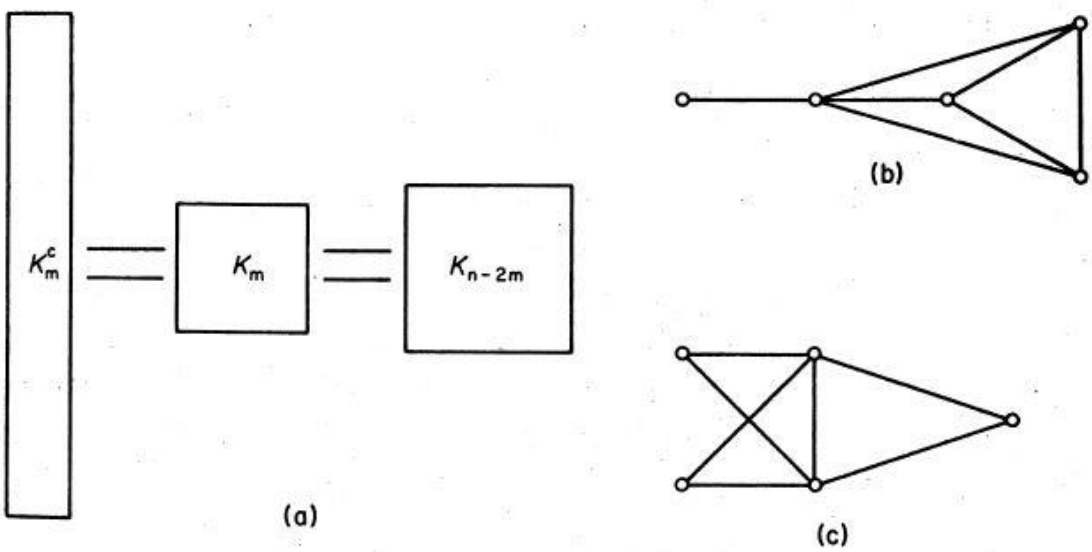


Figure 4.9. (*a*) $C_{m,n}$; (*b*) $C_{1,5}$; (*c*) $C_{2,5}$

(Bondy & Murty 1976, p. 58)

It looks like Comellas et al. have created a new type of graph, which, in many senses, relates to the Circulant Graph, but, in many other senses, does not.

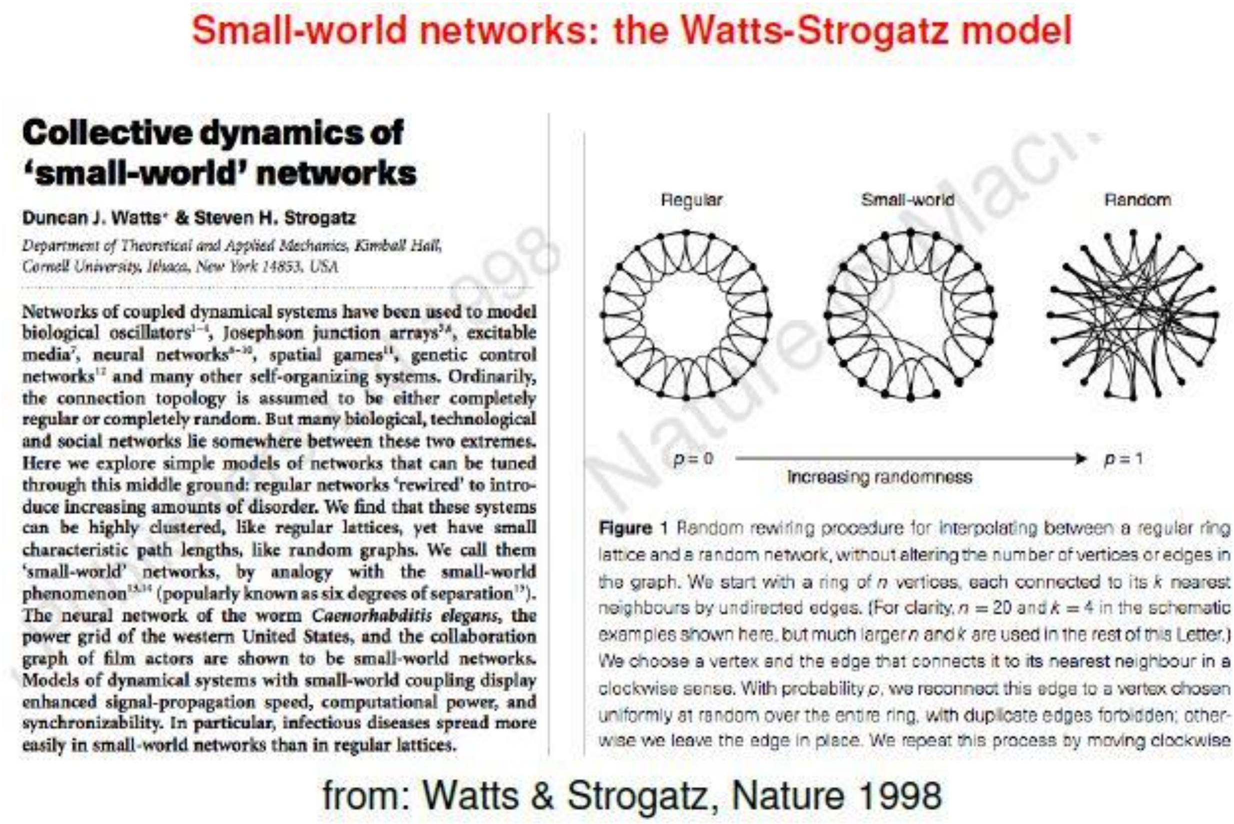
In the sections that follow, we discuss the originality of Comellas et al.'s graph together with the ideas exposed in the mentioned scientific article.

2. Development

Comellas et al. (2000) start this paper, on page 83, with the following paragraph:

"*Small-world networks* were introduced by Watts and Strogatz in a recent paper [9] as models of real world situations including electric power grids, the spread of diseases in populations, the collaboration networks of film actors, and the neural network of the worm *Caenorhabditis elegans* [8,9]."

He was then referring to the article published in Nature:



Collective dynamics of 'small-world' networks

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Networks of coupled dynamical systems have been used to model biological oscillators[1–4], Josephson junction arrays[5,6], excitable media[7], neural networks[8–10], spatial games[11], genetic control networks[12] and many other self-organizing systems. Ordinarily, the connection topology is assumed to be either completely regular or completely random. But many biological, technological and social networks lie somewhere between these two extremes. Here we explore simple models of networks that can be tuned through this middle ground: regular networks 'rewired' to introduce increasing amounts of disorder. We find that these systems can be highly clustered, like regular lattices, yet have small characteristic path lengths, like random graphs. We call them 'small-world' networks, by analogy with the small-world phenomenon[13,14] (popularly known as six degrees of separation[15]). The neural network of the worm *Caenorhabditis elegans*, the power grid of the western United States, and the collaboration graph of film actors are shown to be small-world networks. Models of dynamical systems with small-world coupling display enhanced signal-propagation speed, computational power, and synchronizability. In particular, infectious diseases spread more easily in small-world networks than in regular lattices.

Figure 1 Random rewiring procedure for interpolating between a regular ring lattice and a random network, without altering the number of vertices or edges in the graph. We start with a ring of n vertices, each connected to its k nearest neighbours by undirected edges. (For clarity, $n = 20$ and $k = 4$ in the schematic examples shown here, but much larger n and k are used in the rest of this Letter.) We choose a vertex and the edge that connects it to its nearest neighbour in a clockwise sense. With probability p, we reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden; otherwise we leave the edge in place. We repeat this process by moving clockwise

Small-world networks were actually introduced by Milgram (1969) - an American psychologist who lived from 1933 to 1984 - in his paper, An Experimental Study of the Small World Problem.

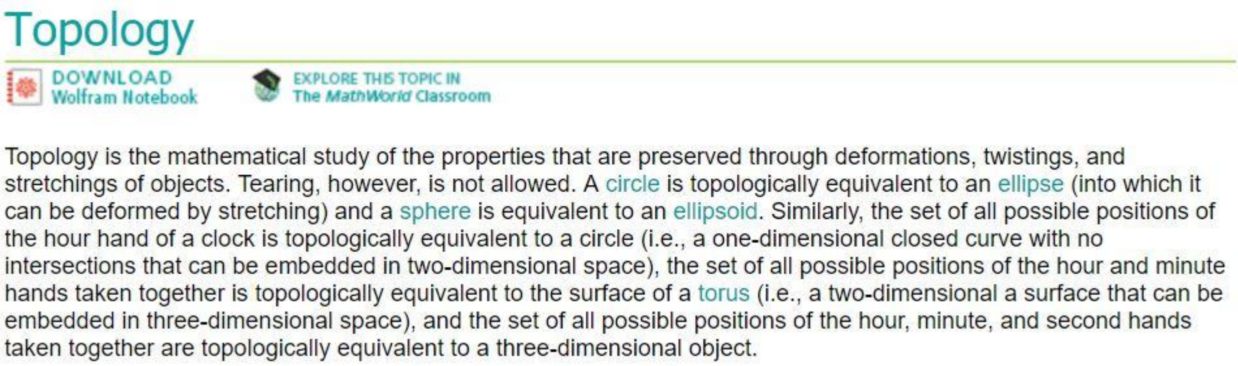
Watts and Strogatz (1998) did mention the small-world phenomenon in their article with Nature about 14 years after the death of Milgram, but the novelty then consisted of the sigmatoids (Pinheiro 2018, p. 38) appearing together, small-world network, or of the world reference (Pinheiro 2018, p. 38) for the expression, now a particular mathematical/statistical model.

Many classes of *structured*[1] networks, including the restricted class of *circulant graphs* studied in [9] and subsequent papers, have strong local *clustering* (nodes have many mutual neighbors), but large *average distances* between pairs of nodes. The opposite extreme is *random* networks which have small average distances but exhibit very little clustering. Networks between these two extremes can be constructed by starting with a structured network and randomly

[1] In [9] and subsequent papers, the term "regular" is used informally to refer to these networks. We use "structured" instead and use "regular" in its usual graph-theoretic sense.

Still from the first page of Comellas et al.'s article (page 83) is the extract above: in this case, the observation is that if one is structured, then the other should be non-structured instead of random, and here we are just following what could be called 'the peasant's logic'.

Random networks could make sense from a generative point of view: computers generating those connections randomly. Notwithstanding, once the networks are formed, and we are able to photograph those, it seems undeniable that non-structured or irregular would be more appropriate. At the same time, why would 'structured' make more sense than regular even in normal language? All models have structure.



Topology

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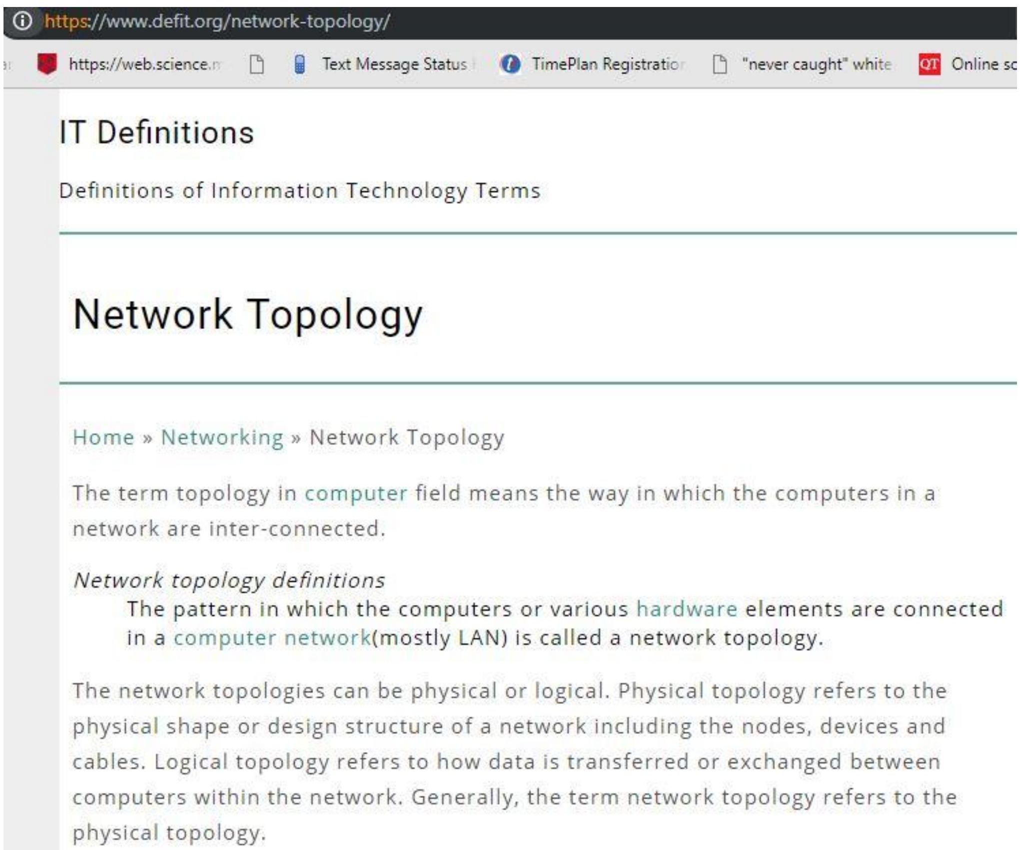
Topology is the mathematical study of the properties that are preserved through deformations, twistings, and stretchings of objects. Tearing, however, is not allowed. A circle is topologically equivalent to an ellipse (into which it can be deformed by stretching) and a sphere is equivalent to an ellipsoid. Similarly, the set of all possible positions of the hour hand of a clock is topologically equivalent to a circle (i.e., a one-dimensional closed curve with no intersections that can be embedded in two-dimensional space), the set of all possible positions of the hour and minute hands taken together is topologically equivalent to the surface of a torus (i.e., a two-dimensional a surface that can be embedded in three-dimensional space), and the set of all possible positions of the hour, minute, and second hands taken together are topologically equivalent to a three-dimensional object.

(Wolfrang Research, Inc. 2018, para. 1)



https://www.defit.org/network-topology/

IT Definitions

Definitions of Information Technology Terms

Network Topology

Home » Networking » Network Topology

The term topology in computer field means the way in which the computers in a network are inter-connected.

Network topology definitions

The pattern in which the computers or various hardware elements are connected in a computer network(mostly LAN) is called a network topology.

The network topologies can be physical or logical. Physical topology refers to the physical shape or design structure of a network including the nodes, devices and cables. Logical topology refers to how data is transferred or exchanged between computers within the network. Generally, the term network topology refers to the physical topology.

(Brainasoft 2018, para. 1-3)

Circulant Graphs may be studied from inside of Pure Geometry (Carlson (2018). There, the terms regular, and irregular seem to be proper.

It is important to understand that Professor Comellas, and his group started precursor work: they started to change what was exclusively of statistical nature into what would be exclusively of mathematical nature (Pinheiro 2012, p. 261).

As a consequence, everything was restarted, and therefore it takes a great amount of reflection to build the best lingo, objects, definitions, and all else.

> In this paper, we replace the probabilistic models with deterministic small-world networks and non-random interconnection patterns.

(Comellas, Ozon, & Peters 2000, p. 84)

The main issue is that small-world connects to the research of Professor Milgram, and his definition of small world is 64 in 296 chains present an average of six edges of distance (Pinheiro 2016, p. 270) or around 64 nodes (also smaller), which means being able to go from one end to another through only a few edges (his average was 5.2) or having around 64 nodes in the network.

The network itself does not change: what changes is the way of studying or describing it. One way is statistical, and therefore is a way to deal with the random events of human life, and this world when predicting future changes. Another way is mathematical, and therefore is a way to deal with the networks as they are at the moment of the analysis, so without considering that things may change in the near future, that is, that connections may be added, may disappear, and so on. In this case, deterministic means mathematical, and therefore, if we are working from inside of the World of Mathematics, the expression 'deterministic small-world networks' can be replaced with 'small-world networks'.

If it is inside of Mathematics, 'networks of maximum diameter 6, and at most 64 nodes' would be a good choice. Then, again, why would they not be just networks?

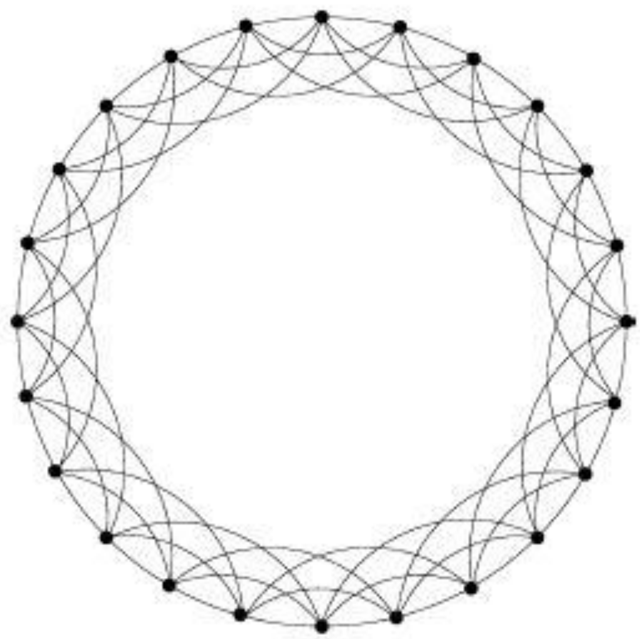


Fig. 1. $C_{n,\Delta}$ with $n = 24$ and $\Delta = 6$, a circulant graph with steps $\pm 1; \pm 2; \pm 3$.

This picture appears on page 84 of Comellas et al.'s article. It then lets us know that his circulant does not comply with the definition of circulant that existed until that year, 2000. Notice that nodes are 24, but only 6 links exist for each node.

In this case, Comellas et al. did not notice, but they effectively created a new model to describe, and study networks, a variation of the circulant graph that needs an original name.

In the section number 5, Comellas et al. make comparisons between the analytical, and the numerical approaches. They then conclude that adding star graphs instead of double loops is wiser if the objective is creating something similar to a small-world real-life situation.

The issue that appears here is that the decision should be clear: if the intention was changing all that was probabilistic, and therefore more of statistical, or real-life nature, into what is deterministic, and therefore more of mathematical, or machine-like nature, then there is no need to 'approach the real-life data', since works depart from the final situation.

Perhaps the value of this part of their study is in attempting to observe, and control the dynamics involved when the network goes from not-so-connected to small-world, and therefore understanding things from a sociological point of view: how are these people connecting to each other as time advances?

The problem then becomes that they are computer simulations, not real data.

There are also criticisms of mathematical nature to this piece.

The extract below came from (Pinheiro 2007, pp. 1063-1064):

3. Limiting δ (new result)

In [1], Comellas et al. assume that nodes i and $(i+j)$ have $\delta-(j+1)$; $1 \leqslant j \leqslant \frac{\delta}{2}$ common neighbors. Hence, when i and j are $\frac{\delta}{2}$ apart, they will have $\left(\frac{\delta}{2}-1\right)$ neighbors.

One of the counter-examples to the above theory is $C_{9,6}$; any vertices n and $(n+3)$ should have two common neighbors but, in fact, they have three common neighbors.

Lemma 2. *If δ is the maximum degree that a vertex may have in the circulant graph C_n (C being allowed to be either regular or random), and n is the number of vertices in $C_{n,\delta}$, $\delta < \frac{2n}{3}$.*

Proof. If i and j have common neighbors to the right of i, for instance, there should not be common neighbors to its left. Analogously, when i and j have common neighbors to the left of i, there should not be common

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neighbors to its right, and the proof of one case is analogous to the other. We shall, therefore, consider just the first case in our proof. According to Comellas et al., $0 < (j-i) \leqslant \frac{\delta}{2}$. The situation to be excluded, taking into account our first supposition regarding neighborhood is, therefore, $(i-u)(\mathrm{mod}\, n) = (j+v)(\mathrm{mod}\, n)$ where

$$0 < u, v \leqslant (j-i), \quad \{u, v\} \subset N$$

what implies that

$$0 < (j-i+u+v) \leqslant \left[\frac{\delta}{2} + 2(j-i)\right]$$

what implies that

$$0 < (j-i+u+v) \leqslant \frac{3\delta}{2}$$

finally implying that we should exclude the cases where

$$0 < kn \leqslant \frac{3\delta}{2}$$

what brings us to our constraint once the lowest value for k is 1. □

3. Conclusions

Professor Comellas, and his group seem to have found another graph, different from the Circulant Graph, which was also part of Algebra through the Circulant Matrix. They use similar elements to describe, and work with their own graphs, but the graphs are distinct, since the number of columns, and rows of the matrices does not equate the number of vertices of the graph.

The expression ‘deterministic small-world networks’ does not make much sense because ‘deterministic’ comes from changing what was statistical into what is mathematical, and there, in Mathematics, all is deterministic.

The expression ‘small-world networks’ cannot make much sense in Mathematics: all should be absolutely determined, so perhaps we say networks of size of about 64 nodes, and with a maximum diameter of 6.

In this case, since the Mathematics used is the same (small or large networks), it looks like we should simply refer to the graph that is formed instead: Comellas’ Circulant.

We then have regular, and irregular Comellas’ circulants: regular circulants of this type would bring vertices that have, all of them, the same degree.

Some small technical details are worth the note: Comellas et al. missed a constraint that is necessary for their theorems, and lemmas to be valid. If, in $C_{n,\delta}$, n is the number of vertices, and δ is the number of links per vertex, then $\delta < 2n/3$.

Comparing numerical results with analytical data cannot make sense, or it is only part of the needed work, because they are both purely abstract in nature: we need to compare one of those results (numerical or analytical), or even both, to the studies on the Bacon number (Wolfram Demonstrations Project & Contributors 2019, para. 1-2) or Milgram's letters (Milgram 1969). The whole interest seems to then be to the side of defining small worlds in a mathematical way (Pinheiro 2018b), and studying step by step the process of 'wiring' between humans from a selected group.

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