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Free Vibration Analysis of Beams Considering Different Geometric Characteristics and Boundary Conditions

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Abstract In this study, free vibration of square cross-sectioned aluminum beams is investigated analytically and numerically under four different boundary conditions: Clamped-Clamped (C-C), Clamped-Free (C-F), Clamped-Simply Supported (C-SS) and Simply Supported-Simply Supported (SS-SS). Analytical solution is carried out using Euler-Bernoulli beam theory and Newton Raphson Method. First, the equations of motion are provided. Then, solutions including the effects of the geometric characteristics, and boundary conditions are obtained and discussed for the natural frequencies of the first three modes. To confirm the reliability of the vibration analysis carried out in the present paper as well, all the analytical results are checked with the corresponding numerical results obtained from the finite-element-method (FEM) based software called ANSYS. Numerical and analytical results are found to be good agreement.

Keywords Free Vibration, Beam, Natural Frequency, Boundary Conditions, Geometric Characteristics

1. Introduction

A beam is a slender horizontal structural member that resists lateral loads by bending, and this important element of engineering structures appears in various forms and comprises various artifacts, such as supporting members in high-rise buildings, railways, long-span bridges, flexible satellites, gun barrels, robot arms, airplane wings, etc. [1, 2].

In many engineering applications, beams are subjected to dynamic loads, which can excite beam structural vibrations and cause durability concerns or discomfort because of the resulting noise and vibration. In addition, if the vibration exceeds certain limits, there is the danger of beam breakage or failure [3-6].

Due to beams are important structural elements, vibration analysis has been a vital task in their design for engineers and researchers for more than a century. Early investigations of the theory of vibration were given in Refs [7-9]. Then an increasing interest has been observed regarding the vibration of beams, and several studies have appeared in the general literature, some of which are provided in the Refs [10-25].

In this study, the free vibration of square cross-sectioned aluminum beams is investigated analytically and numerically under four different boundary conditions. Analytical solution is carried out using Euler-Bernoulli beam theory, in which material is assumed to be linear-isotropic, and Newton Raphson Method. This method is based on the simple idea of linear approximation, and used for finding the roots of equations. It is particularly useful for transcendental equations, composed of mixed trigonometric and hyperbolic terms. Such equations occur in vibration analysis. An example is the calculation of natural frequencies of continuous structures [26]. Solutions including the effects of the geometric characteristics, i.e., length and cross sectional area, and boundary conditions are obtained and discussed for the natural frequencies of the first three modes. Furthermore, to confirm the reliability of the vibration analysis carried out in the present paper as well, all the analytical results are checked with the corresponding numerical results obtained from Finite Element Method (FEM)-based software called ANSYS [27], where the method is established on the idea of building a complicated object with simple blocks, or, dividing a complicated object into smaller and manageable pieces [28].

Present analysis can be used as a comparative study or data for the different solution methods of future works in the related field.

2. Basic Equations

Consider an elastic beam of length L, Young's modulus E, and mass density ρ with uniform cross section A, as shown in Figure 1.

Using Euler-Bernoulli beam theory, one can obtain the equation of motion of a beam with homogeneous material properties and constant cross section as follows [5]:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=0; \quad 0\le x\le L \tag{1}$$

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where I is the area moment of inertia of the beam cross section, w is the transverse displacement, and t is time.

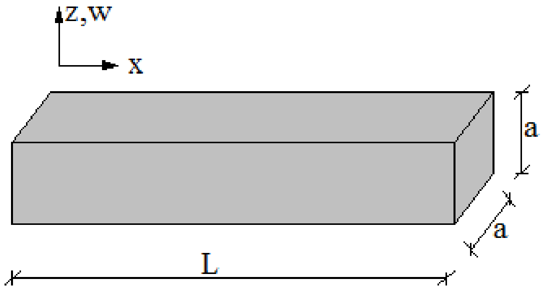

Figure 1. Geometry of the beam

Eq. (1) can be rearranged as follows:

$$EI\frac{\partial^4 w}{\partial x^4}+\kappa\frac{\partial^2 w}{\partial t^2}=0 \tag{2}$$

where $\kappa=\rho A$ is the linear mass density of the beam.

The solution of the Eq. (2) is sought by separation of variables. Assume that the displacement can be separated into two parts: one is depending on the position and the other is depending on time, as follows:

$$w(x,t)=\Lambda(x)\Psi(t) \tag{3}$$

where Λ and Ψ are independent of time and position, respectively.

Substituting Eq. (3) into Eq. (2) and after some mathematical rearrangements, the following equation is obtained:

$$-\frac{EI}{\kappa\Lambda(x)}\frac{\partial^4\Lambda(x)}{\partial x^4}=\frac{1}{\Psi(t)}\frac{\partial^2\Psi(t)}{\partial t^2} \tag{4}$$

As observed from Eq. (4), the left side depends on the variable x, and the right side depends on the variable t, as previously noted. Consequently, the variables have been separated, and each side of (4) must equal a constant, denoted $-\omega^2$ to have simple harmonic motion in the system.

$$-\frac{EI}{\kappa\Lambda(x)}\frac{\partial^4\Lambda(x)}{\partial x^4}=\frac{1}{\Psi(t)}\frac{\partial^2\Psi(t)}{\partial t^2}=-\omega^2 \tag{5}$$

If the position variable is separated

$$\frac{\partial^4\Lambda(x)}{\partial x^4}-\delta^4\Lambda(x)=0 \tag{6}$$

where

$$\delta^4=\omega^2\frac{\kappa}{EI} \tag{7}$$

If the time variable is separated

$$\frac{\partial^2\Psi(t)}{\partial t^2}+\omega^2\Psi(t)=0 \tag{8}$$

Eq. (6) is solved as follows:

$$\Lambda(x)=C_1\sinh\delta x+C_2\cosh\delta x+C_3\sin\delta x+C_4\cos\delta x \tag{9}$$

where $C_1,\ldots,C_4$ are constants, and sinh and cosh are the hyperbolic $\sin e$ and $\cos e$ functions, respectively.

Eq. (8) is solved as follows:

$$\Psi(t)=C_5\sin\omega t+C_6\cos\omega t \tag{10}$$

where C_5 *and* C_6 are constants.

Thus, if Eq. (9) is multiplied by Eq. (10) to obtain $w(x,t)$, it yields eight combined constants as:

$$w(x,t)=\left(C_1\sinh\delta x+C_2\cosh\delta x+C_3\sin\delta x+C_4\cos\delta x\right)\times\left(C_5\sin\omega t+C_6\cos\omega t\right) \tag{11}$$

where the constants C_1,C_2,C_3,C_4 can be obtained from the boundary conditions, and C_5,C_6 can be obtained from the initial conditions

Finally, using Eq. (7) the natural frequency $f_n\,(Hz)$ of the beam is found as follows:

$$f_n=\frac{\omega}{2\pi} \tag{12}$$

3. Solution of the Basic Equations

3.1. Particular Solution for C-C Beam

The boundary conditions satisfied by a C-C beam are as follows:

$$w\big|_{x=0}=0 \quad \text{and} \quad \frac{\partial w}{\partial x}\bigg|_{x=0}=0 \tag{13}$$

$$w\big|_{x=L}=0 \quad \text{and} \quad \frac{\partial w}{\partial x}\bigg|_{x=L}=0 \tag{14}$$

When Eqs. (13)-(14) are considered in Eq. (9), after some mathematical operations, the coefficient matrix is obtained as follows:

$$\begin{bmatrix}\sinh\delta L-\sin\delta L & \cosh\delta L-\cos\delta L\\ \cosh\delta L-\cos\delta L & \sinh\delta L+\sin\delta L\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{15}$$

The non-trivial solution of the determinant of the coefficient matrix is as follows:

$$\cos\delta_n\cosh\delta_n L=1 \tag{16}$$

where the subscript is an integer index.

Because the first three roots of Eq. (16) are calculated using the Newton-Raphson method, the following eigenvalues are obtained:

$$\delta L = 4.73004 \quad for\ n=1$$
$$\delta L = 7.85321 \quad for\ n=2 \qquad (17)$$
$$\delta L = 10.9956 \quad for\ n=3$$

where, δL, is the natural frequency parameter of the beam.

3.2. Particular Solution for C-F Beam

The boundary conditions satisfied by a C-F beam are as follows:

$$w\big|_{x=0} = 0 \text{ and } \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \qquad (18)$$

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0 \text{ and } \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = 0 \qquad (19)$$

When Eqs. (18)-(19) are considered in Eq. (9), after some mathematical rearrangements, the following coefficient matrix is obtained:

$$\begin{bmatrix} \sinh\delta L + \sin\delta L & \cosh\delta L + \cos\delta L \\ \cosh\delta L + \cos\delta L & \sinh\delta L - \sin\delta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (20)$$

The non-trivial solution of the determinant of the coefficient matrix is as follows:

$$\cos\delta_n L \cosh\delta_n L = -1 \qquad (21)$$

When the first three roots of Eq. (21) are calculated using Newton-Raphson method, the following eigenvalues are obtained:

$$\delta L = 1.87510 \quad for\ n=1$$
$$\delta L = 4.69409 \quad for\ n=2 \qquad (22)$$
$$\delta L = 7.85340 \quad for\ n=3$$

3.3. Particular Solution for C-SS Beam

The boundary conditions satisfied by a C-SS beam are as follows:

$$w\big|_{x=0} = 0 \text{ and } \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \qquad (23)$$

$$w\big|_{x=L} = 0 \text{ and } \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0 \qquad (24)$$

When Eqs. (23)-(24) are considered in Eq. (9), after some mathematical rearrangements, the following coefficient matrix is obtained:

$$\begin{bmatrix} \sinh\delta L - \sin\delta L & \cosh\delta L - \cos\delta L \\ \sinh\delta L + \sin\delta L & \cosh\delta L + \cos\delta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (25)$$

The solution of the determinant of the coefficient matrix is as follows:

$$\tanh\delta_n L = \tan\delta_n L \qquad (26)$$

Because the first three roots of Eq. (26) are calculated using Newton-Raphson method, the following eigenvalues are obtained:

$$\delta L = 3.9266 \quad for\ n=1$$
$$\delta L = 7.0686 \quad for\ n=2 \qquad (27)$$
$$\delta L = 10.2102 \quad for\ n=3$$

3.4. Particular Solution for SS-SS Beam

The boundary conditions satisfied by a SS-SS beam are as follows:

$$w\big|_{x=0} = 0 \text{ and } \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0 \qquad (28)$$

$$w\big|_{x=L} = 0 \text{ and } \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0 \qquad (29)$$

When Eqs. (28)-(29) are considered in Eq. (9), after some mathematical rearrangements, the following coefficient matrix is obtained:

$$\begin{bmatrix} \sinh\delta L & \sin\delta L \\ \sinh\delta L & -\sin\delta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (30)$$

The non-trivial solution of the determinant of the coefficient matrix is as follows:

$$\sin\delta_n \sinh\delta_n L = 0 \qquad (31)$$

Because the first three roots of Eq. (31) are calculated using Newton-Raphson method, the following eigenvalues are obtained:

$$\delta L = 3.1415 \quad for\ n = 1$$

$$\delta L = 6.2832 \quad for\ n = 2 \tag{32}$$

$$\delta L = 9.4248 \quad for\ n = 3$$

4. Numerical Results and Discussion

4.1. Comparative Study

In this subsection, a comparative study was performed to validate the present numerical results. The analytical results were compared with the results of the FEM -based software called ANSYS [27]. In finite-element modeling, the beam type element is applied and meshed with 50 elements.

For the first three modes $n = 1;2;3$, the analytical natural frequencies, $f_{nAnaly}\ (Hz)$, and the FEM natural frequencies, $f_{nFEM}\ (Hz)$, differ by 0.223%, 0.842%, and 1.782% under C-C boundary conditions, 0.083%, 0.592%, and 1.381% under C-F boundary conditions, 0.213%, 0.784%, and 1.698% under C-SS boundary conditions, and 0.183%, 0.722%, and 1.605% under SS-SS boundary conditions, respectively. The percentages are calculated as follows: $\left(\dfrac{f_{nAnaly} - f_{nFEM}}{f_{nAnaly}}\right) \times 100$. As shown in Figure 2, the numerical results of both methods are consistent, which shows the accuracy of the present formulation.

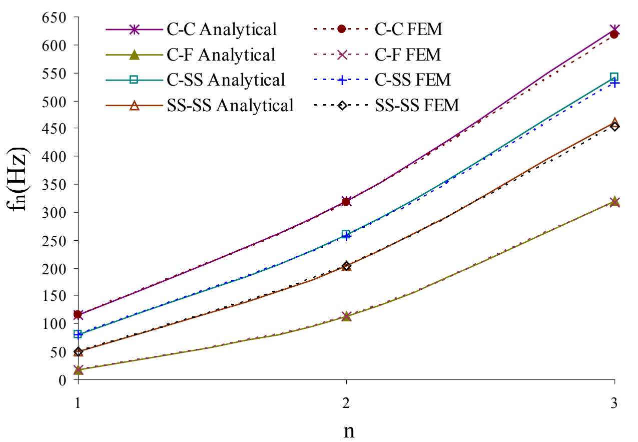

Figure 2. Comparisons between the analytical natural frequencies and those obtained using FEM -based software called ANSYS versus mode number, n $(E = 70 \times 10^9\ \mathrm{N/m^2}; \rho = 2700\,\mathrm{kg/m^3};\ \mathrm{L} = 3\,\mathrm{m};\ \mathrm{A} = 0.04\ \mathrm{m^2})$

4.2. Free Vibration Analyses of Beams

In this section, four studies were performed to investigate the free-vibration behaviors of square cross-sectioned aluminum beams with different geometric characteristics under four different boundary conditions. The natural frequencies were obtained and discussed for the first three modes (n=1, 2, 3), including the effects of the geometric characteristics, i.e., length and cross sectional area, and the boundary conditions.

Study 1:

In Figure 3, the variations in natural frequencies, $f_n\ (Hz)$, of square cross-sectioned aluminum beams versus the first three modes (n=1, 2, 3) under four different cross sections are plotted, where $\mathrm{A1;A2;\ A3}$ denote $\mathrm{A} = 0.0225\,\mathrm{m^2};\ 0.04\,\mathrm{m^2}; 0.0625\,\mathrm{m^2}$, and $\mathrm{L1;\ L2;\ L3}$ denote $\mathrm{L} = 2.75\,\mathrm{m};\ 3\,\mathrm{m}; 3.25\,\mathrm{m}$, respectively. Figure 3 shows that the natural frequencies increase with the increase in mode number. The natural frequency of the first mode number $(n = 1)$ differs from that of the second and third mode numbers $(\mathrm{n} = 2,\ 3)$ by -176% and -440% under the C-C boundary conditions, -527% and -1654% under C-F boundary conditions, -224% and -576% under C-SS boundary conditions, and -300% and -800% under SS-SS boundary conditions, respectively. The percentages were calculated as follows: $\left(\dfrac{f_{n1} - f_{ni}}{f_{n1}}\right) \times 100;\ (i = 2,3)$. Therefore, the variation in mode number has the largest effect on the natural frequency under C-F boundary conditions and the smallest effect under C-C boundary conditions.

Study 2:

Figure 3 shows that the beam has the highest natural frequencies, $f_n\ (Hz)$, under C-C boundary conditions and the lowest frequencies under C-F boundary conditions. To investigate the effect of the boundary condition on the natural frequencies, $f_n\ (Hz)$, of the beam versus the mode number, (n=1,2,3), the C-C boundary conditions were compared with the other boundary conditions. From this comparison, the following results were obtained: for n = 1, 2, and 3, i) the differences between C-C and C-F boundary conditions are 84%, 64%, and 49%, ii) the differences between C-C and C-SS boundary conditions are 31%, 19%, and 14%, and iii) the differences between C-C and SS-SS boundary conditions are 56%, 36%, and 27%, respectively. The percentages were calculated as follows: $\left(\dfrac{f_{nC-C} - f_{ni}}{f_{nC-C}}\right) \times 100;\ (i = C-F; C-SS;\ SS-SS)$. Therefore, the effect of the type of the boundary condition decreases with the increase in mode number, n.

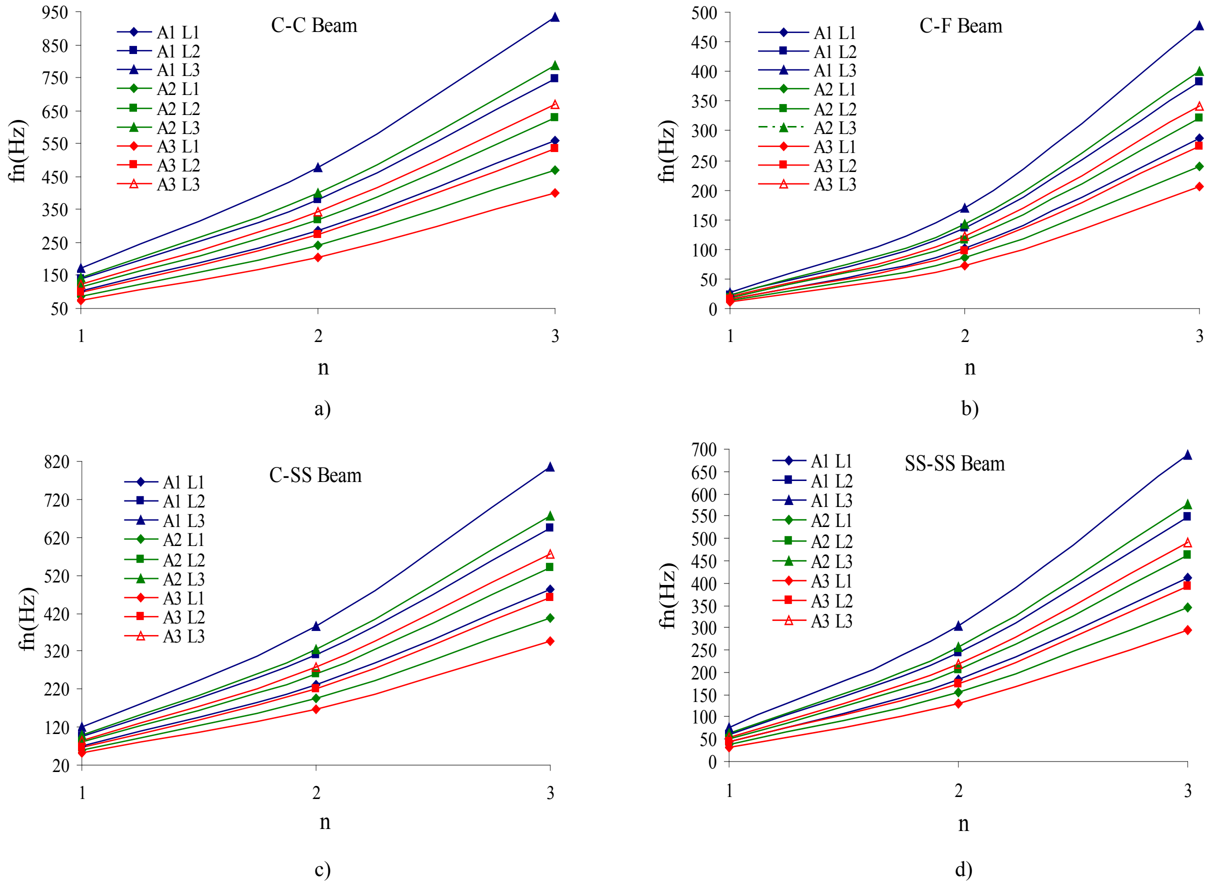

Figure 3. Variations of the natural frequencies, $f_n\,(Hz)$, of square cross-sectioned aluminum beams for the first three modes, (n=1,2,3) $\left(E = 70\times10^9\ \mathrm{N/m^2}; \rho = 2700\,\mathrm{kg/m^3}\right)$

Study 3:

Figure 4 shows the variations in natural frequency, $f_n\,(Hz)$, of square cross-sectioned aluminum beams versus the length, L (m), under four different boundary conditions, where A1;A2; A3 denotes $A = 0.0225\,\mathrm{m^2}$; $0.04\,\mathrm{m^2}; 0.0625\,\mathrm{m^2}$, and n1; n2; n3 denote $n = 1;2;3$, respectively. Figure 4 obviously shows that the natural frequencies, $f_n\,(Hz)$, decrease with the increase in length, L (m). To examine the effect of the variation in length of the beam on the natural frequencies, beams with identical cross sectional areas were compared. The result indicated that the variation in length had a constant effect on the natural frequencies: 15.97% and 28.40% for all cross sectional areas ($A = 0.0225\,\mathrm{m^2};\ 0.04\,\mathrm{m^2}; 0.0625\,\mathrm{m^2}$) under four different boundary conditions. The percentages were calculated as follows: $\left(\dfrac{f_{nL=2.75} - f_{Li}}{f_{nL=2.75}}\right)\times100; \quad (i = 3,\ 3.25)$. Thus, the change in length of the beam has a constant effect on its natural frequencies for various boundary conditions and cross sectional areas.

Study 4:

Figue 5 shows the variations in natural frequencies of square cross-sectioned aluminum beams versus the cross sectional area, A ($\mathrm{m^2}$), under four different cross sections, where L1;L2; L3 denote $L = 2.75$ m;3 m;3.25 m, and n1; n2; n3 denote $n = 1;\ 2;3$, respectively. Figure 5 shows that the natural frequencies increase with the increase in cross sectional area. To investigate the effect of the variation in cross sectional area of the beam on the natural frequencies, beams with identical lengths were compared. The result indicated that variation of the cross sectional area had a constant effect on the natural frequencies: -33.33% and -66.67% for $L = 2.75$ m;3 m;3.25 m under four different boundary conditions. The percentages were calculated as follows: $\left(\dfrac{f_{nA=0.0225} - f_{Ai}}{f_{nA=0.0225}}\right)\times100;$ $(i = 0.04,\ 0.0625)$. Thus, the change in cross sectional area of the beam has a constant effect on the natural frequencies of the beam for various boundary conditions and lengths.

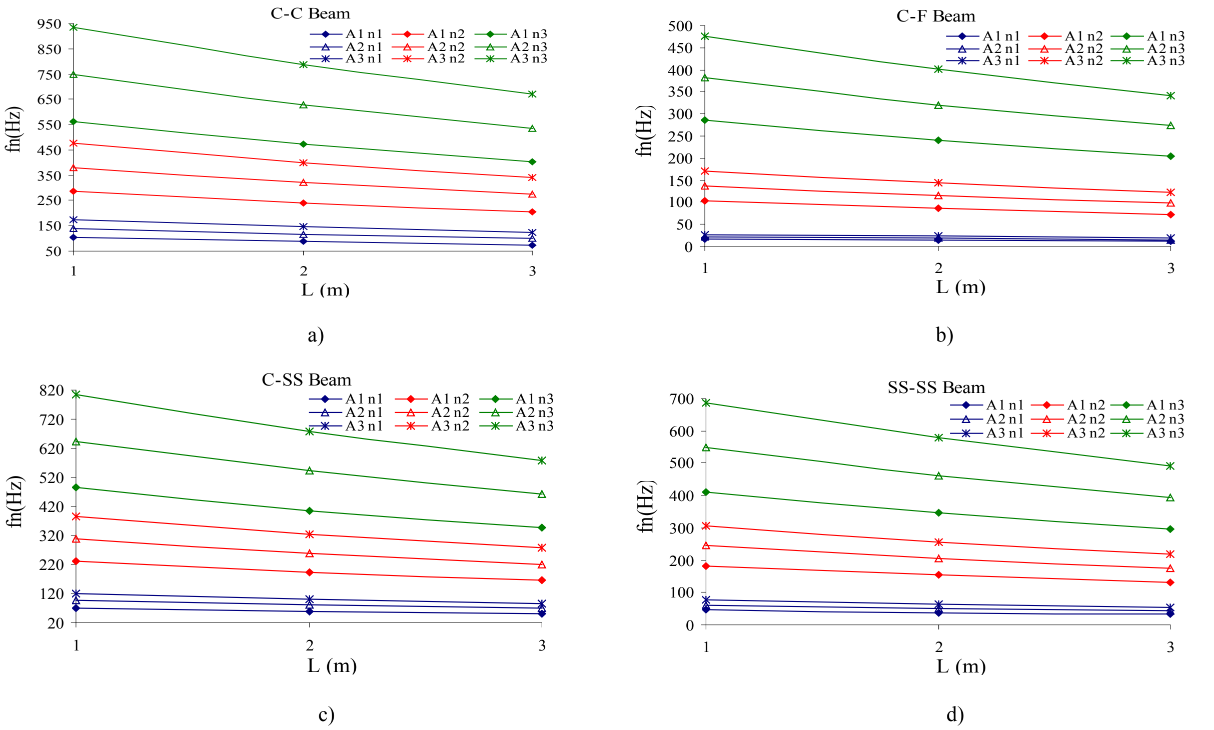

Figure 4. Variations of the natural frequencies, $f_n\,(Hz)$, of square cross-sectioned aluminum beams versus the length, L (m) $\left(E = 70\times10^9\ \text{N/m}^2 ; \rho = 2700\,\text{kg/m}^3\right)$

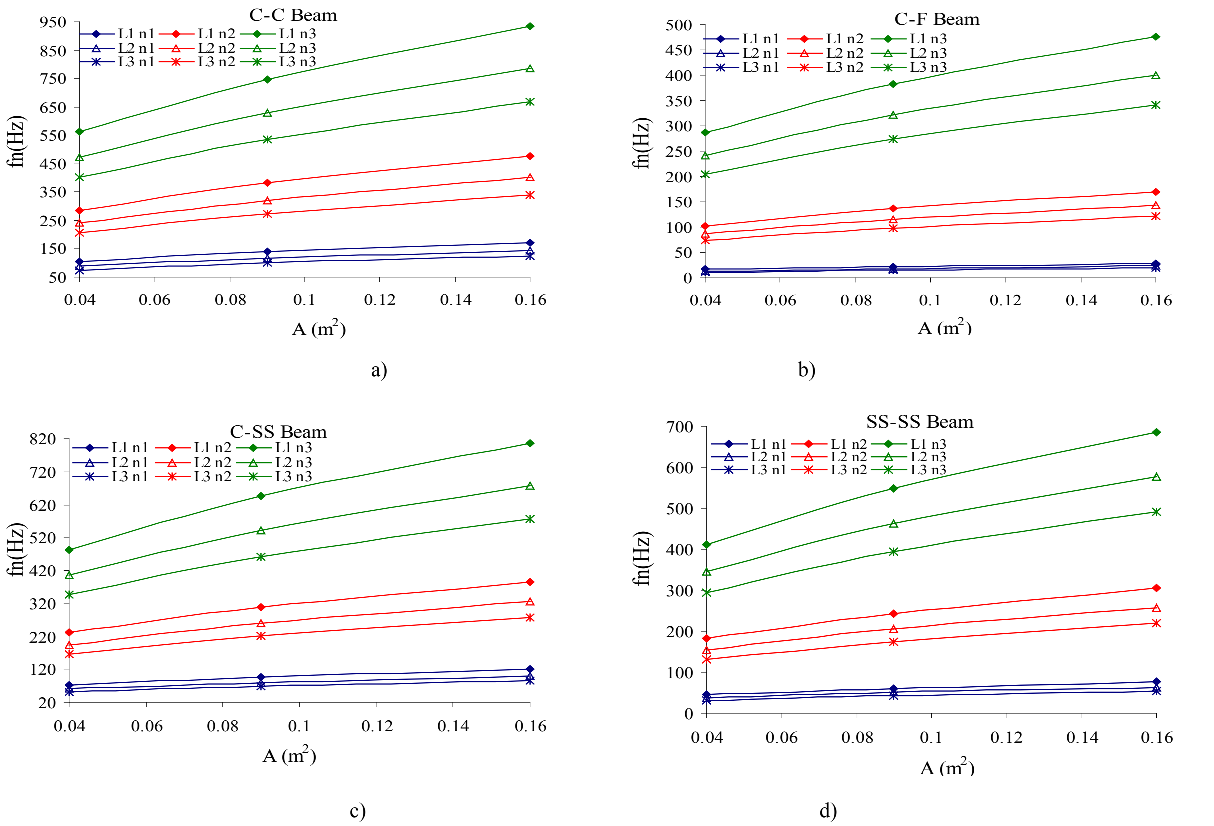

Figure 5. Variations of the natural frequencies of square cross-sectioned aluminum beams versus the cross sectional area, A (m^2) $\left(E = 70\times10^9\ \text{N/m}^2 ; \rho = 2700\,\text{kg/m}^3\right)$

5. Conclusions

In this study, the free vibration of square cross-sectioned aluminum beams is investigated analytically and numerically under four different boundary conditions. Analytical solution is carried out using Euler-Bernoulli beam theory, in which material is assumed Solutions including the effects of the geometric characteristics, i.e., length and cross sectional area, and boundary conditions are obtained and discussed for the natural frequencies of the first three modes.

The following results were obtained:

(1) The natural frequencies increase with the increase in mode number.

(2) The change in mode numbers has the largest effect on the natural frequency under C-F boundary conditions and the smallest effect under C-C boundary conditions.

(3) The beam has the highest natural frequencies under C-C boundary conditions and the lowest frequencies under C-F boundary conditions.

(4) The effect of the type of the boundary condition decreases with increasing mode number.

(5) The natural frequencies of the beam decrease with increasing length.

(6) The change in length of the beam has a constant effect on its natural frequencies for various boundary conditions and cross sectional areas.

(7) The natural frequencies of the beam increase with increasing cross sectional area.

(8) The change in cross sectional area of the beam has a constant effect on the natural frequencies for various boundary conditions and lengths.

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