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Duffing Oscillator with Heptic Nonlinearity under Single Periodic Forcing

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Abstract In this paper, the Homotopy Analysis Method (HAM) is used to obtain an accurate analytical two-term approximate solution to the positively damped cubic-quintic-heptic Duffing equation with algebraically decaying amplitude as well as a single periodic forcing. This paper also presents the interesting behavior of the non-zero auxiliary parameter which provides a convenient way to adjust and control the convergence of the approximations. Our analysis shows that neither the strength of the damping nor that of the forcing exerts any influence on the auxiliary parameter of the nonlinearity. We *observe* also that the degree of damping is elicited by the degree of nonlinearity and the initial guesses of the time constants

Keywords Homotopy Analysis, Duffing Oscillator, Heptic Nonlinearity, Periodic Forcing, Damping

1. Introduction

The Duffing equations have been presented in the literature considering various types of nonlinearity. Pirbodaghi et al[1] have worked on the Duffing equations with cubic and quintic nonlinearities. Some authors[2],[3],[4],[5][6],[7] and[8] have investigated many kinds of nonlinear oscillatory systems in physics, mechanics and engineering. Duffing equations with cubic and quintic nonlinearities have not been extensively studied as the one with cubic nonlinearity because of its complexity[9, 10, 11, 12]. The cubic nonlinear Duffing oscillator had been studied extensively and has been used as model for seismic analysis[13]) and for *earthquake prediction[5]. Recently the cubic-quintic nonlinearity in Duffing* oscillator has been engaging attention[14, 6, 15] and very interesting features have been detected which were non-existent in the cubic nonlinear model. Interesting results have been obtained from the geometric and analytic studies of the un-damped and unforced cubic-quintic Duffing equation using different methods[1, 16, 17]. It has been observed that the cubic-quintic nonlinearity gives vent to damping in a morepronounced way than the cubic nonlinearity. *We therefore in this paper investigate the influence of higher nonlinearity particularly heptic nonlinearity on the damping of Duffing oscillator.*

The nth-order Duffing equation with viscous damping and periodic forcing can be generally expressed as

$$\ddot{u}+\delta\dot{u}+\sum_{n=0}^{N}a_n u^{2n+1}=\sum_{m=0}^{M}\left(F_m\cos\omega t+X_m\sin\omega t\right) \quad (1.0)$$

Where δ is the damping coefficient, and a_n, F_m, X_m are arbitrary constants, $N, M<\infty$

In this paper the homotopy analysis method[18, 19,20]which has been effectively applied to a wide variety of problems in applied mathematics, physics and engineering is applied to the positively damped cubic-quintic-heptic Duffing equation with a single sinusoidal forcing governed by

$$\ddot{u}(t)=f[\dot{u}(t),u(t),F],\ t\geq 0 \quad (1.1)$$

and subject to the initial conditions

$$u(0)=a,\quad \dot{u}(0)=b \quad (1.2)$$

The cubic-quintic-heptic Duffing equation can be used to model a classical particle in a double-well potential. It was used in[21] to model the dynamic behavior of a cargo system under crossover indirect tie-down.

As was noted in[20], the homotopy analysis method which was proposed by introducing an auxiliary parameter g to construct a new kind of homotopy in a more general form has the following advantages:

1). It is valid even if a given nonlinear problem does not contain any small/large parameters at all.

2). It equips us with a convenient way to adjust and control convergence regions of series of analytic approximations.

3). It can be efficiently employed in approximating a nonlinear problem by choosing different sets of base functions.

We investigate here factors that influence the non-zero auxiliary parameter g.

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Free oscillations of a positively damped system have two different time scales. One is related to the frequency of oscillationandthe other to the decaying amplitude of oscillation. It is clear that the free oscillation of positively damped systems can be expressed by the set of base functions

$$\left\{(1+\zeta t)^{-m}\sin(n\omega t),(1+\zeta t)^{-m}\cos(n\omega t)\middle|m\geq 1,n\geq 0\right\} \quad (1.3)$$

where the two unknown time-constants $\omega and \zeta$ relate to the two time scales

$$\tau_1=\omega t,\ \tau_2=\zeta t \quad (1.4)$$

respectively.

Having noted this, it will be reasonable to say that the set of base functions given by (1.3) can as well be used to express the forced oscillation of a positively damped system as long as the forcing is a function of one of the two time scales. In this regard, we suggest a periodicforcing with amplitude ω and given by

$$F\cos\omega t \quad (1.5)$$

Under the transformation (1.4) one has $u(t) = u(\tau_1,\tau_2)$, and the original governing equation (1.1) becomes

$$\omega^2\frac{\partial^2 u}{\partial\tau_1^2}+2\omega\zeta\frac{\partial^2 u}{\partial\tau_1\partial\tau_2}+\zeta^2\frac{\partial^2 u}{\partial\tau_2^2}=f\left[\omega u_{\tau_1}+\zeta u_{\tau_2},F(\tau_1)\right] \quad (1.6)$$

subject to the initial conditions

$$u(\tau_1,\tau_2)=a,\ \omega\frac{\partial u(\tau_1,\tau_2)}{\partial\tau_1}+\zeta\frac{\partial u(\tau_1,\tau_2)}{\partial\tau_2}=b,$$

when

$$\tau_1=\tau_2=0 \quad (1.7)$$

2. Basic Ideas

A real function can be efficiently represented by a better set of base functions. Hence, the need to approximate a given nonlinear problem by a proper set of base functions equips us with the initiative to apply the homotopy analysis method. Clearly, according to (1.3) and the definition (1.4), the considered problem can be represented by (1.3) such that

$$u(\tau_1,\tau_2)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(1+\zeta t)^{-m}\left[\eta_{m,n}\sin n\omega t+\vartheta_{m,n}\cos n\omega t\right] \quad (2.0)$$

Where $\eta_{m,n},\vartheta_{m,n}$ arecoefficients. This provides us with a rule, called the Rule of Solution Expression[15].

As was noted in[20], HAM is based on continuous variations from the initial guesses to the exact solution of a considered problem. For the problem under consideration, one constructs the continuous mapping $u(\tau_1,\tau_2)\to U(\tau_1,\tau_2,q),\omega\to\Omega(q),\zeta\to\Lambda(q)$ in such a way that $U(\tau_1,\tau_2,q),\Omega(q),\Lambda(q)$ vary from their initial guesses $u_0(\tau_1,\tau_2),\omega_0,\zeta_0$ to their exact solutions $u(\tau_1,\tau_2),\omega,\zeta$ respectively as q (the embedding parameter) increases from 0 to 1. In line with the above reasons one constructs a family in q of nonlinear differential equations

$$(1-q)\left\{L\left[U(\tau_1,\tau_2,q)\right]-L_0\left[u_0(\tau_1,\tau_2)\right]\right\}=gpN\left[U(\tau_1,\tau_2,q)\right] \quad (2.1)$$

with initial conditions

$$U(\tau_1,\tau_2,q)=\varphi,\tau_1=\tau_2=0 \quad (2.2)$$

$$\text{And}\quad \omega\frac{\partial u}{\partial\tau_1}+\zeta\frac{\partial u}{\partial\tau_2}=\psi,\ \tau_1=\tau_2=0 \quad (2.3)$$

where $u_0(\tau_1,\tau_2)$ is an initial guess of $u(\tau_1,\tau_2)$, N is a nonlinear operator defined by

$$N[\cup(\tau_1,\tau_2,q)]=\Omega^2(q)\frac{\partial^2 u}{\partial^2\tau_1}+2\Omega(q)\Lambda(q)\frac{\partial^2 u}{\partial\tau_1\partial\tau_2}+\Lambda^2(q)\frac{\partial^2 u}{\partial^2\tau_2}-f\left[\Omega(q)\frac{\partial u}{\partial\tau_1}+\Lambda(q)\frac{\partial u}{\partial\tau_2},u,F(\tau_1)\right] \quad (2.4)$$

$Land\, L_0$are two auxiliary linear operators defined by

$$L[\cup(\tau_1,\tau_2,q)]=\Omega^2\frac{\partial^2\cup}{\partial^2\tau_1}+2\Omega\Lambda(1+\tau_2)\frac{\partial^2\cup}{\partial\tau_1\partial\tau_2}+\Lambda^2(1+\tau_2)^2\frac{\partial^2\cup}{\partial^2\tau_2}+\lambda_1\left[\Omega\frac{\partial\cup}{\partial\tau_1}+\Lambda(1+\tau_2)\frac{\partial\cup}{\partial\tau_2}\right]+\lambda_2 \quad (2.5)$$

and

$$L_0[u_0(\tau_1,\tau_2)]=\omega_0{}^2\frac{\partial^2 u_0}{\partial^2\tau_1}+2\omega_0\zeta_0(1+\tau_2)\frac{\partial^2 u_0}{\partial\tau_1\partial\tau_2}+\zeta_0{}^2(1+\tau_2)^2\frac{\partial^2 u_0}{\partial^2\tau_2}+\lambda_1\left[\omega_0\frac{\partial u_0}{\partial\tau_1}+\zeta_0(1+\tau_2)\frac{\partial u_0}{\partial\tau_2}\right]+\lambda_2 u_0 \quad (2.6)$$

in which ω_0 and ζ_0 are initial guesses of the time-constants $\omega and \zeta$ respectively, and λ_1,λ_2 are two constants to be determined later in terms of ζ_0 and ω_0 respectively. The auxiliary non-zero parameter g equips us with a convenient way to adjust and control the convergence of approximations such that a properly chosen g guarantees the convergence of the resulting series to be given later at $q = 1$.

Under the Rule of Solution Expression[20], one chooses

$$u_0(\tau_1,\tau_2)=a_0\cos\tau_1+(a_1\cos\tau_1+b_1\sin\tau_1)(1+\tau_2)^{-1} \quad (2.7)$$

as the initial guess of $u(\tau_1,\tau_2)$, where $a_0, a_1 and b_1$ are unknown constants to be determined later.

When $q = 0$, Eq. (2.1) has the solution (2.7) with

$$\Omega(0) = \omega_0, \quad \Lambda(0) = \zeta_0 \tag{2.8}$$

and when $q = 1$, Eqs. (2.1)-(2.3) becomes exactly (1.6) and (1.7) as long as $\Omega(1) = \omega and \Lambda(1) = \zeta$.

Thus,

$$\cup(\tau_1, \tau_2, 1) = u(\tau_1, \tau_2) \tag{2.9}$$

Therefore as q increases from 0 to 1, $\cup$ truly varies from the initial trial u_0 to the exact solution u of the original equations (1.6) and (1.7); so do Ω and Λ vary from the initial guesses to the time-constants $\omega and \zeta$ respectively. Equations (2.1)-(2.3) are the zeroth-order deformation equations.

With the nature of (2.8), $\Omega(q)$, $\cup(\tau_1, \tau_2, q) and \Lambda(q)$ can be expanded in power series of q by Taylor's theorem as:

$$\Omega(q) = \omega_0 + \sum_{k=1}^{+\infty} \frac{\omega_0^{[k]}}{k!} q^k \tag{2.10}$$

$$\Lambda(q) = \zeta_0 + \sum_{k=1}^{+\infty} \frac{\zeta_0^{[k]}}{k!} q^k \tag{2.11}$$

$$\cup(\tau_1, \tau_2, 1) = u_0(\tau_1, \tau_2) + \sum_{k=0}^{+\infty} \frac{u_0^{[k]}}{k!} q^k \tag{2.12}$$

where,

$$\omega_0^{[k]} = \frac{d^k \Omega(q)}{dq^k}, q = 0 \quad \zeta_0^{[k]} = \frac{d^k \Lambda(q)}{dq^k}, q = 0 \quad u_0^{[k]} = \frac{\partial^k \cup}{\partial q^k}, q = 0 \tag{2.13}$$

are the kth-order deformation derivatives. The auxiliary non-parameter g influences the convergence of the series (2.10) – (2.12).

A convergent series given by the HAM (at$q = 1$) must be an exact solution of the considered problem as was proved in[8]. Hence, one obtains

$$\omega = \omega_0 + \sum_{k=1}^{+\infty} \omega_k \tag{2.14}$$

$$\zeta = \zeta_0 + \sum_{k=1}^{+\infty} \zeta_k \tag{2.15}$$

$$u(\tau_1, \tau_2) = u_0(\tau_1, \tau_2) + \sum_{k=0}^{+\infty} u_k(\tau_1, \tau_2) \tag{2.16}$$

where

$$\omega_k = \frac{\omega_0^{[k]}}{k!}, \zeta_k = \frac{\zeta_0^{[k]}}{k!}, u_k(\tau_1, \tau_2) = \frac{u_0^{[k]}(\tau_1, \tau_2)}{k!} \tag{2.17}$$

If we differentiate Equations. (2.1) – (2.3) k times with respect to q and then set $q = 0$ and finally divide them by $k!$. We obtain the so-called kth-order deformation equation[2]

$$L_0[u_k(\tau_1, \tau_2) - \chi_k u_{k-1}(\tau_1, \tau_2)] = gR_k(\tau_1, \tau_2) - W_k(\tau_1, \tau_2) + \chi_k W_{k-1}(\tau_1, \tau_2) \tag{2.18}$$

subject to the corresponding initial conditions at $(\tau_1 = 0, \tau_2 = 0)$

$$u_k(\tau_1, \tau_2, q) = \varphi \tag{2.19}$$

$$\text{And } \omega \frac{\partial u_k}{\partial \tau_1} + \zeta \frac{\partial u_k}{\partial \tau_2} = 0 \tag{2.20}$$

where

$$W_k(\tau_1, \tau_2) = \sum_{n=1}^{k} \Big[\big(\sum_{j=0}^{n} \omega_j \omega_{n-j}\big) \frac{\partial^2 u_{k-n}}{\partial \tau_1^2} + 2\big(\sum_{j=0}^{n} \omega_j \zeta_{n-j}\big)(1 + \tau_2) \frac{\partial^2 u_{k-n}}{\partial \tau_1 \partial \tau_2} + \big(\sum_{j=0}^{n} \zeta_j \zeta_{n-j}\big)(1 + \tau_2)^2 \frac{\partial^2 u_{k-n}}{\partial \tau_2^2} +$$

$$\lambda 1 \omega n \partial uk - n \partial \tau 1 + \zeta n 1 + \tau 2 \, \partial uk - n \partial \tau 2 \tag{2.21}$$

and

$$R_k(\tau_1, \tau_2) = \frac{1}{(k-1)!} \frac{d^{k-1} N[\cup]}{dq^{k-1}}, \ at \ q = 0 \tag{2.22}$$

in which $N[\cup]$ is given by (2.4) and

$$\chi_k = \begin{cases} 0, k \leq 1 \\ 1, k > 1 \end{cases} \tag{2.23}$$

from (2.20), the initial condition becomes

$$\left(\sum_{k=0}^{m} \omega_k \frac{\partial u_k}{\partial \tau_1} + \sum_{k=0}^{m} \zeta_k \frac{\partial u_k}{\partial \tau_2}\right) \left(\sum_{j=0}^{k} u_j(\tau_1, \tau_2)\right) = \psi \tag{2.24}$$

Now to determine $\lambda_1 and \lambda_2$, we demand that for any constants $P_1 and P_2$ the equation

$$L_0[(1 + \tau_2)^{-1}(P_1 \sin \tau_1 + P_2 \cos \tau_1)] = 0 \tag{2.25}$$

holds.

Then

$$\lambda_1 = 2\zeta_0, \quad \lambda_2 = \omega_0^2 \tag{2.26}$$

Hence, the initial guesses ω_0, ζ_0 play a crucial role in determining the initial guess $u_0(\tau_1, \tau_2)$ and the auxiliary linear operators$L and L_0$.

However we must take note of the existence of the terms $(1 + \tau_2)^{-1} \sin(\tau_1)$ and $(1 + \tau_2)^{-1} \cos(\tau_1)$ on the right hand side of (2.18). *Their existence goes contrary to the Rule of Solution Expression, which is clearly described by (2.0). Therefore, for a uniformly valid solution, one has to set the coefficients of these two terms to zero.*

We must as well note that for un-damped systems,

$$W_k(\tau_1, \tau_2) = W_{k-1}(\tau_1, \tau_2) = \delta = \zeta = 0 \tag{2.27}$$

It is very important not to forget that for the particular problem (positively-damped with algebraically decaying amplitude), we must select suitable a_0, ω_0 $and \zeta_0$ initially.

3. Application of HAM

Writing (1.1) explicitly under the transformations given in (1.4), we obtain

$$\omega^2 \frac{\partial^2 u}{\partial^2 \tau_1} + 2\omega\zeta \frac{\partial^2 u}{\partial \tau_1 \partial \tau_2} + \zeta^2 \frac{\partial^2 u}{\partial^2 \tau_2} + \delta \left[\omega \frac{\partial u}{\partial \tau_1} + \zeta \frac{\partial u}{\partial \tau_2}\right] + \alpha u + \beta u^3 + \mu u^5 + \rho u^7 = F \cos \tau_1 \quad (3.0)$$

subject to the corresponding initial conditions

$$u(\tau_1, \tau_2) = a, \quad \omega \frac{\partial u(\tau_1,\tau_2)}{\partial \tau_1} + \zeta \frac{\partial u(\tau_1,\tau_2)}{\partial \tau_2} = 0, \quad when \tau_1 = \tau_2 = 0 \quad (3.1)$$

Equation (3.0) can also be obtained from (1.0) by setting $M = 0, X_m = 0, N = 3, a_0 = \alpha, a_1 = \beta, a_2 = \mu, and a_3 = \rho$, after which the transformation depicted by (1.4) is used.

We construct such a family of equations as described by (2.1) where the auxiliary linear operator is given by

$$L_0[u_0(\tau_1, \tau_2)] = \omega_0{}^2 \frac{\partial^2 u_0}{\partial^2 \tau_1} + 2\omega_0\zeta_0(1 + \tau_2) \frac{\partial^2 u_0}{\partial \tau_1 \partial \tau_2} + \zeta_0{}^2(1 + \tau_2)^2 \frac{\partial^2 u_0}{\partial^2 \tau_2} + 2\zeta_0 \left[\omega_0 \frac{\partial u_0}{\partial \tau_1} + \zeta_0(1 + \tau_2) \frac{\partial u_0}{\partial \tau_2}\right] + \omega_0{}^2 u_0 \quad (3.2)$$

and the nonlinear operator with the help of (3.0) is given by

$$N[\cup(\tau_1, \tau_2, q)] = \omega^2 \frac{\partial^2 \cup}{\partial^2 \tau_1} + 2\omega\zeta \frac{\partial^2 \cup}{\partial \tau_1 \partial \tau_2} + \zeta^2 \frac{\partial^2 \cup}{\partial^2 \tau_2} + \delta \left[\omega \frac{\partial \cup}{\partial \tau_1} + \zeta \frac{\partial \cup}{\partial \tau_2}\right] + \alpha \cup + \beta \cup^3 + \mu \cup^5 + \rho \cup^7 - F \cos \tau_1 \quad (3.3)$$

Employing (2.0) and (3.2), we choose our initial guess of $u(\tau_1, \tau_2)$ as

$$u_0(\tau_1, \tau_2) = a(1 + \tau_2)^{-1} \left[\cos \tau_1 + \frac{\zeta_0}{\omega_0} \sin \tau_1\right] \quad (3.4)$$

We set $k = 1$ in (2.18) to obtain

$$L_0[u_1(\tau_1, \tau_2) - u_0(\tau_1, \tau_2)] = g R_1(\tau_1, \tau_2) - W_1(\tau_1, \tau_2) \quad (3.5)$$

For uniformly valid solution, two linear algebraic equations were obtained and solved after setting to zero the coefficients of $(1 + \tau_2)^{-1} \sin(\tau_1)$and $(1 + \tau_2)^{-1} \cos(\tau_1)\, in$ (3.5). The results obtained respectively are as follows:

$$\zeta_1 = -\frac{g}{(\omega_0{}^2 + \zeta_0{}^2)} \left[\frac{\zeta_0 \omega_0{}^2}{2} + \frac{B \omega_0 \zeta_0{}^2}{1+\tau_2} - \frac{\delta \omega_0 \zeta_0 B}{2} - \frac{\omega_0{}^2}{2\zeta_0} - \frac{\delta \omega_0{}^2}{2} + \frac{\zeta_0 \omega_0{}^2}{1+\tau_2} + \frac{F\zeta_0(1+\tau_2)}{2a} + \frac{5B\omega_0 \mu a^4}{4(1+\tau_2)^4} + \frac{7B\rho a^6[29 - 50B^2 + 30B^4 + 2B^6]}{128(1+\tau_2)^6}\right] \quad (3.6)$$

$$\omega_1 = g\left[-\frac{\omega_0}{2} - B\zeta_0(1 + \tau_2)^{-1} + B\zeta_0(1 + \tau_2)^{-2} + \frac{\delta B}{2} - \frac{\delta B(1+\tau_2)^{-1}}{2} + \frac{\alpha}{2\omega_0} - \frac{F(1+\tau_2)}{2a\omega_0} + \frac{3\beta a^2 (1+\tau_2)^{-2}}{8\omega_0}[1 + B^2] + \frac{5\mu a^4 (1+\tau_2)^{-4}}{8\omega_0}\left[\frac{1}{2} + \right.\right.$$

B2+B42+ρa61+τ2−6128ω021+35B2+140B4+35B6 (3.7)

where $B = \frac{\zeta_0}{\omega_0}$.

After eliminating terms that brings non-uniformity equation (3.5) becomes

$$L_0[u_1(\tau_1, \tau_2)] = g \times TCST\ (terms not containing secular terms) \quad (3.8)$$

Equation (3.8) has the solution

$$u_1(\tau_1, \tau_2) = u_{1P} + (1 + \tau_2)^{-1}[a_1 \cos \tau_1 + b_1 \sin \tau_1] \quad (3.9)$$

where

$$u_{1P}(\tau_1, \tau_2) = g[(P_1 K_1 + P_2 K_2 + P_3 K_3) \cos 3\tau_1 + (P_4 K_4 + P_5 K_5 + P_6 K_6) \sin 3\tau_1 + (P_7 K_7 + P_8 K_8) \cos 5\tau_1 + (P_9 K_9 +$$

P10K10sin5τ1+P11K11cos7τ1+P12K12sin7τ1 (3.10)

$$P_1 = \frac{\beta a^3(1-3B^2)(1+\tau_2)^{-3}}{4}, \ P_2 = \frac{5\mu a^5(1-2B^2-3B^4)(1+\tau_2)^{-5}}{16}, \ P_3 = \frac{21\rho a^7(1-9B^2-5B^4-3B^6)(1+\tau_2)^{-7}}{64} \quad (3.11)$$

$$K_1 = \frac{(6\zeta_0{}^2 - 8\omega_0{}^2 + 12\omega_0\zeta_0)}{(6\zeta_0{}^2 - 8\omega_0{}^2)^2 + 144\omega_0{}^2\zeta_0{}^2}, \ K_2 = \frac{(20\zeta_0{}^2 - 8\omega_0{}^2 + 24\omega_0\zeta_0)}{(20\zeta_0{}^2 - 8\omega_0{}^2)^2 + 576\omega_0{}^2\zeta_0{}^2}, K_3 = \frac{(42\zeta_0{}^2 - 8\omega_0{}^2 + 36\omega_0\zeta_0)}{(42\zeta_0{}^2 - 8\omega_0{}^2)^2 + 1296\omega_0{}^2\zeta_0{}^2} \quad (3.12)$$

$$P_4 = \frac{\beta a^3 B(3-3B^2)(1+\tau_2)^{-3}}{4}, \ P_5 = \frac{5B\mu a^5(3+2B^2-B^4)(1+\tau_2)^{-5}}{16}, \ P_6 = \frac{21B\rho a^7(-10+30B^2-3B^4-3B^6)(1+\tau_2)^{-7}}{64} \quad (3.13)$$

$$K_4 = \frac{(6\zeta_0{}^2 - 8\omega_0{}^2 - 12\omega_0\zeta_0)}{(6\zeta_0{}^2 - 8\omega_0{}^2)^2 + 144\omega_0{}^2\zeta_0{}^2}, \ K_5 = \frac{(20\zeta_0{}^2 - 8\omega_0{}^2 - 24\omega_0\zeta_0)}{(20\zeta_0{}^2 - 8\omega_0{}^2)^2 + 576\omega_0{}^2\zeta_0{}^2}, K_6 = \frac{(42\zeta_0{}^2 - 8\omega_0{}^2 - 36\omega_0\zeta_0)}{(42\zeta_0{}^2 - 8\omega_0{}^2)^2 + 1296\omega_0{}^2\zeta_0{}^2} \quad (3.14)$$

$$P_7 = \frac{\mu a^5(1-10B^2+5B^4)(1+\tau_2)^{-5}}{16}, P_8 = \frac{7\rho a^7(1-12B^2-5B^4+5B^6)(1+\tau_2)^{-7}}{64}, K_7 = \frac{(20\zeta_0{}^2 - 24\omega_0{}^2 + 40\omega_0\zeta_0)}{(20\zeta_0{}^2 - 24\omega_0{}^2)^2 + 1600\omega_0{}^2\zeta_0{}^2} \quad (3.15)$$

$$K_8 = \frac{(42\zeta_0{}^2 - 24\omega_0{}^2 + 60\omega_0\zeta_0)}{(42\zeta_0{}^2 - 24\omega_0{}^2)^2 + 3600\omega_0{}^2\zeta_0{}^2}, P_9 = \frac{\mu B a^5(5-10B^2+5B^4)(1+\tau_2)^{-5}}{16}, P_{10} = \frac{7B\rho a^7(5-5B^2-9B^4+B^6)(1+\tau_2)^{-7}}{64} \quad (3.16)$$

$$K_9 = \frac{(20\zeta_0{}^2 - 24\omega_0{}^2 - 40\omega_0\zeta_0)}{(20\zeta_0{}^2 - 24\omega_0{}^2)^2 + 1600\omega_0{}^2\zeta_0{}^2}, K_{10} = \frac{(42\zeta_0{}^2 - 24\omega_0{}^2 - 60\omega_0\zeta_0)}{(42\zeta_0{}^2 - 24\omega_0{}^2)^2 + 3600\omega_0{}^2\zeta_0{}^2}, P_{11} = \frac{7\rho a^7\left(\frac{1}{7} - 3B^2 + 5B^4 - B^6\right)(1+\tau_2)^{-7}}{64} \quad (3.17)$$

$$K_{11} = \frac{(42\zeta_0{}^2 - 48\omega_0{}^2 + 84\omega_0\zeta_0)}{(42\zeta_0{}^2 - 48\omega_0{}^2)^2 + 7056\omega_0{}^2\zeta_0{}^2}, P_{12} = \frac{7B\rho a^7\left(1 - 5B^2 + 3B^4 - \frac{B^6}{7}\right)(1+\tau_2)^{-7}}{64}, K_{12} = \frac{(42\zeta_0{}^2 - 48\omega_0{}^2 - 84\omega_0\zeta_0)}{(42\zeta_0{}^2 - 48\omega_0{}^2)^2 + 7056\omega_0{}^2\zeta_0{}^2} \quad (3.18)$$

From (3.1), *we obtain*

$$a_1 = g[P_7K_7 + P_8K_8 + P_{11}K_{11} + P_1K_1 + P_2K_2 + P_3K_3], \qquad where\tau_1 = \tau_2 = 0 \tag{3.19}$$

$$b_1 = \frac{\zeta_1}{\omega_1}[a + 2g(P_7K_7 + P_8K_8 + P_{11}K_{11} + P_1K_1 + P_2K_2 + P_3K_3)]$$
$$-\left[a\frac{\zeta_0}{\omega_0} + 5g(P_9K_9 + P_{10}K_{10}) + 7gP_{12}K_{12} + 3g(P_4K_4 + P_5K_5 + P_6K_6)\right], where\tau_1 = \tau_2 = 0 \tag{3.20}$$

Following the same procedure, one can obtain $\omega_2, \zeta_2, u_2(\tau_1, \tau_2)$ and so on.

We must note that for the application done above, we have chosen$a_0 = 0$. This choice is not mandatory, but a_0 must be chosen to be so close to zero or zero. The general rule for choosing a_0 was also given in[20].

4. Results

In this section we use the results given above to obtain the solutions to

1). Positively damped cubic-quintic Duffing equation with a single periodic forcing.

2). Positively damped and unforced cubic-quintic-heptic Duffing equation.

3). Un-damped cubic-quintic-heptic Duffing equation with a single periodic forcing.

4). Un-damped and unforced cubic-quintic-heptic Duffing equation.

5). Un-damped cubic-quintic Duffing equation with a single periodic forcing.

6). Un-damped and unforced cubic-quintic Duffing equation.

After applying the transformation in (1.4) and subject to the initial conditions prescribed in (3.1), the positively damped cubic-quintic Duffing equation with a single periodic forcing is obtained from (1.0) by setting $M = 0, X_m = 0, N = 2, a_0 = \alpha, a_1 = \beta, a_2 = \mu$ *giving*

$$\omega^2\frac{\partial^2 u}{\partial^2\tau_1} + 2\omega\zeta\frac{\partial^2 u}{\partial\tau_1\partial\tau_2} + \zeta^2\frac{\partial^2 u}{\partial^2\tau_2}$$
$$+\delta\left[\omega\frac{\partial u}{\partial\tau_1} + \zeta\frac{\partial u}{\partial\tau_2}\right] + \alpha u + \beta u^3 + \mu u^5 = F\cos\tau_1 \tag{4.0}$$

Setting $\rho = 0$ in (3.9), (3.7) and (3.6) one obtains the first-order approximate solution to (4.0) with the same initial guess given in (3.4). Consequently, other higher-order approximate solutions to (4.0) can as well be obtained provided $\rho = 0$.

After applying the transformation in (1.4) and subject to the initial conditions prescribed in (3.1) the positively damped and unforced cubic-quintic-heptic Duffing equation is obtained by setting $M = 0, X_m = 0, F_m = 0, N = 3, a_0 = \alpha, a_1 = \beta, a_2 = \mu, and\ a_3 = \rho$ *in (1.0) as*

$$\omega^2\frac{\partial^2 u}{\partial^2\tau_1} + 2\omega\zeta\frac{\partial^2 u}{\partial\tau_1\partial\tau_2} + \zeta^2\frac{\partial^2 u}{\partial^2\tau_2} + \delta\left[\omega\frac{\partial u}{\partial\tau_1} + \zeta\frac{\partial u}{\partial\tau_2}\right]$$
$$+\alpha u + \beta u^3 + \mu u^5 + \rho u^7 = 0 \tag{4.1}$$

Setting $F = 0$ in (3.7) and (3.6), we obtain the first-order approximate solution to (4.1) as given in (3.9) with the same initial guess function given in (3.4). The other higher-order approximate solutions to (4.1) can also be obtained as long as F remains zero.

Setting $\delta = \zeta = 0$ in (3.9), (3.7), (3.6) and (3.4) one obviously obtains the initial guess function ,the first approximation to its frequency and the first-order approximate solution to the un-damped cubic-quintic-heptic Duffing equation with a single periodic forcing which can be obtained from (1.0) by taking $M = 0, X_m = 0, N = 3, a_0 = \alpha, a_1 = \beta, a_2 = \mu, and\ a_3 = \rho$ and given by

$$\omega^2\frac{d^2u}{d\tau^2} + \alpha u + \beta u^3 + \mu u^5 + \rho u^7 = F\cos\tau_1 \tag{4.2}$$

having applied the transformation in (1.4) and subject to the corresponding initial conditions

$$u(\tau) = a, \quad \omega\frac{du(\tau)}{d\tau} = 0, \ when\tau = 0 \tag{4.3}$$

Similarly one can obtain the higher-order approximate solutions to (4.2).

From (1.0) as well as (4.2), one can obtain the un-damped and unforced cubic-quintic-heptic Duffing equation subject to the initial conditions given by (4.3) and its initial guess function ,first frequency approximation and the first-order approximate solution employing (3.9), (3.7), (3.6) and (3.4), provided $\delta = \zeta = 0$by taking $M = 0, X_m = 0, F_m = 0, N = 3, a_0 = \alpha, a_1 = \beta, a_2 = \mu, and\ a_3 = \rho$. Higher-order approximate solutions can as well be obtained.

Setting $\delta = \zeta = 0$ in (3.9), (3.7), (3.6) and (3.4) one also obtains the initial guess function,the first frequency approximation and the first-order approximate solution to the un-damped cubic-quintic Duffing equation with a single periodic forcing which can be obtained from (1.0) by taking $M = 0, X_m = 0, N = 2, a_0 = \alpha, a_1 = \beta, and\, a_2 = \mu$ and given by

$$\omega^2\frac{d^2u}{d\tau^2} + \alpha u + \beta u^3 + \mu u^5 = F\cos\tau_1 \tag{4.4}$$

having applied the transformation in (1.4) and subject to the corresponding initial conditions

$$u(\tau) = a, \quad \omega\frac{du(\tau)}{d\tau} = 0, \ when\tau = 0 \tag{4.5}$$

Similarly one can obtain the higher-order approximate solutions to (4.4).

Finally the un-damped and unforced cubic-quintic Duffing equation can also be obtained from (1.0) by setting $M = 0, X_m = 0, F_m = 0, N = 2, a_0 = \alpha, a_1 = \beta, a_2 = \mu, and\ \delta = 0$ and is given by

$$\omega^2\frac{d^2u}{d\tau^2} + \alpha u + \beta u^3 + \mu u^5 = 0 \tag{4.6}$$

where $\zeta = 0$, after applying the transformation in (1.4) and subject to the initial conditions prescribed in (4.3).

By enforcing$\zeta = \delta = F = 0$ in (3.9),(3.7),(3.6) and (3.4), one as well obtains the first-order approximate solution and the first frequency approximation to (4.6) as given in (3.9). The other higher-order approximate solutions to (4.1) can also be obtained as long as F remains zero. We note that the first frequency approximation and the first-order approximate solution to (4.6) obtained by setting some parameters above equal to zero is exactly the same results obtained in[22].

Below are presented some of the simulations we did, in **Figure One** and **Figure Two**, we observe the behavior of the damped and forced cubic-quintic-heptic Duffing oscillator. **Figure Three** and **Figure Four** depict the behavior of the damped and forced cubic-quintic Duffing oscillator. We also did a lot of simulations using different values for our parameters to observe the behavior of the auxiliary non-zero parameter which controls the convergence of our approximations. Interestingly neither the strength of the damping nor that of the forcing exert any influence on g for all the different cases we treated. It was rather the initial guesses of the time constants ω_0 and ζ_0, and of course as was noted in[20] the strength of the nonlinearity that determines and influences g. Furthermore, just as was noted in[20], problems with stronger nonlinearity demands that we choose g such that its absolute value lies very close to zero but must not be zero.

The table below depicts some of these interesting behaviors of g.

Table 1. Behavior of the auxiliary parameter g

α	β	μ	ρ	δ	a	ζ_0	F	ω_0	g
1	1	1	1	1	1	1	1	1	(Good approx.) $< \pm\frac{1}{10} \geq$ (not good)
1	1	1	1	4	1	1	1	1	”
1	1	1	1	1	1	1	4	1	(Good approx.) $< \pm\frac{1}{80} \geq$ (not good)
1	1	1	1	4	1	1	4	1	”
1	1	1	1	1	1	$\frac{1}{2}$	1	1	(Good approx.) $< \pm 1 \geq$ (not good)
1	1	1	1	4	1	$\frac{1}{2}$	1	1	(Good approx.) $< \pm\frac{8}{10} \geq$ (not good)
1	1	1	1	1	1	$\frac{1}{2}$	4	1	”
1	1	1	1	4	1	$\frac{1}{2}$	4	1	”
1	1	1	0	1	1	1	1	1	(Good approx.) $< \pm\frac{1}{4} \geq$ (not good)
1	1	1	0	4	1	1	1	1	”
1	1	1	0	1	1	1	3	1	”
1	1	1	0	4	1	1	4	1	(Good approx.) $< \pm\frac{1}{10} \geq$ (not good)
1	1	1	0	1	1	$\frac{1}{2}$	1	1	(Good approx.) $< \pm\frac{1}{2} \geq$ (not good)
1	1	1	0	4	1	$\frac{1}{2}$	1	1	”
1	1	1	0	1	1	$\frac{1}{2}$	4	1	”
1	1	1	0	4	1	$\frac{1}{2}$	4	1	”
1	1	1	1	0	1	0	1	1	(Good approx.) $< \pm\frac{1}{25} \geq$ (not good)
1	1	1	0	0	1	0	1	1	(Good approx.) $< \pm\frac{1}{30} \geq$ (not good)
1	1	1	1	0	1	0	1	$\frac{1}{2}$	(Good approx.) $< \pm\frac{1}{190} \geq$ (not good)
1	1	1	0	0	1	0	1	$\frac{1}{2}$	(Good approx.) $< \pm\frac{1}{116} \geq$ (not good)

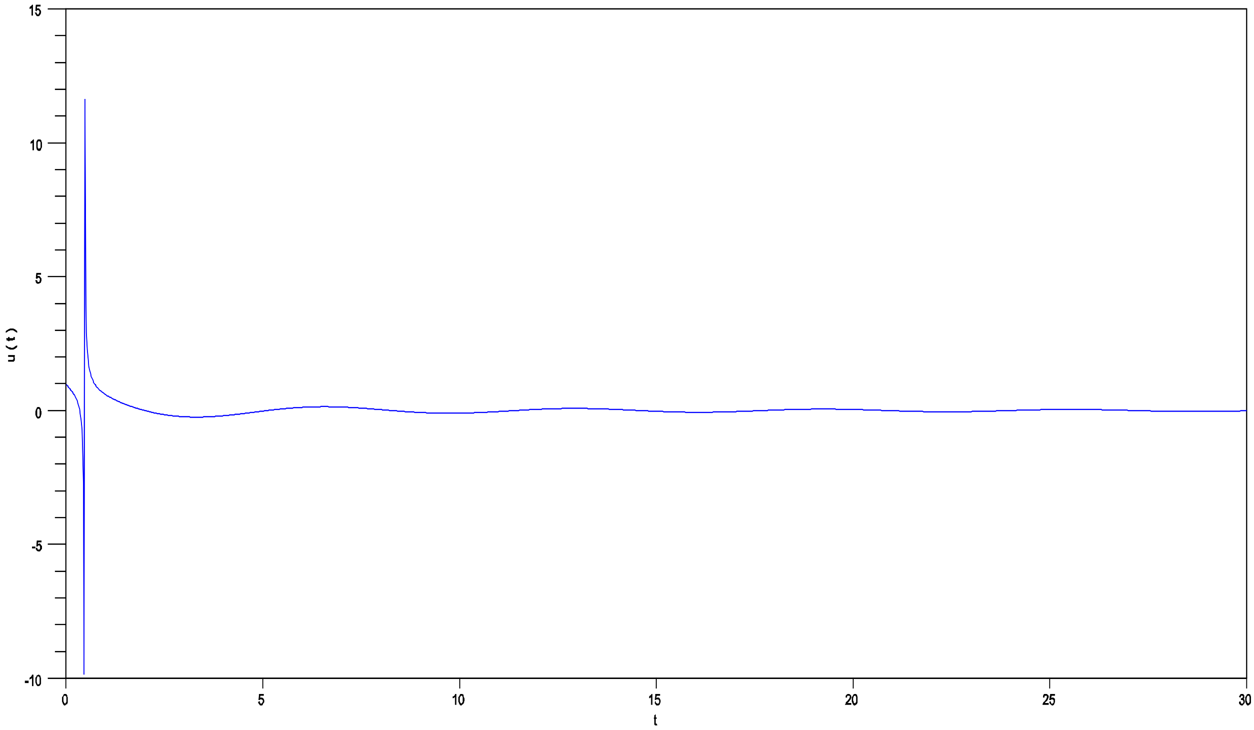

Figure 1. $\alpha = \beta = \mu = \rho = a = \delta = \omega_0 = \zeta_0 = 1$. Behavior of the displacement of the damped and forced cubic-quintic-heptic Duffing oscillator as time increases for $\omega_0 = \zeta_0 = 1$

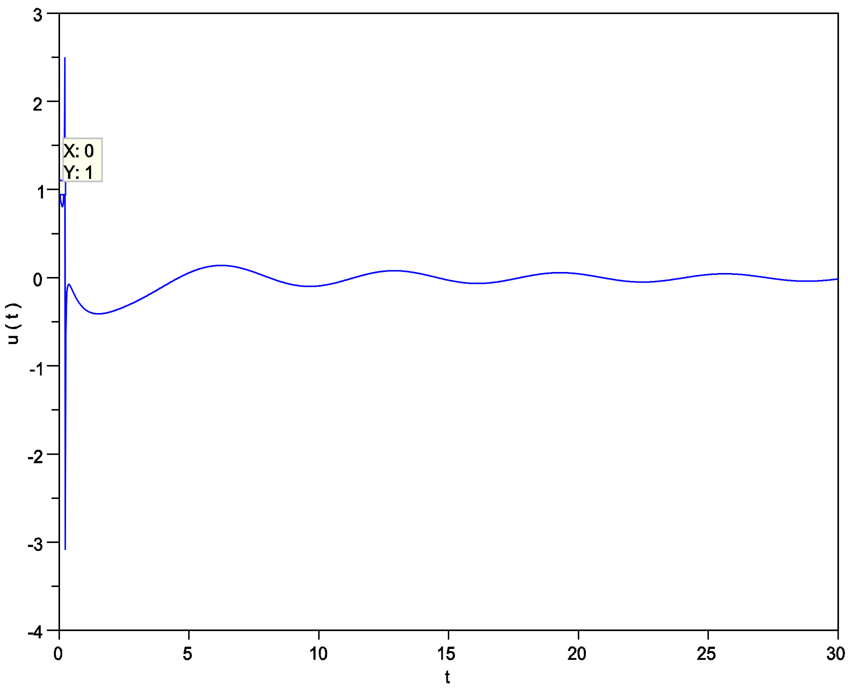

Figure 2. $\alpha = \beta = \mu = \rho = a = \delta = \omega_0 = 1, \zeta_0 = \frac{1}{2}$.Behavior of the displacement of the damped and forced cubic-quintic-heptic Duffing oscillator as time increases for $\omega_0 = 1\ and\ \zeta_0 = 1/2$

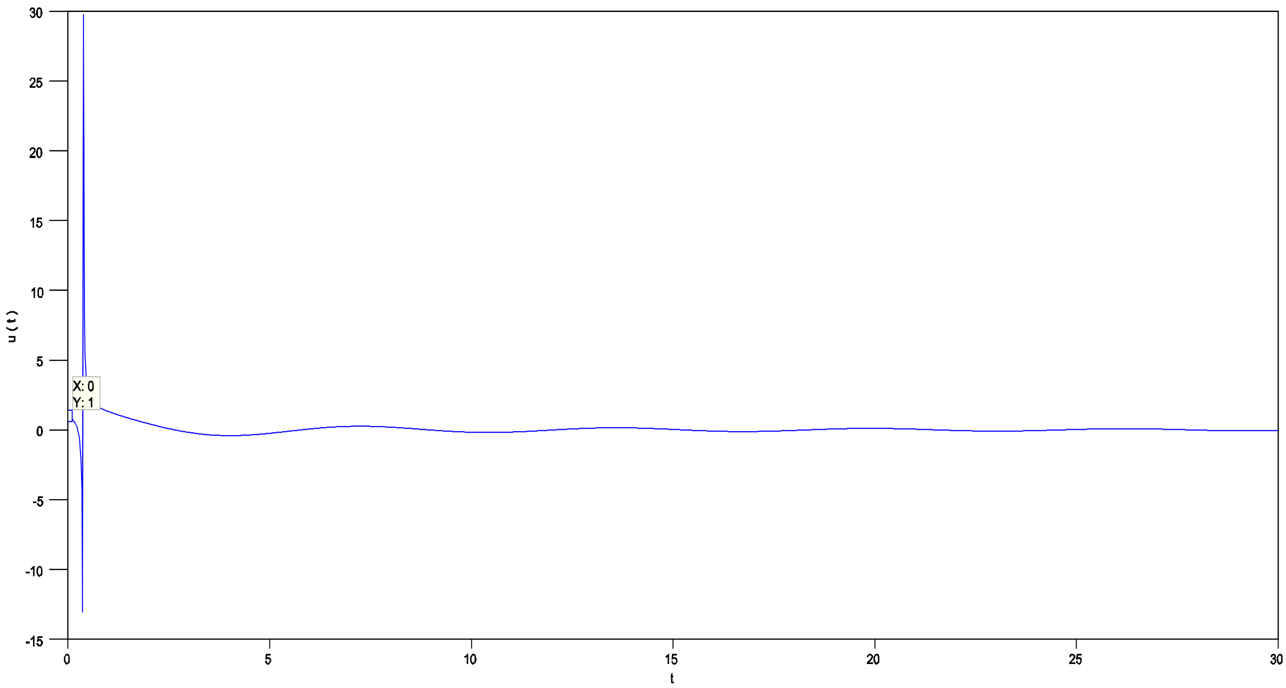

Figure 3. $\alpha = \beta = \mu = a = \delta = \omega_0 = \zeta_0 = 1$.Behavior of the displacement of the damped and forced cubic-quintic Duffing oscillator as time increases for $\omega_0 = \zeta_0 = 1$

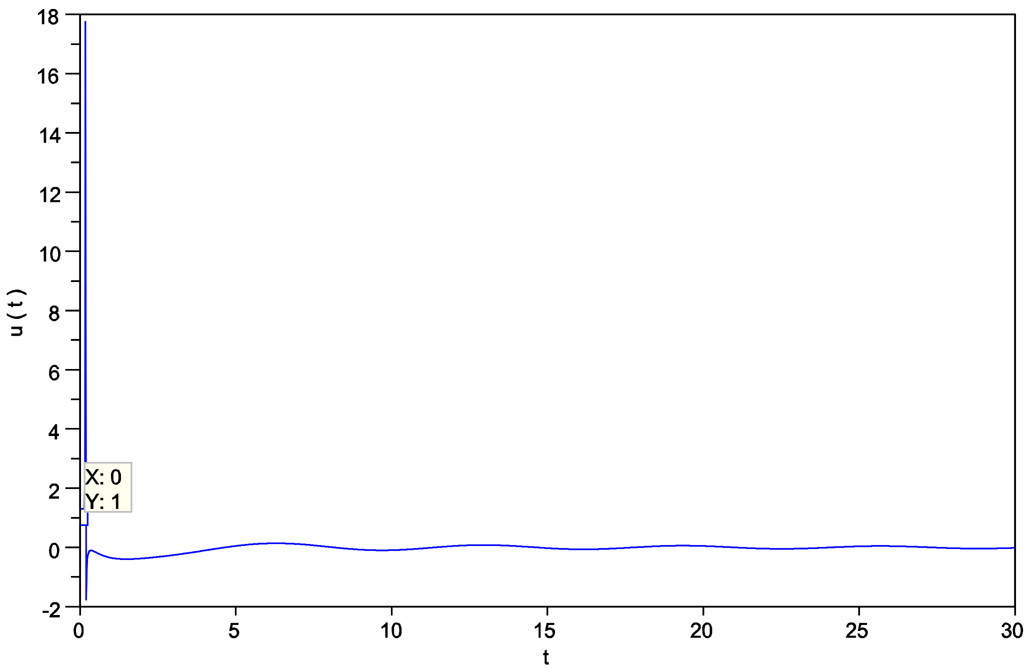

Figure 4. $\alpha = \beta = \mu = a = \delta = \omega_0 = 1, \zeta_0 = \frac{1}{2}$.Behavior of the displacement of the damped and forced cubic-quintic Duffing oscillator as time increases for $\omega_0 = 1$ $and \zeta_0 = 1/2$

5. Conclusions

In this paper, the homotopy analysis method (HAM) was used to obtain analytic and uniformly-valid approximate solutions to the damped and driven, free oscillating, as well as the un-damped and un-driven Duffing oscillator equations with different nonlinearities.It was observed as noted in[6] that the degree of damping is elicited by the degree of nonlinearity as can be seen in **Figures 1-4**. Hence one can use increased nonlinearity to reduce the effect of external forcing as well as negative damping. We also observed that apart from the strength of the nonlinearity of a given problem, the initial guesses ω_0 and ζ_0, chosen for any given problem aids in determining a suitable auxiliary non-zero parameter such that, given $g_c < |g| \geq g_d$, where $1 \gg |g_c| > 0$ and $g_d \gg 0$. It is also observed that every chosen g satisfying

$g_d > |g| \geq g_c$ gives a good approximation while every chosen g satisfying $g_d \geq |g| > g_c$ does not give a good approximation. *There is need to consider and investigate the degree of nonlinearity that can reduce the effect of external forcing for Duffing oscillator with multiple forcing functions and investigate the influence and the range of validity of the auxiliary parameter. Stability analysis also should engage attention in subsequent work.*

Highlights

1. Neither the strength of the damping nor that of the forcing exerts any influence on the auxiliary parameter of the nonlinearity.
2. Factors that affect the nonzero auxiliary parameter (which controls the convergence of the approximate solutions obtained) in HAM are shown and discussed.
3. The degree of damping is elicited by the degree of nonlinearity and the initial guesses of the time constants

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REFERENCES

[1] Pirbodaghi, T., Hoseni, S. H., Ahmadian, M. T., Farrahi, G. H. (2009) Duffing equations with cubic and quintic nonlinearities. J. of Computers and Mathematics with Applications 57, 500-506.

[2] Nayfeh, A. H., Mook, D. T. (1979) Nonlinear oscillations, John Wiley &Sons Inc., New York.

[3] Sedighi, H. M., Shirazi, K. H., Zare, J. (2012) An analytic solution of transversal oscillation of quintic nonlinear beam with homotopy analysis method. Int. J. of Nonlinear Mechanics 47, 777-784.

[4] Guckenheimer, J. and Philip Holmes (1983)Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Number 42 in Applied Mathematical Sciences, Springer-Verlag, New York, NY.

[5] Oyesanya, M. O. (2008) Duffing Oscillator as a model for predicting earthquake occurrence 1, J. of NAMP 12, 133-142.

[6] Oyesanya, M. O., Nwamba, J. I. (2012) Stability Analysis of damped cubic-quintic Duffing Oscillator, World Journal of Mechanics (In Press).

[7] Rand, R. H. (2003) Lecture notes on nonlinear vibrations, a free online book available at http://www.tam.cornell.edu/randdocs/nlvibe45.pdf.

[8] Mickens, R. E. (1996) Oscillations in Planar Dynamic Systems, World Scientific, Singapore.

[9] Wu, B. S., Sun, W. P., Lim, W. C. (2006) An analytic approximate technique for a class of strongly nonlinear oscillators, Int. Journal of Nonlinear Mechanics, 41, 766-774.

[10] Lin, J. (1999) A new approach to Duffing equation with strong and high nonlinearity, Communications in Nonlinear Science and Numerical Simulation, 4, 132-135.

[11] Hamidan, M. N., Shabaneh, N. H. (1997) On the large amplitude free vibration of a restrained uniform beam carrying an intermediate lumped mass, Journal of Sound and Vibration, 199, 711-736.

[12] Lai, S. K., Lim, C. W., Wu, B. S., Wang, S., Zeng, Q. C., He, X. F. (2009) Newton-harmonic balancing approach for accurate solutions to nonlinear cubic-quintic Duffing oscillators, Journal of Applied Mathematical Modelling, 33, 852-866.

[13] Correig, A. M. and Urquizu (2002) Some dynamical aspects of microseismtime series, Geophys. J. Int. 149, 589-598

[14] Belandez, A., Bernabeu, G., Frances, J., Mendez, D. I., Marini, S. (2010) An accurate approximate solution for the quintic Duffing oscillator equation. J. of Mathematical and Computer Modelling 52, 637-641.

[15] Chua, V. (2003) Cubic-Quintic Duffing Oscillators, www.its.caltech.edu/~mason/research/duf.pdf.

[16] Kargar, A., Akbarzade, M. (2012) An analytic solution of nonlinear cubic-quintic Duffing equation using Global Error Minimization Method, J. Adv. Studies Theor. Phys., 6 (10), 467-471.

[17] Farzaneh, Y., Tootoonchi, A. A. (2010) Global error minimization method for solving strongly nonlinear oscillator differential equations. J. of Computers and Mathematics with Applications 59, 8, 2887-2895

[18] Liao, S. J. (1992) The proposed homotopy analysis techniques for the solutions of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University.

[19] Liao, S. J. (1999) (a) A simple way to enlarge the convergence region of perturbation approximations, Int. J. Nonlinear Dynamics, 19 (2), 91-110.

[20] Liao, S. J. (2003) An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude, Int. J. Nonlinear Mech. 38, 1173-1183.

[21] Lesage, J. C., Liu, M. C. (2008) On the investigation of a restrained cargo system modeled as a Duffing oscillator of various orders, Proceedings of ECTC (Early Career Technical Conference) ASME, Maimi, Florida, USA.

[22] Ganji, S. S., Ganji, D. D., Babazadeh, H., Karimpour, S. (2008) Variational approach Method for nonlinear oscillations of the motion of a Rigid Rod rocking back and cubic-quintic Duffing oscillators, Progress inElectromagnetic Research M, 4, 23-32.