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Spectroscopy of the Quarkonium Systems for Heavy Quarks

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Abstract The study of the spectroscopy of bound states of quarkonium systems like $c\bar{c}, b\bar{b}$ and B_c meson in the quark model framework with phenomenological potentials is motivated by quantum chromodynamics (QCD). It is found that analysis of the mass spectra of these systems is effectively given by the nonrelativistic Schrödinger's equation. There are several methods which are used to solve Schrödinger's equation with a general polynomial potential one of them is the Nikiforov-Uvarov (NU) method. It's one of the effective methods which gives the energy eigenvalues and eigenstates for our potential. The results obtained are in good agreement with the experimental data and are better than previous theoretical studies.

Keywords Quarkonium spectroscopy, B_C Meson Mass Spectrum

1. Introduction

In quark and anti-quark system, the quantitively description is given by (quantum chromodynamics (QCD) spectroscopy and the standard model theory) [1-4] is important to specify the mechanism of that system and its nature of being bound systems.

The Schrödinger's equation describes quarkonium systems [5-9] with a heavy quark and anti-quark interaction (two-body problem). We solve the Schrödinger equation in a spherical-symmetric coordinate and using radial potentials which can be described by the asymptotic limits of QCD which has been qualitatively verified by Lattice QCD calculations [2-4].

The main purpose of this paper is that an interaction potential in the quark-antiquark bound system is taken as a general polynomial to get the general eigenvalue and eigenfunction solution then choosing a specific potential according to the description of the physical mechanism of the system.

The Nikiforov-Uvarov (NU) method [10-15], gives asymptotic expressions for the eigenfunctions and eigenvalues of the Schrödinger's equation. Hence one can calculate the energy eigenvalues and eigenstates for the spectrum of the quarkonium systems [17-26].

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2. The Schrödinger Equation with Polynomial Potential

The Schrödinger equation reads

$$\frac{d^2Q}{dr^2} + \left[\frac{2\mu}{\hbar^2}\big(E - V(r)\big) - \frac{l(l+1)}{r^2}\right] Q = 0 \qquad (1)$$

We use the generalized potential

$$V(r) = \sum_{m=-2}^{m} A_m\, r^m \;\; , m = -2, -1, 0, 1, 2, 3, 4, \dots \qquad (2)$$

$$\sum_{m=-2}^{m} A_m\, r^m = A_{-2}\, r^{-2} + A_{-1}\, r^{-1} + A_0 \; + A_1\, r^1 + A_2\, r^2 + A_3\, r^3 + A_4\, r^4 + \cdots \qquad (3)$$

By substituting in Schrödinger equation (1), we get

$$\frac{d^2Q}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E - \sum_{m=-2}^{m} A_m\, r^m\right) - \frac{l(l+1)}{r^2}\right] Q = 0 \qquad (4)$$

$$\frac{d^2Q}{dr^2} + \left[\sum_{m=-2}^{m} (a,b)_m\, r^m\right] Q = 0 \qquad (5)$$

Where

$$\sum_{m=-2}^{m} (a,b)_m\, r^m = -[b_{-2}r^{-2} + a_{-1}r^{-1} + b_0 + a_1 r^1 + a_2 r^2 + a_3 r^3 + \cdots] \qquad (6)$$

and

$$\frac{2\mu}{\hbar^2} A_m = a_m \, , b_{-2} = l(l+1) + a_{-2} \, , \quad b_0 = a_0 - \tfrac{2\mu}{\hbar^2} E = a_0 - \epsilon_0 \qquad (7)$$

Let $r = \frac{1}{x}$, and substituting in equation (5), we get

$$x^4 \frac{d^2Q}{dx^2} + 2x^3 \frac{dQ}{dx} + \left[-b_{-2}x^2 - a_{-1}x - b_0 - \sum_{m=1}^{m} a_m \left(\frac{1}{x}\right)^m\right] Q = 0 \quad (8)$$

Because of singularity, we expand the polynomial function by Taylor's series around x=0

Let $x = y + \delta$

$$\sum_{m=1}^{m} a_m \left(\frac{1}{x}\right)^m = \sum_{m=1}^{m} \frac{a_m}{(y+\delta)^m} = \sum_{m=1}^{m} \frac{a_m}{\delta^m}\left[1 + \frac{y}{\delta}\right]^{-m} \quad (9)$$

$$let \qquad f(y) = \left[1 + \frac{y}{\delta}\right]^{-m} \quad (10)$$

Neglecting higher order terms, we get

$$f(y) = \left[1 - \frac{m}{\delta} y + \frac{m(m+1)}{2\delta^2} y^2\right] \quad (11)$$

By substituing $x - \delta = y$, we get

$$f(x) = \left[\left[\frac{(m+2)(m+1)}{2}\right] - \left[\frac{m(m+2)}{\delta}\right] x + \frac{m(m+1)}{2\delta^2} x^2\right] \quad (12)$$

By substituting in equation (9), we obtain

$$\sum_{m=1}^{m} a_m \left(\frac{1}{x}\right)^m = \sum_{m=1}^{m} \frac{a_m}{\delta^m}\left[1 + \frac{y}{\delta}\right]^{-m} \cong \sum_{m=1}^{m} \frac{a_m}{\delta^m}\left[\left[\frac{(m+2)(m+1)}{2}\right] - \left[\frac{m(m+2)}{\delta}\right] x + \frac{m(m+1)}{2\delta^2} x^2\right] \quad (13)$$

$$\sum_{m=1}^{m} a_m \left(\frac{1}{x}\right)^m \cong \sum_{m=1}^{m} \left[\frac{(m+2)(m+1)a_m}{2\delta^m} - \frac{m(m+2)a_m}{\delta^{m+1}} x + \frac{m(m+1)a_m}{2\delta^{m+2}} x^2\right] \quad (14)$$

By substituting in equation (8), we get

$$x^4 \frac{d^2Q}{dx^2} + 2x^3 \frac{dQ}{dx} + \left[-b_{-2}x^2 - a_{-1}x - b_0 - \sum_{m=1}^{m} \left[\frac{(m+2)(m+1)a_m}{2\delta^m} - \frac{m(m+2)a_m}{\delta^{m+1}} x + \frac{m(m+1)a_m}{2\delta^{m+2}} x^2\right]\right] Q = 0 \quad (15)$$

We rearrange equation (15), and divide by x^4 where $x \neq 0$ to obtain,

$$\frac{d^2Q}{dx^2} + \frac{2x}{x^2}\frac{dQ}{dx} + \frac{1}{x^4}\left[-\left(b_0 + \sum_{m=1}^{m} \frac{(m+2)(m+1)a_m}{2\delta^m}\right) + \left(\sum_{m=1}^{m} \frac{m(m+2)a_m}{\delta^{m+1}} - a_{-1}\right) x - \left(b_{-2} + \sum_{m=1}^{m} \frac{m(m+1)a_m}{2\delta^{m+2}}\right) x^2\right] Q = 0 \quad (16)$$

Let $$\left(b_0 + \sum_{m=1}^{m} \frac{(m+2)(m+1)a_m}{2\delta^m}\right) = q,\ \left(\sum_{m=1}^{m} \frac{m(m+2)a_m}{\delta^{m+1}} - a_{-1}\right) = w,\ \left(b_{-2} + \sum_{m=1}^{m} \frac{m(m+1)a_m}{2\delta^{m+2}}\right) = z \quad (17)$$

Equation (16) becomes

$$\frac{d^2Q}{dx^2} + \left(\frac{2x}{x^2}\right)\frac{dQ}{dx} + \frac{1}{x^4}[-q + wx - zx^2]Q = 0 \quad (18)$$

We use the Nikiforov-Uvarov (NU) method [12,14] as mentioned before,

$\tilde{\tau}, \sigma, \tilde{\sigma}, \pi(x), k, \tau(x), \Delta, \lambda, \lambda_n, \rho(x), \Psi(r), \varphi(r), \Upsilon(r)$ are symbols used in the Nikiforov-Uvarov (NU) method.

$$\tilde{\tau} = 2x,\ \sigma = x^2,\ \tilde{\sigma} = -q + wx - zx^2 \quad (19)$$

$$\pi(x) = \frac{2x - 2x}{2} \pm \sqrt{\left(\frac{2x - 2x}{2}\right)^2 + q - xw + x^2 z + x^2 k} \quad (20)$$

$$\pi(x) = \pm\sqrt{q - xw + x^2 z + x^2 k} \quad (21)$$

We chose the value of k which make the square root in equation (21) as a quadratic term

$$\Delta = b^2 - 4ac = 0,\ w^2 - 4(k+z)q = 0 \quad (22)$$

$$(k+z) = \frac{w^2}{4q} \rightarrow k = \frac{w^2}{4q} - z \quad (23)$$

Substituting in equation (21), we get

$$\pi(x) = \pm\sqrt{\left(\frac{w^2}{4q}\right)x^2 - wx + q} \tag{24}$$

$$\pi(x) = \pm\sqrt{\left(\frac{w}{2\sqrt{q}}x - \sqrt{q}\right)^2} \tag{25}$$

We take the negative value

$$\pi(x) = -\left(\frac{w}{2\sqrt{q}}x - \sqrt{q}\right) = -\frac{w}{2\sqrt{q}}x + \sqrt{q} \tag{26}$$

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \tag{27}$$

$$\tau(x) = \left[2 - \frac{w}{\sqrt{q}}\right]x + 2\sqrt{q}\ and\ \frac{w}{\sqrt{q}} > 2 \tag{28}$$

$$ƛ\ = k + \dot{\pi}(x) = \frac{w^2}{4q} - z - \frac{w}{2\sqrt{q}} = \frac{w^2}{4q} - \frac{w}{2\sqrt{q}} - z \tag{29}$$

$$ƛ_{\ n} = -n\dot{\tau}(x) - \frac{n(n-1)}{2}\ddot{\sigma}(x) \tag{30}$$

$$ƛ_{\ n} = -n\left[2 - \frac{w}{\sqrt{q}}\right] - n(n-1) = -2n + \frac{w}{\sqrt{q}}n - n^2 + n \tag{31}$$

By equating equations (29) and (31), we get the eigen value equation

$$ƛ_{\ n} = ƛ\ = \frac{w^2}{4q} - \frac{w}{2\sqrt{q}} - z = -n + \frac{w}{\sqrt{q}}n - n^2 \tag{32}$$

$$\left[\frac{w}{2\sqrt{q}} - \left(n + \frac{1}{2}\right)\right]^2 = z + \frac{1}{4} \rightarrow \frac{w}{2\sqrt{q}} = \sqrt{z + \frac{1}{4}} + \left(n + \frac{1}{2}\right) \tag{33}$$

$$q = \left[\frac{w}{2n + 1 + 2\sqrt{z + \frac{1}{4}}}\right]^2 \tag{34}$$

By substituting equation (17) in equation (34), we get

$$b_0 + \sum_{m=1}^{m}\frac{(m+2)(m+1)a_m}{2\delta^m} = \left[\frac{\sum_{m=1}^{m}\frac{m(m+2)a_m}{\delta^{m+1}} - a_{-1}}{2n + 1 + 2\sqrt{b_{-2} + \sum_{m=1}^{m}\frac{m(m+1)a_m}{2\delta^{m+2}} + \frac{1}{4}}}\right]^2 \tag{35}$$

By substituting equation (7) in equation (35), we get

$$\epsilon_0 = a_0 + \sum_{m=1}^{m}\frac{(m+2)(m+1)a_m}{2\delta^m} - \left[\frac{\sum_{m=1}^{m}\frac{m(m+2)a_m}{\delta^{m+1}} - a_{-1}}{2n + 1 + 2\sqrt{b_{-2} + \sum_{m=1}^{m}\frac{m(m+1)a_m}{2\delta^{m+2}} + \frac{1}{4}}}\right]^2 \tag{36}$$

Putting $\delta = \frac{1}{r_0}$ and substituting in equation (36), we obtain

$$\epsilon_0 = a_0 + \frac{1}{2}\sum_{m=1}^{m}(m+2)(m+1)\ a_m\ {r_0}^m - \left[\frac{\sum_{m=1}^{m} m(m+2)\ a_m {r_0}^{m+1} - a_{-1}}{2n + 1 + 2\sqrt{l(l+1) + a_{-2} + \frac{1}{2}\sum_{m=1}^{m} m(m+1)a_m {r_0}^{m+2} + \frac{1}{4}}}\right]^2 \tag{37}$$

So, the total energy eigen value is

$$E = A_0 + \frac{1}{2}\sum_{m=1}^{m}(m+2)(m+1)\,A_m\,{r_0}^m \;-\frac{2\mu}{\hbar^2}\left[\frac{\sum_{m=1}^{m} m(m+2)\,A_m {r_0}^{m+1} - A_{-1}}{2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\hbar^2}A_{-2}+\frac{2\mu}{\hbar^2}\sum_{m=1}^{m}\frac{m(m+1)}{2}A_m {r_0}^{m+2}}}\right]^2 \tag{38}$$

To find the eigenfunctions for the general potential form

$$\mathrm{Q(x)} = \varphi(\mathrm{x})\Upsilon(\mathrm{x}) \tag{39}$$

First, we calculate $\varphi(\mathrm{x})$

$$\frac{1}{\varphi(\mathrm{x})}\frac{d\varphi(\mathrm{x})}{dx} = \frac{\pi(x)}{\sigma(x)} \tag{40}$$

$$\int\frac{d\varphi(\mathrm{x})}{\varphi(\mathrm{x})} = \int\left[-\frac{\frac{w}{2\sqrt{q}}}{x}+\frac{\sqrt{q}}{x^2}\right]dx \tag{41}$$

$$\varphi(\mathrm{x}) = x^{-\frac{w}{2\sqrt{q}}}\,e^{-\frac{\sqrt{q}}{x}} \tag{42}$$

Second, we calculate $\Upsilon(\mathrm{x})$

$$\frac{1}{\rho(\mathrm{x})}\frac{d\rho(\mathrm{x})}{dx} = \frac{\tau(x)-\dot{\sigma}(x)}{\sigma(x)} \tag{43}$$

$$\int\frac{d\rho(\mathrm{x})}{\rho(\mathrm{x})} = \int\left[-\frac{\frac{w}{\sqrt{q}}}{x}+\frac{2\sqrt{q}}{x^2}\right]\mathrm{dx} \tag{44}$$

$$\rho(\mathrm{x}) = x^{-\frac{w}{\sqrt{q}}}\,e^{-\frac{2\sqrt{q}}{x}} \tag{45}$$

$$\Upsilon(\mathrm{x}) = \Upsilon_n(x) = \frac{j_{nl}}{\rho(\mathrm{x})}\frac{d^n}{dx^n}\left[\sigma^n(x)\rho(\mathrm{x})\right] \tag{46}$$

$$\Upsilon(\mathrm{x}) = \Upsilon_n(x) = j_n x^{\frac{w}{\sqrt{q}}}\,e^{\frac{2\sqrt{q}}{x}}\,\frac{d^n}{dx^n}\left[x^{2n-\frac{w}{\sqrt{q}}}\;e^{-\frac{2\sqrt{q}}{x}}\right] \tag{47}$$

By substituting in equation (39), we obtain

$$\mathrm{Q(x)} = \varphi(\mathrm{x})\Upsilon(\mathrm{x}) = N_n\;x^{\frac{w}{2\sqrt{q}}}\,e^{\frac{\sqrt{q}}{x}}\;\frac{d^n}{dx^n}\left[x^{2n-\frac{w}{\sqrt{q}}}\;e^{-\frac{2\sqrt{q}}{x}}\right] \tag{48}$$

Where N_{nl} is a normalization constant.
So, the radial wavefunction of Schrödinger equation (1) for a polynomial potential is

$$\mathrm{Q(r)} = N_n\;(-1)^n\;n!\;r^{-\frac{1}{2}-\sqrt{z+\frac{1}{4}}}\,e^{\sqrt{q}\,r}\sum_{k=1}^{n}\frac{a_k}{k!}\;r^k\;\frac{d^k}{dr^k}\left[r^{1+2\sqrt{z+\frac{1}{4}}}\;e^{-2\sqrt{q}\,r}\right] \tag{49}$$

To find the normalization constant

$$\int_0^\infty |Q(\mathrm{r})|^2\,r^2\,dr = 1 \tag{50}$$

The angular part of the spherical symmetric potential is

$$Y_{lm}(\theta,\varphi) = (-1)^{|m|}\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi\,(l+|m|)!}}\;P_{lm}(cos\theta)\;e^{im\varphi} \tag{51}$$

So, the total wavefunction in spherical symmetric potential is

$$Ж_{nlm}(r,\theta,\varphi) = Q(\mathrm{r})_{nl}Y_{lm}(\theta,\varphi) =$$
$$(-1)^{n+|m|}\,N_n\;n!\;\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}\sum_{k=1}^{n}\frac{a_k}{k!}\;r^{\left[k-\frac{3}{2}-\sqrt{z+\frac{1}{4}}\right]}\,e^{\sqrt{q}\,r}\;\frac{d^k}{dr^k}\left[r^{1+2\sqrt{z+\frac{1}{4}}}\;e^{-2\sqrt{q}\,r}\right]\;P_{lm}(cos\theta)\,e^{im\varphi} \tag{52}$$

In our interquark potential, and using the natural units

$$V(r) = \frac{b}{r} + ar + dr^2 + pr^4 \tag{53}$$

Putting

$$A_1 = a,\ A_2 = d,\ A_4 = P,\ A_{-1} = b \text{ and the rest of equation (3) equals zero} \tag{54}$$

The energy eigen values equation according to such potentials become

$$E = 3ar_0 + 6dr_0{}^2 + 15Pr_0{}^4 - 2\mu\left[\frac{3ar_0{}^2 + 8dr_0{}^3 + 24Pr_0{}^5 - b}{2n + 1 + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + 2\mu * [ar_0{}^3 + 3dr_0{}^4 + 10Pr_0{}^6]}}\right]^2 \tag{55}$$

3. Results and Discussion

In this section, we will calculate the spectra for the bound states of heavy quarks such as charmonium, bottomonium and B_C meson. The mass spectra equation is

$$M = m_q + m_{\bar{q}} + E \tag{56}$$

By substituting equation (55) in equation (56), we get

$$M = m_q + m_{\bar{q}} + 3ar_0 + 6dr_0{}^2 + 15Pr_0{}^4 - 2\mu\left[\frac{3ar_0{}^2 + 8dr_0{}^3 + 24Pr_0{}^5 - b}{2n + 1 + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + 2\mu * [ar_0{}^3 + 3dr_0{}^4 + 10Pr_0{}^6]}}\right]^2 \tag{57}$$

Equation (57) depends on the potential parameters $(a, b, d, p \text{ and } r_0)$ which will be obtained from the experimental data. In the case of charmonium $[\psi = c\bar{c}]$, the rest mass equation is

$$M = 2.55 + 3ar_0 + 6dr_0{}^2 + 15Pr_0{}^4 - 2\mu\left[\frac{3ar_0{}^2 + 8dr_0{}^3 + 24Pr_0{}^5 - b}{2n + 1 + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + 2\mu * [ar_0{}^3 + 3dr_0{}^4 + 10Pr_0{}^6]}}\right]^2 \tag{58}$$

And we get the following masses in GeV

The charmonium system $r_0 = 0.9302$, $a = 25.6245$, $b = -7.11$, $d = -28.7354$, $p = 7.1026$

n	L	type	Present	Ref (27)	Ref (28)	Ref (29)	Ref (30)	Ref (31)	PDG (25)
1	0	1S	2.98301	2.989	2.981	2.984	2.925	2.979	2.984
2	0	2S	3.67657	3.681	3.635	3.679	3.676	3.673	3.686
3	0	3S	4.01076	4.129	4.039	4.030	3.803	4.022	4.039
4	0	4S	4.19694	4.514	4.427	4.281	-	4.273	4.421
5	0	5S	4.31118	4.799	4.811	4.459	-	4.446	--
6	0	6S	4.38627	5.124	5.155	-	-	4.595	--
1	1	1P	3.40565	3.428	3.413	3.415	3.323	3.433	3.415
2	1	2P	3.87319	3.955	3.870	3.848	3.833	3.937	3.927
3	1	3P	4.11774	4.296	4.301	4.146	-	4.131	--
4	1	4P	4.26148	4.653	4.698	-	-	-	--
5	1	5P	4.35306	4.983	-	-	-	-	--
1	2	1D	3.79208	3.755	3.813	3.808	3.869	3.799	-
2	2	2D	4.07279	4.176	4.220	4.112	3.806	4.103	--
3	2	3D	4.23402	4.549	4.574	4.340	-	4.331	--
4	2	4D	4.33508	4.89	4.920	-	-	-	--
1	3	1F	4.04628	3.99	4.041	-	-	-	-
2	3	2F	4.21806	4.378	4.361	-	-	-	--
3	3	3F	4.32474	4.73	-	-	-	-	--

In the following we draw the radial wave functions as a function of the radius,

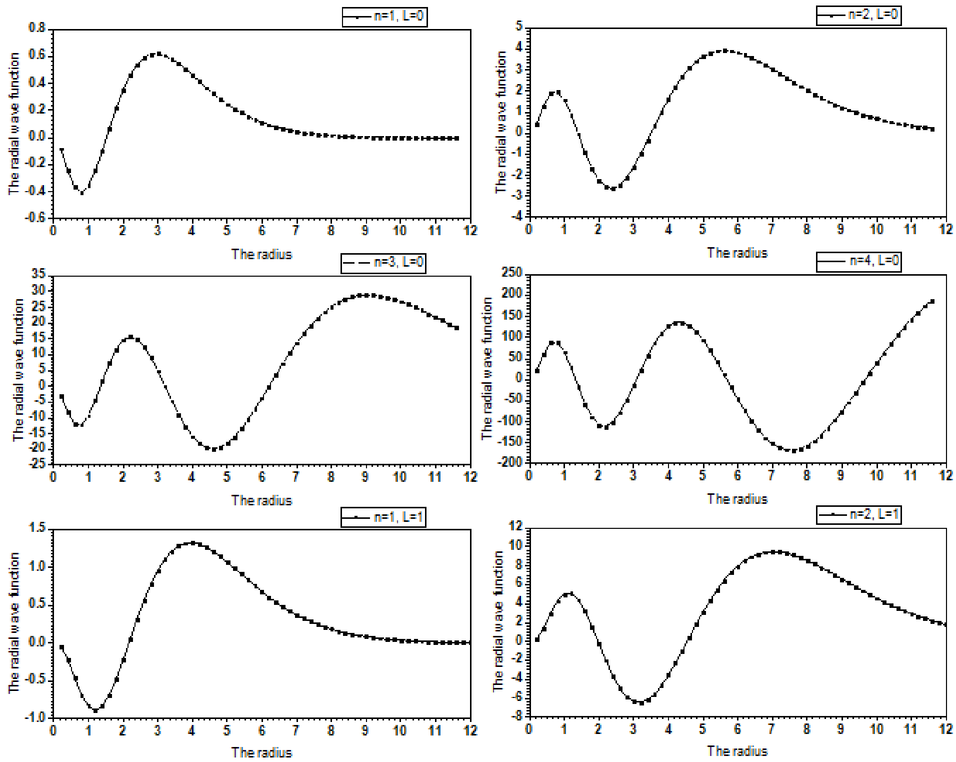

Figure 1. Graphs represent the relation between the radial wave function and the radius in different n, L states according to experimental energy states in *the charmonium system*

In the case of bottomonium $[\Upsilon = b\bar{b}]$, the rest mass equation is

$$M = 8.842 + 3ar_0 + 6dr_0^2 + 15Pr_0^4 - 2\mu\left[\frac{3ar_0^2 + 8dr_0^3 + 24Pr_0^5 - b}{2n + 1 + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + 2\mu * [ar_0^3 + 3dr_0^4 + 10Pr_0^6]}}\right]^2 \tag{59}$$

And we get the following masses in GeV

The bottomonium system $r_0 = 0.8313\,, a = 8.2234\,, b = -2.843\,, d = -8.8278\,, p = 2.532$

n	L	type	Present	Ref (27)	Ref (28)	Ref (30)	Ref (29)	Ref (32)	PDG (25)
1	0	1S	9.39858	9.428	9.398	9.390	9.414	9.389	9.398
2	0	2S	10.05295	9.979	10.023	10.015	10.089	10.016	10.023
3	0	3S	10.35403	10.359	10.355	10.343	10.327	10.351	10.355
4	0	4S	10.51698	10.683	10.586	10.597	-	10.611	10.579
5	0	5S	10.61499	10.975	10.869	10.811	-	10.831	10.876
6	0	6S	10.67848	11.243	11.088	10.988	-	11.023	11.019
1	1	1P	9.83834	9.806	9.859	9.864	9.815	9.865	9.859
2	1	2P	10.24859	10.205	10.233	10.220	10.254	10.226	10.232
3	1	3P	10.45758	10.54	10.521	10.490	-	10.502	--
4	1	4P	10.57828	10.84	10.781	-	-	10.732	--
5	1	5P	10.65423	11.115	-	-	-	10.933	--
1	2	1D	10.19392	10.075	10.161	10.153	10.145	10.151	10.163
2	2	2D	10.42783	10.423	10.449	10.436	-	10.442	--
3	2	3D	10.56034	10.733	-	-	-	10.680	--
4	2	4D	10.64259	11.015	-	-	-	10.886	--
1	3	1F	10.4114	10.283	10.343	10.338	-	-	-
2	3	2F	10.55055	10.604	10.610	-	-	-	--
3	3	3F	10.6363	10.894	-	-	-	-	--

In the following we draw the radial wave functions as a function of the radius,

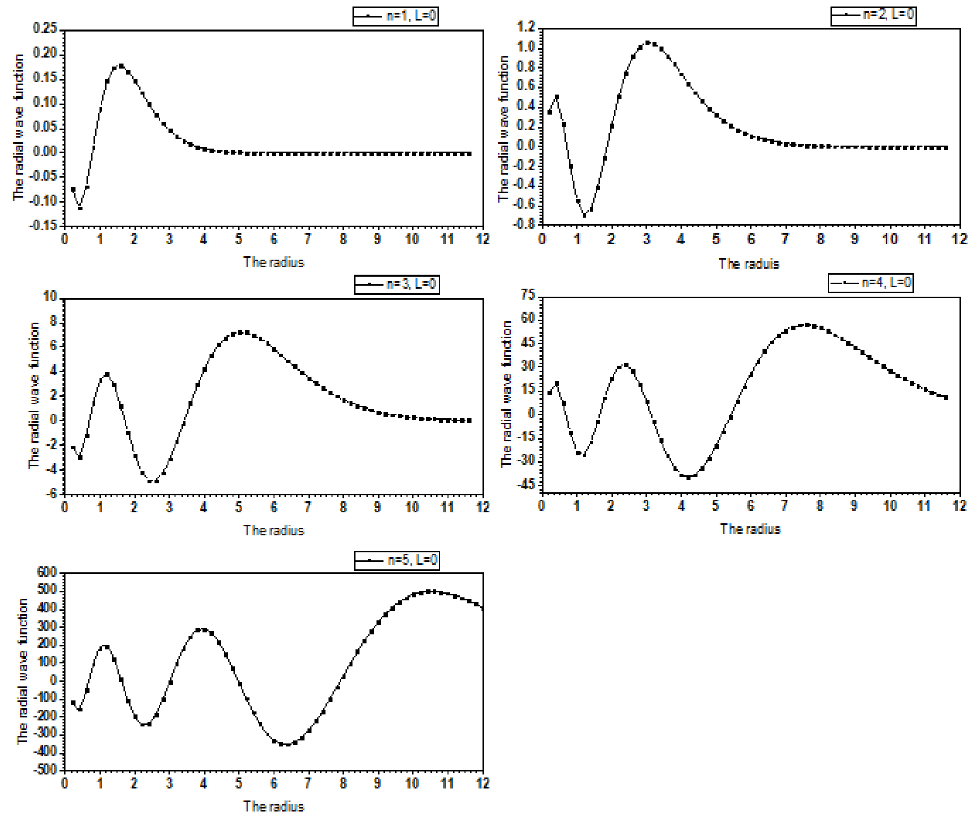

Figure 2. Graphs represent the relation between the radial wave function and the radius in different n, L according to experimental energy states in *the bottomonium system*

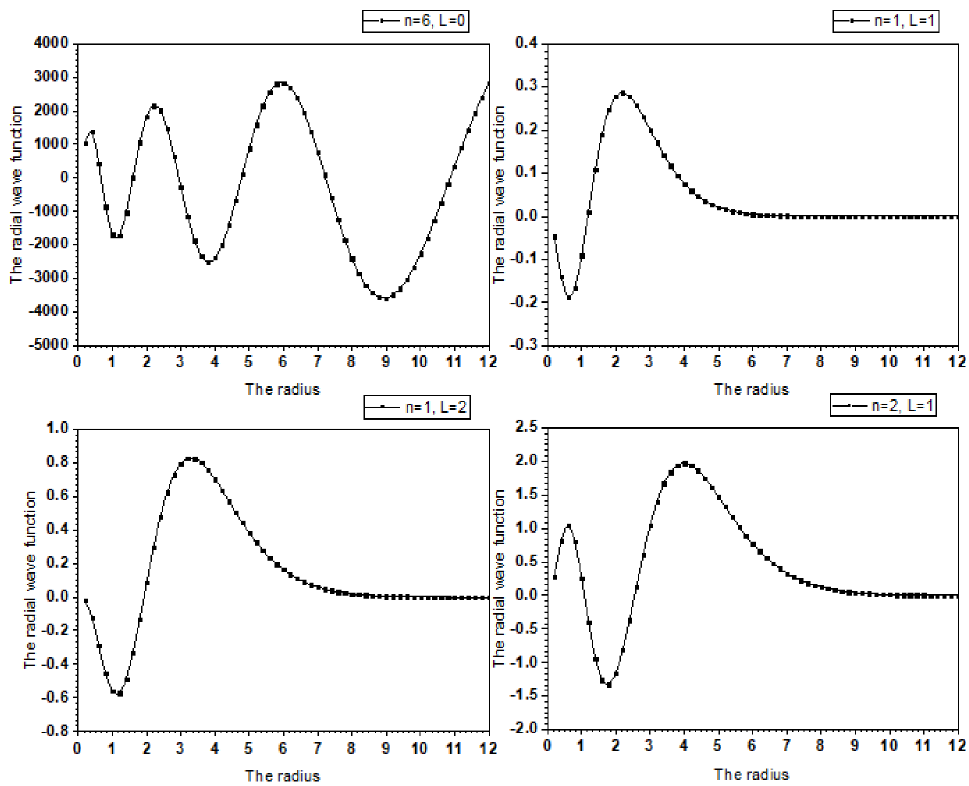

Figure 3. Graphs represent the relation between the radial wave function and the radius in different n, L according to experimental energy states in the bottomonium system

In the case of the meson $[B_c = b\bar{c}]$, the rest mass equation is

$$M = 5.696 + 3ar_0 + 6dr_0^2 + 15Pr_0^4 - 2\mu\left[\frac{3ar_0^2 + 8dr_0^3 + 24Pr_0^5 - b}{2n + 1 + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + 2\mu * [ar_0^3 + 3dr_0^4 + 10Pr_0^6]}}\right]^2 \quad (60)$$

And we get the following masses in GeV

The meson B_c system $r_0 = 0.2235, a = 2.2486, b = -2.5912, d = -0.17658, p = 4.2183$

n	L	Type	Present	Ref (27)	Ref (28)	Ref (33)	Ref (34)	Ref (35)	PDG (25)
1	0	1S	6.27543	6.272	6.272	6.278	6.271	6.275	6.275
2	0	2S	6.84121	6.864	6.842	6.863	6.855	6.838	6.842
3	0	3S	7.04335	7.306	7.226	7.244	7.250	-	-
4	0	4S	7.13794	7.684	7.585	7.564	-	-	-
5	0	5S	7.18967	8.025	7.928	7.852	-	-	--
6	0	6S	7.22101	8.340	-	8.120	-	-	--
1	1	1P	6.83016	6.686	6.699	6.748	6.706	6.672	-
2	1	2P	7.03865	7.146	7.094	7.139	7.122	6.914	-
3	1	3P	7.13552	7.536	7.474	7.463	-	-	--
4	1	4P	7.18827	7.885	7.817	-	-	-	--
5	1	5P	7.22012	8.207	-	-	-	-	--
1	2	1D	7.03763	6.990	7.029	7.026	7.045	6.980	-
2	2	2D	7.135	7.399	7.405	7.363	-	-	--
3	2	3D	7.18796	7.761	7.750	-	-	-	--
4	2	4D	7.21993	8.092	-	-	-	-	--
1	3	1F	7.13477	7.234	7.273	-	7.269	-	-
2	3	2F	7.18783	7.607	7.618	-	-	-	--
3	3	3F	7.21985	7.946	-	-	-	-	--

In the following we draw the radial wave functions as a function of the radius,

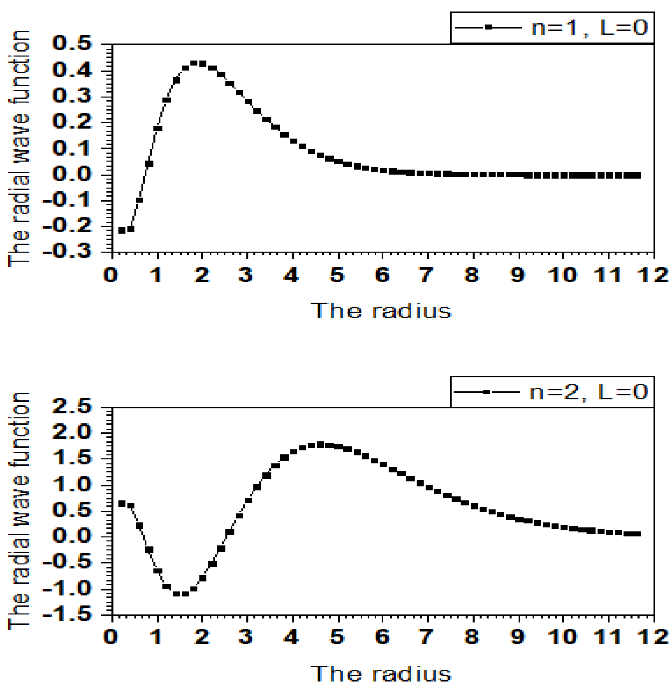

Figure 4. Graphs represent the relation between the radial wave function and the radius in different n, L according to experimental energy states in the meson $[B_c = b\bar{c}]$ system

In conclusion, form the tables, we found that our theoretical work is comparable with the experimental data and explains the behavior of the quarkonium systems. The difference between the experimental data and theoretical work may be because we neglect the spin terms, so, the spin can also be considered if one uses relativistic corrections and the appropriate relativistic Schrödinger's equation. From the figures which represent the quarkonium radial state wave functions, one can calculate physical parameters like the decay parameter.

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