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Quantum Theory of Circuit Systems – I. Aspects of Symmetry Principles and a Generalization of the Nonlinear Schrödinger Equation in Charge Space

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Abstract The quantization of circuits finds many interesting applications, e.g. quantum computer, molecular and biophysics. The method can be extended to nuclear physics, if the exchange interactions between nuclear particles are described by currents. A system of mutually coupled circuits can be treated by the linear Schrödinger equation yielding symmetries such as SU_2, SU_3, SU_4, etc. A generalization of the principles to a nonlinear/nonlocal Schrödinger is presented; the nonlocal exchange between elementary particles is mediated by a Gaussian kernel (integral transform) in the charge space. The SU_3 (or SU_4, if four charges $Q_1,\ldots,Q_4$ are accounted for) are used as the basic structure of the circuits. In nonlocal fields these symmetries hold, in the same fashion, too, but the energy levels are not equidistant as it the case at circuit quantization, equivalent to the 3D harmonic oscillator. The Gaussian kernel assumes zero value in the positive half-plane with E > 0. To avoid this hindrance a kernel is proposed, which only uses terms up to the second order of the Gaussian kernel, and a self-interacting 3D oscillator keeps excitations of arbitrary order to indemnify the confinement of quarks.

Keywords Linear/nonlinear Schrödinger equation, Quantization of circuits, Charge space, Symmetry principles

1. Introduction

The past decades have emerged novel application fields of quantum mechanics, namely the quantization of electromagnetic circuits. Thus an essential goal of these developments is the novel access to problems of radiation-, molecular- and biophysics. In particular, we mention the so-called '***Qubi***t' properties considered in molecular electronic devices, which make the requests the quantum computer feasible [1-10]. It although should be remembered that the first study to tread π-electrons of aromatic molecules by electric circuits now appeared about 70 years ago [11]. The basic principles of the quantization procedure are rather easily founded and well-known from textbooks in physics. Thus the Lagrangian L of such a single oscillator reads:

$$\left.\begin{aligned} \mathrm{L} &= \frac{\mathrm{L}}{2}\cdot\dot{\mathrm{Q}}^2 - \frac{\mathrm{L}\cdot{\omega_0}^2}{2}\cdot\mathrm{Q}^2 \\ \omega_0^2 &= 1/(\mathrm{LC}) \end{aligned}\right\} \tag{1}$$

L is the inductance and C the capacitance. Subjecting eq. (1) to the well-known Lagrange formalism we obtain the canonical momentum (magnetic flux):

$$P = \frac{\partial \mathrm{L}}{\partial \dot{Q}} = L\cdot\dot{Q} \tag{2}$$

With the help of eqs. (1, 2) the Hamiltonian H assumes the shape:

$$H = \frac{P^2}{2\cdot L} + \frac{L}{2}\cdot\omega_0^2\cdot Q^2 \tag{3}$$

The canonical commutation relations yield either the Schrödinger equation or the creation- and annihilation representation of the Hamiltonian. The Schrödinger equation reads:

$$E\psi = -\frac{\hbar^2}{2L}\frac{\partial^2}{\partial Q^2}\psi + \frac{L}{2}\cdot\omega_0^2\cdot Q^2\psi \tag{4}$$

Instead of solving eq. (4) via Gaussian and Hermite polynomials it is more convenient to introduce creation- and annihilation operators:

$$\left.\begin{aligned} b &= \alpha\cdot Q + i\beta\cdot P \\ b^+ &= \alpha\cdot Q - i\beta\cdot P \end{aligned}\right\} \tag{5}$$

The parameters α and β are given by:

$$\alpha^2 = \frac{L\omega_0}{2\cdot\hbar};\quad \beta^2 = \frac{1}{2\cdot\hbar\cdot L\cdot\omega_0} \tag{6}$$

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By eqs. (5, 6) the commutation relation reads:

$$b b^{+} - b^{+} b = 1 \tag{7}$$

The associated Hamiltonian results by:

$$\left.\begin{aligned} H &= \frac{\hbar\omega_0}{2}[b b^{+} + b^{+} b] \\ E_n &= \hbar\omega_o (n + 1/2) \\ n &= 0, 1, 2, ... \end{aligned}\right\} \tag{8}$$

With respect to the problem of Ohm's resistance in electromagnetic systems it should be noted that the similar situation, namely the friction of classical particles, has been studied basically on a quantum mechanical level. Thus it turned out that this aspect has to be handled with a logarithmic nonlinear Schrödinger equation [12 - 16], since the violation of the uncertainty relation by friction is a severe problem. It is the goal of this study to extend the circuit to coupled circuits and the related symmetry properties. However, a serious problem arises by eq. (8), which provides equidistant energy levels. This is also true for coupled oscillators, but in molecular as well as nuclear physics we have not to deal with equidistant excitation energies. By that, we also study nonlinear extensions in the charge space.

2. Systems of Coupled Circuits and the Generalization to a Nonlinear Schrödinger Equation in the Charge Space

2.1. Coupled Oscillators and Symmetry Properties

Figure 1 shows a layer with 3 mutually coupled oscillators with magnetic coupling $\mathbf{M_i}$. In a formal way we are able to write systems of N oscillators:

$$\left.\begin{aligned} & L\ddot{Q}_k + M_i \sum_{l=1, l\neq k}^{N} \ddot{Q}_l + Q_k / C = 0 \\ & k = 1, .., N \end{aligned}\right\} \tag{9}$$

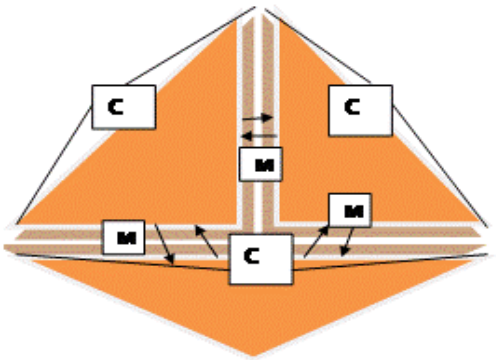

Figure 1. 3 coupled oscillators with mutually magnetic interaction

However, a topographical representation of a system with more than 4 mutually coupled oscillators is rather difficult to perform.

The eigen-frequencies of the coupled oscillators (normal modes) are given by:

N = 3:

$$\left.\begin{aligned} \omega_k^{\,2} &= \frac{1}{C\cdot(L - M_i)}) \; (k = 1, 2) \\ \omega_3^{\,2} &= \frac{1}{C\cdot(L + 2M_i)} \end{aligned}\right\} \tag{10}$$

N = 4:

$$\left.\begin{aligned} \omega_k^{\,2} &= \frac{1}{C\cdot(L - M_i)}) \; (k = 1, 3) \\ \omega_4^{\,2} &= \frac{1}{C\cdot(L + 3M_i)} \end{aligned}\right\} \tag{11}$$

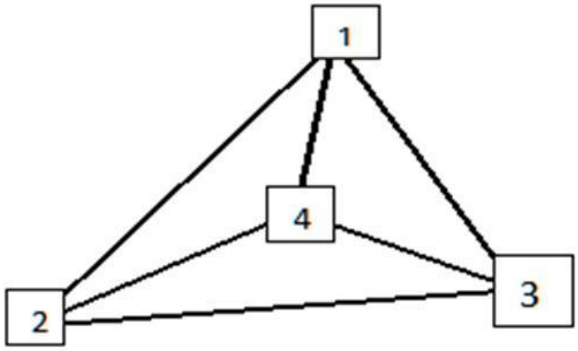

Figure 2. 4 mutually coupled oscillators can be schematically shown by a tetrahedron

The cases N = 3 and 4 assume a central basis; the coupling scheme of N = 4 yielding formally a tetrahedron is shown in Figure 2. We use the abbreviations $\mathbf{L_k = L\text{-}M_i}$ with k = 1, N-1; $\omega_k{}^2 = 1/(C\cdot L_k)$, and k = N: $\mathbf{L_N = L+(N\text{-}1)\cdot M_i}$, $\omega_N{}^2 = 1/(C\cdot L_N)$. Now we obtain:

$$\left.\begin{aligned} b_k &= \alpha_k \cdot Q_k + i\beta_k \cdot P_k \\ b_k^{+} &= \alpha_k \cdot Q_k - i\beta_k \cdot P_k \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \alpha_k^{\,2} &= \frac{L_k\omega_k}{2\hbar}; \;\; \beta_k^{\,2} = \frac{1}{2\hbar L_k\omega_k} \\ \omega_k^{\,2} &= \frac{1}{C\cdot L_k} \end{aligned}\right\} \tag{13}$$

Eq. (13) now corresponds to eqs. (7, 8) with the corresponding indices.

The determination of the energy levels according to the normal modes is identical as previously carried out.

$$\left.\begin{aligned} & b_k b_l^{+} - b_l^{+} b_k = \delta_{kl} \\ & k, l = 1, .., N (N = 3, 4) \end{aligned}\right\} \tag{14}$$

Some comments to symmetry principles:

N identical oscillators with inductances L, capacitances C and mutual magnetic coupling M_i imply in quantum

mechanics the Lie groups of a perturbed symmetry SU_N, if the coupling constant M_i satisfies $M_i << L$. In the present considerations the special cases are SU_3 and SU_4 are rather interesting. With regard to SU_3 there is a similar situation with the quantum mechanical harmonic oscillator in 3 D:

$$H = \frac{1}{2m}\vec{p}^2 + \frac{m}{2}\omega_0{}^2\vec{q}^2 \tag{15}$$

For this purpose, we replace the inductivity L by the mass m and the capacitance C by m/f (f: force constant) to receive the Hamiltonian H in terms of the operator notation:

$$\left.\begin{aligned} &\alpha = \sqrt{m\cdot\omega_0/2\hbar};\ \ \beta = 1/\sqrt{2\cdot m\cdot\omega_0\cdot\hbar} \\ &b_k = \alpha\cdot q_k + i\cdot\beta\cdot p_k \\ &b_k{}^+ = \alpha\cdot q_k - i\cdot\beta\cdot p_k \\ &b_k\cdot b_l{}^+ - b_l{}^+\cdot b_k = \delta_{kl}\quad (k, l=1,3) \end{aligned}\right\} \tag{16}$$

Thus the Hamiltonian can be written as:

$$H = \hbar\omega_0\cdot\sum_{k=1}^{3}(b_k{}^+ b_k + 3/2) \tag{17}$$

With the help of these annihilation- and creation operators the generators of SU_3 can be defined, which may be verified in textbooks of Lie groups. However, in the mechanical case each mode can be excited, and due to the angular momentum operators the quanta can be transferred from one mode to any other without changing the total energy, whereas in the case of circuits we need a coupling between the resonators. Otherwise each circuit can only be excited separately without transferring excitation energy from one oscillator to any other.

2.2. Extension to the Nonlinear Schrödinger Equation (NSE)

The problem arises that neither with mechanical nor electric oscillators the energy differences between the energy levels remain constant. This fact is valid for atomic and molecular physics [16 - 18] as well as in nuclear physics.

In similar fashion as with regard to a nonlinear, nonlocal Schrödinger equation containing a Gaussian kernel in the position space to account for the nonlocal influence (NNSE) we shall now pass to an outstanding generalization of the Schrödinger equation in the charge space. According to previous investigations the 3D version of the NNSE reads [18,19]:

$$\left.\begin{aligned} &-\frac{\hbar^2}{2m}\Delta\psi(\vec{x}) - \lambda\int K(\varepsilon, \vec{x}-\vec{x}')\cdot \\ &\cdot\left|\psi(\vec{x}')\right|^2 d^3x'\psi(\vec{x}) = E\psi(\vec{x}) \end{aligned}\right\} \tag{18}$$

For bounded states the coupling constant has to satisfy $\lambda > 0$. The kernel K incorporates a Gauss transform of the square of the wave-function:

$$\left.\begin{aligned} &K(\varepsilon, \vec{x}-\vec{x}') = \frac{1}{\varepsilon^3}\cdot\frac{1}{(\sqrt{2\pi})^3}\cdot \\ &\cdot\exp(-(\vec{x}-\vec{x}')^2/2\varepsilon^2) \end{aligned}\right\} \tag{19}$$

Thus by taking the limit case $\varepsilon \to 0$ the above equation takes the form of a cubic Schrödinger equation, i.e. the kernel K assumes the shape of δ-kernel:

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{x}) - \lambda\left|\psi(\vec{x})\right|^2\cdot\psi(\vec{x}) = E\psi(\vec{x}) \tag{20}$$

We note that in this and in all following chapters the coupling constant of the nonlinear equations is throughout positive, i.e. $\lambda > 0$. It should be mentioned that eq. (21) is appreciated with regard to the solitonic solutions in one dimension.. In a preceding publication [19] we have developed complete solution functions of eq. (21) in one dimension, which shall be reconsidered in the charge space. There are two solution spectra of the above equation:

$$\left.\begin{aligned} &\psi_{\beta,s} = \sum_{n=\beta}^{\infty} A_n\cdot(\cosh kx)^{-n} \\ &E_\beta = -\hbar^2k^2\beta^2/2m;\ \beta = 1,2,3..... \end{aligned}\right\} \tag{21}$$

The subscript '***s***' stands for the set of symmetric solutions; in the solution set (22) '***as***' stands for the anti-symmetric solutions:

$$\left.\begin{aligned} &\psi_{\beta,as} = \sum_{n=\beta}^{\infty} B_n\sinh(kx)\cdot\cosh(kx)^{-n} \\ &E_{ß} = -\hbar^2k^2(1-\beta)^2/2m;\ \beta = 2,3,... \end{aligned}\right\} \tag{22}$$

The expansion coefficients A_n and B_n according to eqs. (21, 22) have to be determined as a recursive function of the lowest term. As an example we present A_{2n+1} as a function of $A_ß$ with ß = 1.

$$\left.\begin{aligned} &A_{2n+1} = A_1\cdot\sum_{j=0}^{n} R_{j.n}\cdot A_1{}^{2n-2}\cdot \\ &\cdot\frac{(m\cdot(-\lambda))^{n-j}}{4^n\cdot(k^2\cdot\hbar^2)^{n-j}} \end{aligned}\right\} \tag{23}$$

For even j the coefficients of eq. (23) are given by:

$$\left.\begin{aligned} &R_{j,n} = (2^{j/2}/j!)n\cdot(n-1)..(n+1-j/2)\cdot \\ &(2n+1-2j)\cdot(2n-1)(2n-3)..(2n-j+2) \end{aligned}\right\} \tag{24}$$

For odd j the coefficients of eq. (23) are:

$$\left.\begin{aligned} &R_{j,n} = (2^{(j-1)/2}/j!)\cdot n(n-1)(n-2). \\ &\cdot(2n+1-2j)(2n-1)(2n-3).(2n-j+2) \end{aligned}\right\} \tag{25}$$

Since all expansion functions according to eqs. (21, 22) have alternating signs with regard to $\lambda > 0$, the whole procedures are restricted to the convergence criterion of conditional convergence. As previously shown this fact implies for each eigen-function that there exist a maximum and minimum values $\mathbf{k}_{ß,max}$ and $\mathbf{k}_{ß,min}$ in order to reach convergence of the expansion functions. However, this fact indicates that the energies $E_ß(k)$ show upper and lower limit domains, i.e. we have to deal with band structures.

The extension to the 3D case of the cubic Schrödinger equation is not a trivial one, since the substitution of **kx** by $\mathbf{k_1x + k_2y + k_3z}$ in the arguments of the expansion functions.

However, it is easy to verify that the substitution cannot satisfy L_2-integrability, and, by that, they cannot be taken into account. This fact can readily be verified by the substitution $\boldsymbol{u = k_1x+k_2y+k_3z}$; $\mathbf{k_2y}$ and $\mathbf{k_3z}$ remain unchanged. An acceptable 3D version of eqs. (21, 22) is presented in eq. (26). The aspects of the determination the coefficient A_n and B_n do not differ from those already given by eqs. (23 - 25).

$$\left.\begin{aligned}
&\psi_{\beta,s} = \sum_{n=\beta}^{\infty} A_n \cdot [(\cosh k_1 x)\cdot(\cosh k_2 y)\cdot(\cosh k_3 z)]^{-n} \\
&E_\beta = -\hbar^2 \vec{k}^2 \beta^2 / 2m;\ \beta = 1,2,3.. \\
&\psi_{\beta,as} = \sum_{n=\beta}^{\infty} B_n (\sinh k_1 x)\cdot(\sinh k_2 y)\cdot(\sinh k_3 z)\cdot \\
&[(\cosh k_1 x)\cdot(\cosh k_2 y)\cdot(\cosh k_3 z)]^{-n} \\
&E_ß = -\hbar^2 \vec{k}^2 (1-\beta)^2 / 2m;\ \beta = 2,3,....
\end{aligned}\right\} \quad (26)$$

2.2.1. Some Properties of the Gaussian Kernel

Since the main interest is the charge space, we shall now study the nonlinear Schrödinger equation (NSE) and the nonlocal/nonlinear Schrödinger equation (NNSE) within this frame-work, the starting-point is an exchange Hamiltonian. The preceding aspects of convolution in the position space shall now be extended to the charge space, and we recall that the present considerations are valid in many different disciplines. Let $\boldsymbol{\varphi}$ be a distribution function and $\boldsymbol{\Phi}$ a source function, mutually connected by the operator $\mathbf{F_H}$; this represents an operator notation of a canonical ensemble. An exchange Hamiltonian H_{ex} couples the source field Φ with an environmental field φ by F_H, due to the interaction with surrounding charges.

The partition function of a canonical ensemble reads:

$$\left.\begin{aligned}
&F_H = \exp(-H_{ex} / k_B \cdot T) \\
&H_{ex} = \vec{P}^2 / 2L \rightarrow \frac{-\hbar^2}{2L}\Delta
\end{aligned}\right\} \quad (27)$$

The operator notation of F_H acting on the source function Φ in the charge space with 3 different charges is given by:

$$\left.\begin{aligned}
&\phi = F_H \Phi = \exp(-H_{ex} / E_{ex})\Phi \\
&= \exp(\frac{\varepsilon^2}{2}\cdot\frac{\hbar^2}{2L\cdot E_{ex}}\Delta)\Phi \\
&\varepsilon^2 = \frac{\hbar^2}{L\cdot E_{ex}}
\end{aligned}\right\} \quad (28)$$

$\boldsymbol{F_H}$ may formally be expanded in the same fashion as the usual exponential function $\exp(\xi)$; ξ may either be a real or complex number. This expansion is referred to as Lie series of an operator function. This equation can be solved by the spectral theorem and yields the Gaussian kernel in the charge space.

Since the second derivatives in the exponential operator have the dimension 1/charge^2, the parameter ε have also the dimension of an electric charge, and therefore it is possible to identify ε with the electric charge e_0. If we identify the exchange energy $E_{exchange}$ (E_{ex}) with the energy of an exchange boson $M\cdot c^2$, we are able to determine the parameter L, which originally is a macroscopic magnitude, by $\mathbf{L = 2\cdot\hbar^2/(e_0^2\cdot M\cdot c^2)}$. It should be mentioned that $\mathbf{L = 2\cdot\hbar^2/(n\cdot e_0^2\cdot M\cdot c^2)}$ is possible, too, with n = 1, 2, , to a maximum value $\boldsymbol{n_{maximum}}$. However, the magnitude of $n_{maximum}$ cannot be defined explicitly and depends on the kind of application (atomic/molecular physics on the one side, nuclear and particle physics on the other side. Probably is n = 1, 2 a very good chose, if applications in elementary particle problems are account for. In the domain of nuclear physics n > 2 seems reasonable.

$$\left.\begin{aligned}
&\exp(\frac{\varepsilon^2}{2}\cdot\Delta)\frac{1}{\sqrt{2\pi}}\exp(i\vec{k}\vec{Q}) \\
&= \frac{1}{\sqrt{2\pi}}\exp(-\frac{\varepsilon^2}{2}\vec{k}^2)\cdot\exp(i\vec{k}\vec{Q})
\end{aligned}\right\} \quad (29)$$

$$\left.\begin{aligned}
&K(\varepsilon,\ \vec{Q}-\vec{Q}') = \\
&\frac{1}{2\pi}\int\exp(-\frac{\varepsilon^2}{2}\vec{k}^2)\exp(i\vec{k}(\vec{Q}-\vec{Q}'))d^3k \\
&= \frac{1}{\varepsilon^3}\cdot\frac{1}{\sqrt{\pi^3}}\cdot\exp(-(\vec{Q}-\vec{Q}')^2/2\varepsilon^2)
\end{aligned}\right\} \quad (29a)$$

A nice property of F_H is the n-times iterated operator:

$$[\exp(0.5\varepsilon^2\Delta)]^n = \exp(0.5n\varepsilon^2\Delta) \quad (30)$$

By that, the iterated kernel reads:

$$\left.\begin{aligned}
&K(\varepsilon,\vec{Q}-\vec{Q}') = N\cdot\exp(-(\vec{Q}-\vec{Q}')^2/2n\varepsilon^2) \\
&N = \frac{1}{n^{3/2}\varepsilon^3}\cdot\frac{1}{(\sqrt{2\pi})^3}
\end{aligned}\right\} \quad (31)$$

Although we consider here the charge space, there is a connection to statistical mechanics.

The connection to statistical mechanics is as follows: The statistical motion of free particles ($E_k = \hbar^2k^2/2m$), we equate $\epsilon^2 = \hbar^2/ k_BT$, if $E_{ex} = k_BT$. In the continuous case, the summation over k has to be replaced by an integral. With the help of eq.(30), various properties (e.g., the partition function) can be calculated. According to [21], we equate K to the path-integral kernel $K_F(k_BT, u - z)$, useful for the calculation of perturbation problems of statistical mechanics, if the formal substitution $it/\hbar \rightarrow 1/ k_BT$ is carried out. Thus, the operator-function formalism according to eqs. (29, 30) can be regarded as an operator calculus of path-integral kernels. The question now is, which temperature T or exchange energy E_{ex} should be used and whether $E_{ex} = k_BT$ may hold in energy straggling of charged particles (e.g. protons) and, consequently, this parameter may be kept constant along the pathway. In connection F_H we define the operators O and O^{-1}:

$$\left.\begin{aligned} O &= \exp(\frac{s^2}{2}\cdot\Delta) \\ O^{-1} &= \exp(-\frac{s^2}{2}\cdot\Delta) \end{aligned}\right\} \tag{32}$$

Since $O\cdot O^{-1} = 1$ (unit operator) the inverse kernel can be derived from O^{-1}; both operators a determined by their Lie series; for simplification we present the inverse kernel K^{-1} in one dimension in the charge space:

$$\left.\begin{aligned} &K^{-1}(s,\, Q-Q') = \delta(Q-Q') + \\ &\sum_{n=1}^{\infty}(-1)^n\cdot\frac{(2^n-1)\cdot s^{2n}}{n!\cdot 2^n}\cdot\frac{d^{2n}}{dQ^{2n}}K(s,Q-Q') \end{aligned}\right\} \tag{33}$$

The extension to a 3D-version of eq. (33) has previously been given [18]. A recent publication of a modified 3D version based on eq. (33) has been presented [22], where possible applications to gravitational waves scattered at dark matter have been pointed out.

2.2.2. Nonlinear Schrödinger Equation in Charge Space with Local Interaction

In the sense of quantum electrodynamics (QED) is appears quite natural to take the self-interaction of charges into account. In the following, we only consider 3 different charges Q_1, Q_2, Q_3. Since only the NNSE removes the property of equidistant energy levels, and we make use of this analogy and postulate a NNSE in charge space:

$$\left.\begin{aligned} &i\hbar\partial\psi/\partial t = -\frac{\hbar^2}{2L}\Delta\psi \\ &-\lambda_c\int\left|\psi(\vec{Q}')\right|^2 K(\varepsilon,\,\vec{Q}-\vec{Q}')d^3Q'\cdot\psi(\vec{Q}) \\ &\Delta = \frac{\partial^2}{\partial {Q_1}^2}+\frac{\partial^2}{\partial {Q_2}^2}+\frac{\partial^2}{\partial {Q_3}^2} \end{aligned}\right\} \tag{34}$$

If $\varepsilon \rightarrow 0$, the kernel K assumes the shape of a δ-kernel:

$$E\psi = -\frac{\hbar^2}{2L}\Delta\psi - \lambda_c\left|\psi(\vec{Q})\right|^2\psi(\vec{Q}) \tag{35}$$

The eigen-values are given by:

$$\left.\begin{aligned} E_{\beta,s} &= -\hbar^2\vec{k}^2\beta^2/2L;\ \beta = 1,2,3.. \\ E_{ß,as} &= -\hbar^2\vec{k}^2(1-\beta)^2/2L;\ \beta = 2,3,..... \end{aligned}\right\} \tag{36}$$

This is the NSE in the charge space. Eq. (35) only needs a change of the variables of eq. (26) by the substitutions **x, y,** and **z** by $\mathbf{Q_1}$, $\mathbf{Q_2}$, and $\mathbf{Q_3}$ and $\mathbf{m} \rightarrow \mathbf{L}$ to receive the solutions in the charge space. Therefore we have only stated the spectrum of the eigen-values. The principal difference between the expansion (26) and eqs. (35, 36) is the physical contents, that we have deal with a band structure in the charge space. Since the NSE is equivalent to the Ginsburg-Landau theory of superconductivity and phase transitions, the band structure now can be interpreted as a superconducting state, where the charges have undergone transitions to Cooper pairs. The Josephson junction is a rather natural consequence.

2.2.3. Nonlinear/Nonlocal Schrödinger Equation in Charge Space (NNSE)

With regard to the kernel K appearing in eq. (34) it is interesting to consider its generating function:

$$\left.\begin{aligned} &K(\varepsilon,\,\vec{Q}'-\vec{Q}) = \frac{1}{\sqrt{2\pi}^3\cdot\varepsilon^3}\cdot \\ &\cdot\sum_{j=0}^{\infty}\exp(-{Q_1}'^2/2\varepsilon^2)\cdot H_j(Q_1'/\sqrt{2}\cdot\varepsilon)\cdot Q^j/(\varepsilon^j\cdot j!)\cdot \\ &\cdot\sum_{k=0}^{\infty}\exp(-{Q_2}'^2/2\varepsilon^2)\cdot H_k(Q_2'/\sqrt{2}\cdot\varepsilon)\cdot {Q_2}^k/(\varepsilon^k\cdot k!)\cdot \\ &\cdot\sum_{l=0}^{\infty}\exp(-{Q_3}'^2/2\varepsilon^2)\cdot H_l(Q_3'/\sqrt{2}\cdot\varepsilon)\cdot {Q_3}^l/(\varepsilon^l\cdot l!) \end{aligned}\right\} \tag{37}$$

With the help of the generators of Hermite polynomials the NNSE assumes the shape:

$$\left.\begin{aligned} &\Phi_{n1,n2,n3} = N\cdot\frac{1}{n1!}\cdot\frac{1}{n2!}\cdot\frac{1}{n3!}\cdot\frac{1}{\varepsilon^{n1+n2+n3}}\cdot \\ &\int\left|\psi(\vec{Q}')\right|^2\cdot\exp(-(\vec{Q}'^2/2\varepsilon^2))\cdot \\ &\cdot H_{n1}(Q_1'/\sqrt{2}\cdot\varepsilon)\cdot H_{n2}(Q_2'/\sqrt{2}\cdot\varepsilon) \\ &\cdot H_{n3}(Q_3'/\sqrt{2}\cdot\varepsilon)d^3Q' \end{aligned}\right\} \tag{38}$$

Thus the 3D harmonic oscillator reads:

$$\left.\begin{aligned} &E\cdot\psi+\frac{\hbar^2}{2\cdot L}\cdot\Delta\psi=\phi(\vec{Q})\cdot\psi \\ &=-\lambda_C\cdot(1/\sqrt{2\pi}^3\cdot\varepsilon^3)\cdot \\ &\sum_{n1=0}^{\infty}\sum_{n2=0}^{\infty}\sum_{n3=0}^{\infty}\Phi_{n1,n2,n3}\cdot \\ &Q_1{}^{n1}\cdot Q_2{}^{n2}\cdot Q_3{}^{n3}\psi \end{aligned}\right\} \quad (39)$$

The above equation cannot be exactly solved. However, by taking account of inflexion point Q_{inf} of the Gaussian kernels K we able to define the domain where the quadratic terms are the leading contributions, i.e. where the second derivative is vanishing (note that **T** represents tensor forces of second order). If we also neglect the term T the result will be a 3D harmonic oscillator in the charge space. The only difference is that the energy levels cannot be equidistant as in the linear Schrödinger equation. The following approach is valid only, if the solutions are concentrated within the domain Q_{inf}: $\mathbf{Q \le Q_{inf}}$ and $\mathbf{Q_{inf} = \varepsilon}$.

If this condition is satisfied, the remaining terms represent tensor forces and even weak perturbations. In the other case ($Q_{inf} > \epsilon$), higher order perturbations are more significant:

$$\left.\begin{aligned} &E\cdot\psi+\frac{\hbar^2}{2\cdot L}\cdot\Delta\psi=\phi(\vec{Q})\cdot\psi= \\ &-\lambda_C\cdot\frac{1}{\sqrt{\pi}^3\cdot\varepsilon^3}[\Phi_{0,0,0}+\Phi_{0,0,2}(\vec{Q}^{\,2})+T]\cdot\psi \\ &T=\Phi_{2,2,0}\cdot Q_1{}^2\cdot Q_2{}^2+\Phi_{2,0,2}\cdot Q_1{}^2\cdot Q_3{}^2 \\ &+\Phi_{0,2,2}\cdot Q_2{}^2\cdot Q_3{}^2 \end{aligned}\right\} \quad (40)$$

Eq. (40) can be brought to diagonal form by setting $\Phi_{1,0,0} = \Phi_{0,2,0} = \Phi_{0,0,1} = \Phi_g$, which yields a self-interacting harmonic oscillator in charge space, represented by eq. (41). The harmonic oscillator eigen-functions according to eq. (42) are well known, i.e., the eigen-values are more difficult to calculate.

$$\left.\begin{aligned} &E\cdot\psi+\frac{\hbar^2}{2\cdot L}\cdot\Delta\psi=\phi(\vec{Q})\cdot\psi \\ &=\frac{1}{C}[Q_1{}^2+Q_2{}^2+Q_3{}^2]\cdot\psi \\ &C^{-1}=\Phi_g\cdot\lambda_C\cdot\frac{1}{\sqrt{\pi}^3\cdot\varepsilon^3};\ \omega_0{}^2=\frac{1}{LC} \end{aligned}\right\} \quad (41)$$

$$\left.\begin{aligned} &\psi_{n1,n2,n3}=N(n1,n2,n3)\cdot\exp(-L\omega_0^2\vec{Q}^2)\cdot \\ &H_{n1}(\sqrt{L\omega_0/\hbar}\cdot Q_1)\cdot H_{n2}(\sqrt{L\omega_0/\hbar}\cdot Q_2) \\ &H_{n3}(\sqrt{L\omega_0/\hbar}\cdot Q_3) \end{aligned}\right\} \quad (42)$$

A modification is the further addition of an external 3D-harmonic oscillator to the NNSE reads:

$$V_h=\frac{1}{2}\omega_0{}^2\vec{Q}^2-\lambda_C\cdot e_0{}^3 \quad (43)$$

The associated NNSE now assumes the shape:

$$\left.\begin{aligned} &E\psi=-\frac{\hbar^2}{2L}\Delta\psi+V_h- \\ &\lambda_C\int\left|\psi(\vec{Q}')\right|^2K(\varepsilon,\vec{Q}-\vec{Q}')d^3Q'\cdot\psi(\vec{Q}) \end{aligned}\right\} \quad (44)$$

However, the modification of eq. (44) makes the external potential V_h superfluous in the absence of an external magnetic field!

We reconsider the above Gaussian kernel K, which shows disadvantages that perturbation methods only work as already mentioned above. More serve is its vanishing in the domain of positive energies. Since it is the goal to apply the present model to the quark theory, the nonlocal interaction kernel K is not adequate, and in the positive energy domain the model breaks down, and free quarks could never be observed. By taking account of this situation, we expand this kernel up to the second order term and declare a parabola K as valid for the whole energy domain:

$$\left.\begin{aligned} &K(\varepsilon,\vec{Q}-\vec{Q}')\Rightarrow K_p(\varepsilon,\vec{Q}-\vec{Q}')= \\ &1-\frac{1}{(\sqrt{2\pi})^3\cdot\varepsilon^3}\left(1-\frac{(\vec{Q}_1-\vec{Q}_1')^2}{2\cdot\varepsilon^2}\right) \end{aligned}\right\} \quad (45)$$

By the aide of eq. (45) the general NNSE for the quark model, which we shall use for their ground states of protons, neutrons, and deuteron in the following section, assumes the shape:

$$\left.\begin{aligned} &E\psi=-\frac{\hbar^2}{2L}\Delta\psi- \\ &\lambda_C\int\left|\psi(\vec{Q}')\right|^2K_p(\varepsilon,\vec{Q}-\vec{Q}')d^3Q'\cdot\psi(\vec{Q}) \end{aligned}\right\} \quad (46)$$

Eq. (46) is the starting point for the application to protons (***p***), neutrons (***n***) and deuterons ($\boldsymbol{d_2^1}$).

3. Results

3.1. Proton – Neutron – Deuteron Based on Quantized Circuits

The starting-point for ***p*** and ***n*** is the representation both particles as resonators and the assumption of two identical particles with only different iso-spin, i.e. their rest-energy amounts to:

$$E_{isosp}=M_0c^2=\frac{\hbar\omega rest}{2}=\frac{\hbar}{2\sqrt{LC}} \quad (47)$$

Due to the different electromagnetic properties of ***p*** and ***n*** the different rest-energies results from a modification of eq. (47) by the relationship:

$$\left.\begin{aligned} E_p &= 938.257019042969\,\text{MeV} = \frac{\hbar}{2\sqrt{C(L+M_i)}} \\ E_n &= 939.565002441406\,\text{MeV} = \frac{\hbar}{2\sqrt{C(L-M_i)}} \end{aligned}\right\} \quad (48)$$

By some simple numerical manipulations we are able to receive the following results:

$$\left.\begin{aligned} &(E_{isosp}/E_p)^2 = (L+M_i)/L = \alpha^2; \\ &(E_{isosp}/E_n)^2 = (L-M_i)/L = \beta^2 \\ &\alpha^2+\beta^2 = 2;\ \alpha^2-\beta^2 = 2M_i/L \\ &= 2.786169826269544\cdot 10^{-3} \\ &E_{isosp} = \sqrt{2E_p^{\,2}\cdot E_n^{\,2}/(E_p^{\,2}+E_n^{\,2})} \\ &= 938.910327442572\,\text{MeV} \end{aligned}\right\} \quad (49)$$

Since according to eq. (49) the ratio 2·M/L is given, it makes sense to write M_i = factor·L, factor assumes the value factor = **$1.393084913134772\cdot 10^{-3}$.** The differences between E_{isosp} and E_p, E_n are also noteworthy:

$E_{isosp} - E_p$ = 0.653308399603588 MeV;
$E_n - E_{isosp}$ = 0.654674998833912 MeV.

The deuterium model based on resonators is shown by Figure 3. The principal coupling $\mathbf{M_i}$ between the resonators corresponds to eqs. (47 - 49). There are two additional branches referring to the motion of ***p*** and ***n*** denoted by $\mathbf{L_M}$, which are connected to the capacitances.

In order to determine the normal modes we consider the equations according to eq. (49) and the succeeding equations (50) and (51):

$$\left.\begin{aligned} &(L+L_M)\ddot{Q}_1 + (M_i+M_{im})\ddot{Q}_2 \\ &+Q_1/C = 0 \\ &(L+L_M)\ddot{Q}_2 + (M_i+M_{im})\ddot{Q}_1 \\ &+Q_2/C = 0 \end{aligned}\right\} \quad (50)$$

By the abbreviations $\mathbf{L_1 = L + L_M}$, $\mathbf{M_1 = M_i + M_{im}}$, and the substitutions $q_1 = Q_1 + Q_2$, $q_2 = Q_1 - Q_2$ we obtain the normal modes of eq. (50), which is a standard task, and the modes are given by:

$$\left.\begin{aligned} \omega_1'^2 &= \frac{1}{(L_1+M_1)C} \\ \omega_2'^2 &= \frac{1}{(L_1-M_1)C} \end{aligned}\right\} \quad (51)$$

Using the assumptions (simplifications) that (besides factor = M_i/L, already defined) $\mathbf{L_B = \lambda_f \cdot L}$ and $\mathbf{M_{im}}$ = **factor·L_B**, we are readily able to calculate the rest energy of the **deuteron.** The calculated value of this rest energy is E_{calc} = 1875.59007 MeV, but the measured value is 1875.612 928(12) MeV.

By the aid of these assumptions the deuteron calculation can be reduced to the following problem:

$$\left.\begin{aligned} &deuteron(calc) = (E_{proton}+E_{neutron})/ \\ &\sqrt{1+L_B/L} \end{aligned}\right\} \quad (51a)$$

The ratio $\mathbf{L_B/L}$ is determined as **L_B/L = 0.0023568.** With regard to the energy of **45.6 MeV** we have neglected spin-orbit coupling. If we change one above assumption, namely the M_{im} = factor·L_B by M_{im} = 2·factor·L_B, then the calculated rest energy of deuteron approaches the measured value**: E_{calc} = 1875.609434 MeV** and L_B/L are associated with an energy of **46.24 MeV.** Thus only the relation (51a) is slightly more complicated.

The calculation is based on the ground-state level of the quantized version of eq. (51), i.e.:

$$E_{deuteron} = \frac{\hbar}{2}\cdot[\omega_1' + \omega_2'] \quad (51b)$$

Thus **ω_1'** refers to ***p*** and **ω_2'** to ***n.*** The spin-orbit coupling has been neglected, but due to the different sign of ***p*** and ***n*** the errors partially compensate.

With regard to this consideration we should point out that the presented model requires empirical measurement data to determine the parameters of the coupled resonators. The present model is not based on first principles, which would enable to calculate all parameters appearing in eqs. (47 – 51).

In Figure 3 the motion of the proton and neutron is denoted by currents with L_M and mutual coupling between protons – neutrons by M_{im}. The aspect of iso-spin and spin did not play a decisive role in the above considerations. At first we point out that eq. (47) is immediately related to iso-spin, i.e. we have two identical nucleons differing in the related quantum number.

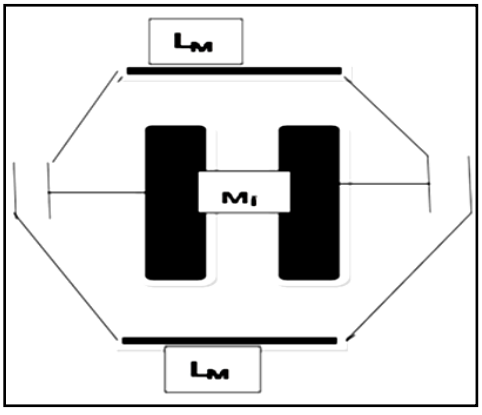

Figure 3. Deuteron model with the constituents *p* and *n*

The deuteron constituents ***p*** and ***n*** are represented by two coupled oscillators; their motions are characterized by

currents denoted by L_M with mutual coupling $\mathbf{M_{im}}$ between the proton - neutron current.

Thereafter we introduce the spin and require that the wave-function obeys the Pauli principle. These requirements lead to the introduction of a spinor with contains **iso-spin x spin (1, -1, 1, 1,)** or (in the absence of a magnetic field and neglect of spin-orbit coupling) **(1, -1, -1, -1)** and to a determinant representation of the wave-function, i.e. :

$$\left.\begin{aligned} &H_0 = \tfrac{1}{2\cdot L} P_1^2 + \tfrac{L}{2}\cdot\omega_0^2\cdot Q_1^2 + \\ &\tfrac{1}{2\cdot L} P_2^2 + \tfrac{L}{2}\cdot\omega_0^2\cdot Q_2^2 \\ &H_o\cdot Det\left|\Phi_{1,1},\Phi_{-1,1}\right| \\ &= E\cdot Det\left|\Phi_{1,1},\Phi_{-1,1}\right| \end{aligned}\right\} \quad (52)$$

In a next step eq. (52) can be modified by using ω_1' and ω_2' instead of ω_0, and L ± M_{im} instead of L. The final version of eq. (52) uses the coupling modifications of eq. (51), and the ground-state energy level of deuterium is given according to eq. (51b) and eq. (52) assumes the shape:

$$\left.\begin{aligned} &H = \frac{1}{2\cdot L_1} P_1^2 + \frac{L_1}{2}\cdot\omega_1'^2\cdot Q_1^2 + \frac{1}{2\cdot L_2} P_2^2 \\ &+ \frac{L_2}{2}\cdot\omega_2'^2\cdot Q_2^2 \\ &H\cdot Det\left|\Phi_{1,1},\Phi_{-1,1}\right| = E\cdot Det\left|\Phi_{1,1},\Phi_{-1,1}\right| \end{aligned}\right\} \quad (52a)$$

It certainly looks rather uncomfortable to treat the two nucleon cases with many circuit parameters, and therefore it is the goal to study it in section 3.2 based on NNSE. By that, the quark structure of ***p*** and ***n*** is accounted for.

3.2. Proton – Neutron – Deuteron via NNSE in Charge Space

3.2.1. Proton - Neutron

It should be mentioned that the proton quarks are given by $[(2/3)e_0, (2/3)e_0, -(1/3)e_0]$ and the neutron quarks by $[[(2/3)e_0, -1/3)e_0, -(1/3)e_0]$, the square of proton quarks yields the effective charge $2\cdot(4/9)\cdot e_0^2 + (1/9)\cdot e_0^2 = e_0^2$, whereas for the neutron we obtain $(4/9)\cdot e_0^2 + 2\cdot(1/9)\cdot e_0^2 = (2/3)\cdot e_0^2$. This fact is important with regard to the normalization.

With respect to the following equations we use the nomenclature: the bold '**1**' refers to proton index and the neutron index is characterized by '**2**'. As already previously point out the basic equation for protons (and neutrons) is given by the following equation:

$$\left.\begin{aligned} &H_1\psi_1 = -\frac{\hbar^2}{2L_p}\Delta_Q\cdot\psi_1 - \lambda_c\int\left|\psi(\vec{Q}_1')\right|^2\cdot \\ &K_{1,p}(\varepsilon_p, \vec{Q}_1 - \vec{Q}_1')d^3Q_1'\cdot\psi(\vec{Q}_1) = E_1\psi_1 \end{aligned}\right\} \quad (53)$$

By this modification we get a 3D harmonic oscillator in the charge space with self-interaction and non-equidistant energy levels in contrast to the results of eq. (18). It should be pointed out that the solutions of eq. (53) are the same as in the previous case, namely the Gaussian multiplied with Hermite polynomials:

$$\left.\begin{aligned} &\psi_{1,n1,n2,n3} = N(n1,n2,n3)\cdot\exp(-\bar{Q}_1^2/2\varepsilon_1^2)\cdot \\ &H_{n1}(Q_{1,1}/\sqrt{2}\cdot\varepsilon_1)\cdot H_{n2}(Q_{1,2}/\sqrt{2}\cdot\varepsilon_1)\cdot \\ &\cdot H_{n3}(Q_{1,3}/\sqrt{2}\cdot\varepsilon_1);\ n1,n2,3 = 0,1,.... \end{aligned}\right\} \quad (54)$$

It should be noted that in contrast to eqs. (16 - 18) linear combinations of eq. (51) do not exist. With regard to the present study we are interested in the ground-state case representing the proton and not excited quark states, i.e.: n1 = n2 = n3 = 0. Inserting the ground state condition of eq. (54) into eq. (53) and, after carrying out simple integrations, collecting all terms ***with and without*** $\mathbf{Q_1^2}$**,** the following connections will be received:

$$\left.\begin{aligned} &\lambda_c = \frac{\hbar^2\sqrt{2}^3\cdot(2\pi)^3\cdot\varepsilon_1^4}{L};\ \varepsilon_1^2 = \frac{\hbar^2}{L}\cdot E_{ex,1} \\ &E_{0,1} = \frac{3\cdot\hbar^2}{2\cdot L\cdot\varepsilon_1^2} - \frac{3\lambda_c}{4\sqrt{2^3}\cdot(2\pi)^3\varepsilon_1^6} \\ &\rightarrow E_{0,proton} = \frac{3\cdot\hbar^2}{4\cdot\varepsilon_1^2\cdot L} = \frac{3}{4}\cdot E_{ex,1} \end{aligned}\right\} \quad (55)$$

This expression is equivalent to the ground state energy **0.5 ħ·ω₀** and identified with calculated rest energy of a proton. On the other hand, if the exchange energy $E_{ex,1}$ is known, we are able to determine the coupling constant λ_c, which is required for the calculation of the binding energy of the deuteron.

When we pass from **p** to **n** we have to deal with the identical equation and only the index '1' is replaced by '2'., In particular, the nonlocal range parameter $\boldsymbol{\epsilon_1}$ must be replaced by $\boldsymbol{\epsilon_2}$, which is slightly different. Thus the result is the last term of eq. (55), which now assumes the shape:

$$E_{0,neutron} = \frac{3\cdot\hbar^2}{4\cdot\varepsilon_2^2\cdot L} = \frac{3}{4}\cdot E_{ex,2} \quad (56)$$

Some common remarks:

Since the half-width parameters $\boldsymbol{\epsilon_n^2}$ and $\boldsymbol{\epsilon_n^2}$ result from the charge normalization of the sum of the quark charges and the related squares, the only unknown parameter is the formal '**inductivity L'** related to the kinetic energy of the quarks in the considered nuclei. Since the present model cannot determine E_{ex} for proton and neutron quarks, we have used the dynamical quark model [21] to give the necessary support.

$$\left.\begin{aligned} E_{ex,\,proton} &= 1250.92853\,MeV \\ E_{0,\,proton} &= 938.257019\ MeV \\ E_{ex,\,neutron} &= 1252.75071\,MeV \\ E_{0,\,neutron} &= 939.56301\,MeV \end{aligned}\right\} \quad (57)$$

It appears that the result is interesting in spite of the neglect of spin-orbit coupling of the quarks. The spin may readily be added according to the section 3.1, and only to avoid unnecessary writing we did not account for this property here.

3.2.2. Deuteron

The basic equation reads:

$$H\cdot\psi = (H_1 + H_2)\cdot\psi + H_{I,\,p\leftrightarrow n}\cdot\psi \quad (58)$$

The overall wave-function ψ can either be chosen as a product of $\boldsymbol{\psi_1\cdot\ \psi_2}$ or a determinant, but the choice is unimportant for the present consideration. The interaction Hamiltonian between ***p*** and ***n*** is assumed by:

$$\left.\begin{aligned} H_{I,p\leftrightarrow n} &= -\lambda c^2 \int\int \left|\psi_1(\vec{Q}_1')\right|^2 \cdot \\ &\cdot \frac{(\vec{Q}_1' - \vec{Q}_2')^2}{\varepsilon_1^{\,2} + \varepsilon_2^{\,2}} \left|\psi_2(\vec{Q}_2')\right|^2 d^3Q_1' d^3Q_2' \end{aligned}\right\} \quad (59)$$

Since all mixed products of the form $\mathbf{Q_1'\cdot Q_2'}$ vanish, if the ground state Gaussian is used, the squares $Q_1'^2$ and $Q_1'^2$ provide the significant correction of the sum of rest energies $\mathbf{E_{0,proton} + E_{0,neutron}}$. Owing to the square of the coupling constant λ_c the correction energy (binding energy) is rather small, and the results of the total energy for deuteron are:

$$E_{deuteron} = 1875.612928\,MeV\ (measured)$$
$$E_{deuteron} = 1875.59340\,MeV\ (eq.59)$$
$$E_{deuteron} = 1875.59007\,MeV\ (eqs.50-52)$$

4. Conclusions

With regard to deuteron we should like to point out that we could not verify of an excited overall state, although the proton/neutron quarks may undergo excitations. By that, excited resonance states are obtained, which are well-known in particle physics. In last time, some debate arose, whether deuterons have excited states, but we are unable to confirm these speculations [23].

It should further be pointed out that in a continuation we consider a nonlinear Dirac equation with self-interaction based on [10]. By that, we shall obtain a rather interesting form of Quantum Chromo-Dynamics (QCD) formulated by quantization of the charge space.

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