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Series Solution of the 1 + 2 Continuous Toda Chain

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Abstract The 1+2 dimensional continuous Toda chain presents a formidable challenge to the construction of solutions. Two variable reductions of the equation are known, but up till now nothing more is known. In this paper a way to solve the equation is presented, and some solutions are explicitly constructed. The method depends upon connecting series solutions of the symmetry equation of the Toda chain with the Toda solutions, as is guaranteed by general theory. Such solutions of the symmetry equation are obtainable a series method. An explicit solution is constructed by this method, and a general procedure is given to realize further solutions, which, however are given only in implicit form.

Keywords Toda chain, Integrable systems and Non-Linear Differential Equations

1. Introduction

The usual form of the equation under consideration is the following one;

$$\rho_{y,x} = (e^{\rho})_{z,z}, \quad (\ln u)_{y,x} = u_{z,z}, \quad u \equiv e^{\rho} \quad (1)$$

Here ρ(x, y, z) is an unknown function of three independent variables. This equation arises as a reduction of the Plebansky equation[1] describing self-dual four dimensional (0 + 4); (2 + 2) gravity. In this connection it was considered in[2] and in literature is known as the Boyer-Finley equation.

Equation (1) can also be obtained as a limit of the discrete Toda chain

$$\rho_{y,x} = e^{\rho_{n+1}} - 2e^{\rho} + e^{\rho_{n-1}}$$

under appropriate rescaling (n→ z). The series solutions of the symmetry equation for the Toda chain was found in[3].

Also, a solution of the two dimensional reduction of (1) (ρ = (z, y + x) was found in implicit form in[4]. Infinite series solutions of the symmetry equation corresponding to (1) were found in[5]. But the connection between series solutions of the symmetry equation with the solution of the initial system (1) has not been discovered. However general theory gives a guarantee that each solution of symmetry equation is connected with an analytical solution of the initial system in explicit or implicit form. The goal of the present paper is to fill this gap and demonstrate a way how an analytical solution of (1) is connected with the solution constructed in[3] of the symmetry equation. In[5], the Plebansky equation was represented in the form of two equations of the first order for two unknown functions, one of which satisfies the Plebansky equation by itself, the second one satisfies the corresponding symmetry equation. It is possible to find **in** an independent way the solution of symmetry equation in recurrence form. As was remarked the above equation under consideration in the present paper is a reduction of the Plebansky equation and so it is possible to try to solve it by the same methods[6].

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2. Preliminary Manipulations. Short Excursion into[3]

Let us rewrite (1) in the form of a system of two equations of the first degree.

$$(\ln u)_y = T_z, \quad u_z = T_x, \quad (\ln u)_{x,y} = u_{zz}, \quad (\frac{T_x}{u})_y = T_{zz})$$

or as the initial equation is symmetrical with respect to exchange of the variables x, y, the following is also a possible form;

$$(\ln u)_x = w_z, \; u_z = w_y, \; (\ln u)_{x,y} = u_{zz}, \; (\frac{w_y}{u})_x = w_{zz} 3)$$

The symmetry equation arises from the initial one after differentiation by an arbitrary parameter and considering this derivative as a new unknown function. In the case under consideration this equation is

$$\dot{u} = S, \; (\frac{S}{u})_{x,y} = S_{z,z}, \; S = T_x(w_y), \; (\frac{T_x}{u})_y = T_{zz}, \; (\frac{w_y}{u})_x = w_{zz}$$

It is necessary to understand the last manipulation in such way, that if we represent the solution of the symmetry equation in the form S = T_x or S = w_y then the last equations in (2), (3) are exactly the symmetry equation by itself. Finally a linear system of equations of first order for the function u function is

$$(\ln u)_y = T_z, \; u_z = T_x, \; (\ln u)_x = w_z, \; u_z = w_y 4)$$

In[3] we have obtained series solutions of the symmetry equation in integro-differential terms of the function u. Thus it is possible use these expressions in the system T, u and obtain two self consistent equation in-stead of only one equation for the function u. It is obvious that in this way we will not be able to obtain a general solution for the equation for u but only its partial soliton like series solutions. Solving the second equation of (2), $u = \theta_X$, $T = \theta_Z$ we rewrite (1) in the form

$$\theta_{y,x} = \theta_x \theta_{z,z}, \quad w_{y,x} = w_y w_{z,z}$$

In[3] it was shown that the solution of the symmetry equation T, may be obtained in terms of α_n functions which satisfy the following recurrence relations

$$\alpha^{n-1} = \int dy \frac{(u^{n+1}\alpha^n)_z}{(n+1)u^n}, \quad \frac{(u^{n+1}\alpha^n)_x}{u^{n+1}} = (n+1)\alpha_z^{n-1}$$

Eliminating α^{n-1} from both equations we arrive at the equation for the functions α^n in the form

$$\left[\frac{(u^{n+1}\alpha^n)_z}{u^n}\right]_z = \left[\frac{(u^{n+1}\alpha^n)_x}{u^{n+1}}\right]_y, \quad (u^{n+2}\alpha_z^n)_z = (u^{n+1}\alpha_y^n)_x$$

The left and right equations are the same. From these expressions it follows that there exists an obvious solution $\alpha^n = 1$ which leads to a finite solution for T. The solution for T becomes

$$T = u\alpha^0 = u\int dy \frac{(u^2\alpha^1)_z}{2u}, \quad \theta_y - \frac{\theta_z^2}{2} = \frac{u^2\alpha^1}{2} \quad (5)$$

The second equality is obtained from the first one after the substitution

$T=\theta_z$, $u= \theta_x$ and differentiation of the subsequent expression $\frac{\theta_z}{\theta_x}$ with respect to the argument y and integration once over z

$$\theta_x \left(\frac{\theta_z}{\theta_x}\right)_y = (\theta_x^2\alpha^1)_z, \quad \theta_{x,y} - \theta_z\theta_{z,z} = (\theta_x^2\alpha^1)_z$$

The last equality is a series of additional conditions on the $\theta_{x,y} = \theta_x \theta_{z,z}$ for the function θ.

3. Generalization of R. Ward's Solution

This section explains why the analytic solutions of Ward exist at all. The simplest solution of the symmetry equation is a linear combination of derivatives of the functions u $S = w_Y = u_Z = au_X + bu_Y + cu_Z$. The solution of Richard Ward corresponds to choosing c = 1, a =- b. In the case c is not equal to zero we have $u_Z = Au_X + Bu_Y$ and the second system under this additional condition becomes

$$u_x = uw_z, \quad u_z = w_y, \quad u_x = u(Aw_x + Bw_y), \quad Au_x + Bu_y = w_y$$

Let us seek a solution of this system in the form

$$x = \theta(u,w), \; y = \sigma(u,w)$$

The system of equations defining derivatives of (u, w) with respect to space coordinates (x, y) is the following one;

$$1 = \theta_u u_x + \theta_w w_x, \quad 0 = \sigma_u u_x + \sigma_w w_x$$

$$0 = \theta_u u_y + \theta_w w_y, \quad 1 = \sigma_u u_y + \sigma_w w_y.$$

After solving the last system,

$$u_x = \frac{\sigma_w}{D}, \quad w_x = -\frac{\sigma_u}{D}, \quad u_y = -\frac{\theta_w}{D}, \quad w_y = \frac{\theta_u}{D}$$

and substitution into the previous one we arrive at a linear system of equations for θ (u, w) and σ(u, w).

$$\sigma_w = u(-A\sigma_u + B\theta_u), \quad A\sigma_w - B\theta_w = \theta_u$$

The last system after eliminating (for instance) the function σ leads to an equation of the second order with separable variables

$$u\theta_{u,u} + \frac{1}{A}\theta_{w,u} + \frac{B}{A}\theta_{w,w} = 0$$

The case considered by Ward corresponds to the limiting case $A\to\infty$, $B\to\infty$, $\frac{A}{B}\to -1$

4. The Zero Order Term of Series Solution to the Symmetry Equation

In the case $\alpha^0 = 1$ from the general formula it follows that T = u or w = u and from the corresponding formulas of the previous section we obtain $u_X = u_Z$ or $u_Y = u_Z$. These are particular cases of the generalized Ward construction of the previous section. The first equations in this **case** lead $u_Y = uu_Z$. This is the well known Monge equation (the equation of Hamilton-Jacobi for free motion in one dimension) with general solution z + y + ux = F (u) or z + x + uy = F (u). It is not difficult to connect these solutions with the generalized Ward solution of the previous section.

5. The First Term of the Symmetry Equation Series Solution

In this case $\alpha^1 = 1$. It is possible represent the solution of the symmetry equation in different variables and in connection of this it will possible to obtain two different solutions (at least known to us) of the continuous Toda chain.

5.1. The first possibility

In connection with the recurrence procedure we obtain $\alpha_0=\int dy u_z$ and the solution for T takes the form

$$T = u\left(\int dy u_z\right)$$

or after passing to the function θ, $u=\theta_x$, $T= \theta_z$, (obeying the equation $\theta_{y,x}= \theta_x\theta_{z,z}$) it appears as

$$\left(\frac{\theta_z}{\theta_x}\right)_y = \theta_{z,x}, \quad \theta_y = \frac{\theta_z^2}{2} + \frac{\theta_x^2}{2}$$

In the transformation from the left to the right hand sides we have used the equation for θ and the once integrated result on z. From the second form of the equation above it follows that θ is a solution of the Monge-Ampere equation of the third order and it is possible to use its known general solution. We will go by more direct way (which is obviously equivalent to the previous one). Now let us introduce the notations

$$\lambda_1 = \theta_x, \quad \lambda_2 = \theta_z, \quad \lambda_3 = \theta_y$$

After differentiation on the right hand equation by x and z respectively and taking into account the above definition of λ we arrive at the system of equations

$$(\lambda_1)_y = \lambda_1(\lambda_1)_x + \lambda_2(\lambda_2)_x, \quad (\lambda_2)_y = \lambda_1(\lambda_1)_z + \lambda_2(\lambda_2)_z$$

The last system is well known and its general solution in implicit form is the following:[7]

$$x + \lambda_1 y = F^1(\lambda_1, \lambda_2), \quad z + \lambda_2 y = F^2(\lambda_1, \lambda_2)$$

where F^1, F^2 are arbitrary functions of their two arguments. But in our case we have additional conditions $(\lambda_1)_z = (\lambda_2)_x$ and this fact will limit the functions F^1, F^2. After differentiation of the two equations by x, z, y respectively and solving a linear system of algebraic equations we obtain all derivatives of the functions (λ_1, λ_2) with respect to these three arguments. The result is the following:

$$(\lambda_1)_x = \frac{F^2_{\lambda_2} - y}{D}, \quad (\lambda_2)_x = -\frac{F^2_{\lambda_1}}{D}, \quad (\lambda_1)_z = -\frac{F^1_{\lambda_2}}{D}, \quad (\lambda_2)_z = \frac{F^1_{\lambda_1 - y}}{D} \qquad (6)$$

where

$$D = (F^1_{\lambda_1} - y)(F^2_{\lambda_2} - y) - F^1_{\lambda_2} F^2_{\lambda_1} \qquad (7)$$

is the determinant of linear system under consideration. From the additional condition above and calculated values of corresponding derivatives of the function λ we obtained in the last equation we have $F^1 = G_{\lambda_1}$; $F^2 = G_{\lambda_2}$. In the same way we obtain for derivatives of the functions with respect the argument y

$$(\lambda_1)_y = \lambda_1 \frac{G_{\lambda_2,\lambda_2} - y}{D} - \lambda_2 \frac{G_{\lambda_2,\lambda_1}}{D}, \quad (\lambda_1)_y = -\lambda_1 \frac{G_{\lambda_2,\lambda_1}}{D} + \lambda_2 \frac{G_{\lambda_1,\lambda_1} - y}{D}$$

The last calculations show that the implicitly defined functions from the equations above satisfy the necessary system of equations and allow us to obtain a further constraint on the function G which arises from the fact that the equation for the function θ must be satisfied.. Namely

$$\theta_{y,x} = \theta_x \theta_{z,z}, \quad (\lambda_1)_y = \lambda_1(\lambda_2)_z$$

After substitution in the last equation of all expressions obtained above we pass to

$$\lambda_1(G_{\lambda_2,\lambda_2} - G_{\lambda_1,\lambda_1}) = \lambda_2 G_{\lambda_1,\lambda_2}$$

After the introduction of new variables $s=\lambda_1 \lambda_2$, $t=\frac{\lambda_2}{\lambda_1}$ the last equations lead to a two dimensional equation with separable variables. Indeed

$$G_{\lambda_1} = \frac{1}{\lambda_1}(sG_s - tG_t), \quad G_{\lambda_1,\lambda_1} = \frac{1}{\lambda_1^2}\left(-(s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}) + (s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t})^2\right)G$$

$$G_{\lambda_2,\lambda_2} = \frac{1}{\lambda_2^2}\left(-(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}) + (s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^2\right)G, \quad G_{\lambda_2,\lambda_1} = \frac{1}{\lambda_2\lambda_1}\left((s\frac{\partial}{\partial s})^2 - (t\frac{\partial}{\partial t})^2\right)G$$

Substituting these expressions into equation for G we seek its solution in the form $G = \int dk s^k t^{-k} g(t,k)$ and pass to

$$g_{tt} + (2k-1)tg_t = (2k-1)2kg$$

Thus for the function G we obtain finally

$$G = \int dk s^k t^{-k}(c_1(k)g_1 + c_2(k)g_2)$$

where c_1, c_2 are arbitrary functions of argument k and g_1, g_2 are two fundamental solutions of the linear equation of second order above. Thus for the solution of the Toda chain $u = \theta_x = \lambda_1$ we obtain an implicit solution for it

$$x + \lambda_1 y = G_{\lambda_1}(\lambda_1, \lambda_2), \quad z + \lambda_2 y = G_{\lambda_2}(\lambda_1, \lambda_2)$$

where G is the general solution of the linear equation above containing two arbitrary functions.

5.1.1. An example

By direct calculations it is not difficult to check that $G = \frac{1}{3}\lambda_2^3 + \frac{1}{2}\lambda_2 \lambda_1^2$ is a partial solution of equation of the subsection above. Thus in connection of the main result

$$x + \lambda_1 y = \lambda_1 \lambda_2, \quad z + \lambda_2 y = \lambda_2^2 + \frac{1}{2}\lambda_1^2$$

and $\lambda_1 = u$ is a solution of the continuous Toda chain. Eliminating λ_2 from the equations above we obtain for u

$$x^2 + uxy - zu^2 + \frac{1}{2}u^4 = 0$$

Calculation of necessary derivatives

$$u_x = -\frac{2x + uy}{xy - 2uz + 2u^3}, \quad (\ln u)_y = -\frac{x}{xy - 2uz + 2u^3}, \quad u_z = \frac{u^2}{xy - 2uz + 2u^3}$$

and finally after straightforward calculations we check that

$$(\ln u)_{x,y} = u_{z,z}$$

5.2. The second possibility

In the case $\alpha_1 = 1$ in connection with the recurrence procedure we obtain $\alpha_0 = \int dyu_z$ and the solution for T takes the form

$$T = u(\int dyu_z), \quad U = \int dyu, \quad u = U_y, \quad T = U_y U_z$$

and the equations which are necessary to solve are the following;

$$U_{y,z} = (U_y U_z)_x, \quad (\ln U_y)_y = (U_y U_z)_z, \quad (\ln U_y)_x = U_{z,z}$$

The first and the last equations above lead to a relation between the derivatives (after integration once with respect to the argument z) in the form $\ln U_y = \frac{1}{2}U_z^2 + U_x$, $U_y = e^{\frac{1}{2}U_z^2 + U_x}$. As in the previous subsection it means that U is a solution of the Monge-Ampere equation of the third order. It is possible to substitute its general solution into the last equation and find some restrictions to determine the arbitrary functions in the solution. We will go more directly.

Let us seek a solution of these equations using the following parameterization

$$x = X(U_x, U_z, y), \quad z = Z(U_x, U_z, y), \quad \beta \equiv U_x, \quad \gamma \equiv U_z$$

from which expressions for second order derivatives follow immediately;

$$U_{x,x} = \frac{Z_\gamma}{D} \quad U_{x,z} = -\frac{Z_\beta}{D} = -\frac{X_\gamma}{D} \quad U_{z,z} = \frac{X_\beta}{D},$$

$X = W_\beta(\beta, \gamma, y)$; $Z = W_\gamma(\beta, \gamma, y)$ and the equation transform to a linear equation of second order with separable variables

$$-\gamma W_{\beta,\gamma} + W_{\gamma,\gamma} = W_{\beta,\beta}, \quad W = \int dk e^{k\beta} G(\gamma, k), \quad -\gamma k G_\gamma + G_{\gamma,\gamma} = k^2 U$$

It is possible determine the dependence of the function W upon its argument after solution of two equations which arise after differentiation of the previous equations by the argument y

$$X_\beta U_{x,y} + X_\gamma U_{z,y} + X_y = 0, \quad Z_\beta U_{x,y} + Z_\gamma U_{z,y} + Z_y = 0$$

Remembering that $U_y = e^{\frac{1}{2}\gamma^2+\beta}$ after trivial manipulations we rewrite the last equations in a form

$$e^{\frac{1}{2}\gamma^2+\beta} + W_{\beta,y} = 0, \quad \gamma e^{\frac{1}{2}\gamma^2+\beta} + W_{\gamma,y} = 0$$

we obtain for W and explicit expressions for x,z

$$W = -ye^{\frac{1}{2}\gamma^2+\beta} + W^L, \quad x = W_\beta = -ye^{\frac{1}{2}U_z^2+U_x} + W_\beta^L \quad z = W_\gamma = -ye^{\frac{1}{2}\gamma^2+\beta} + W_\gamma^L$$

where W^L is solution of the linear equation obtained above, which do not depend on argument y. The solution of the Toda chain of the beginning of this paper is given by connection u = $U_y = e^{\frac{1}{2}\gamma^2+\beta}$ and dependence β, γ functions on independent arguments x, z, y is defined in explicit form by formulas above.

5.3 Second example

This example may help the reader to understand the difficulties in trying to obtain solutions in explicit form. It is easy to check that $W^L = \gamma e^{-\beta}$ is an explicit solution of the linear equation and thus $W = -ye^{\frac{1}{2}U_z^2+U_x} + \gamma e^{-\beta}$. The implicit form of the solution is given by

$$x = W_\beta = -ye^{\frac{1}{2}\gamma_z^2+\beta} - \gamma e^{-\beta}, \quad z = W_\gamma = -y\gamma e^{\frac{1}{2}\gamma_z^2+U_x} + e^{-\beta}, \quad U_z = \gamma, U_x = \beta$$

From these expressions the equation

$$U_{z,z} + 2U_z U_{z,x} - U_{x,x} = 0$$

follows immediately. It is equivalent to our linear system above. With the help of this equation it is not difficult to check that

$$T = U_y U_z = 2e^{\frac{1}{2}U_z^2+U_x}, \quad U_y = e^{\frac{1}{2}U_z^2+U_x}$$

satisfy the equations

$$U_{y,z} = T_x, \quad (\ln U_y)_y = T_z$$

and thus $u = U_y$ is a solution of the equation of title of this paper.

We rewrite equations which define an implicit solution in a form u =

$$U_y = ye^{\frac{1}{2}U_z^2+U_x} \equiv e^{\frac{1}{2}\gamma^2+\beta}$$ in equivalent form

$$x = -yu - \gamma\frac{1}{u}e^{\frac{1}{2}\gamma^2}, \quad z = -\gamma yu + \frac{1}{u}e^{\frac{1}{2}\gamma^2}$$

After eliminating terms without y on the right hand side we arrive at a quadratic equation to determinate the variable γ

$$\gamma^2 + \frac{z}{yu}\gamma + \frac{x}{yu} + 1 = 0$$

Substituting the solution of this equation into the first or second equations we reach an equation determining in implicit form the function u. It is obvious that to obtain this equation is not a very simple problem.

6. Second Step

In the case where $\alpha_2 = 1$ in connection with the recurrence procedure we obtain

$$\alpha_1 = \int dy u_z = U_z, \quad \alpha_0 = \int dy \frac{u^2\alpha_1}{2u} = \int dy(U_{y,z}U_z + \frac{1}{2}U_y U_{z,z}) = \frac{1}{2}(U_z^2 + U_x)$$

and the solution for T takes the form

$$T = \frac{1}{2}U_y(U_z^2 + U_x), \quad U = \int dyu, \quad u = U_y, \quad U_{y,x} = U_y U_{z,z} \tag{8}$$

The equations which are necessary to solve are the following;

$$U_{y,z} = (T)_x = \frac{1}{2}[U_{y,x}(U_z^2 + U_x) + U_y(2U_z U_{z,x} + U_{x,x})], \quad (\ln U_y)_y = (T)_z$$

Let us introduce the definitions

$$U_y = e^c, \quad U_x = a, \quad U_z = b, \quad \alpha = \frac{1}{2}(b^2 + a)$$

Equations above together with the notation introduced lead to the following system of equations

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & b & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_x \equiv L \begin{pmatrix} a \\ b \\ c \end{pmatrix}_x, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}_y = e^c L \begin{pmatrix} a \\ b \\ c \end{pmatrix}_z$$

As in the cases above we will seek solution of these equation by implicit substitution

$$x = X(a,b,c), \quad z = Z(a,b,c), \quad y = Y(a,b,c)$$

After differentiation of these equalities with respect to the independent arguments of the problem and introduction of the matrix

$$V = \begin{pmatrix} X_a & X_b & X_c \\ Z_a & Z_b & Z_c \\ Y_a & Y_b & Y_c \end{pmatrix}$$

we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_x = V^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}_z = V^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}_y = V^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Substituting these expressions in linear system equations of the first order we obtain

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = VLV^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e^c VLV^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The last equations allow us to reconstruct the explicit form matrix VLV^{-1}

$$VLV^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^c \\ 1 & 0 & be^c \\ 0 & e^{-c} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-c} \end{pmatrix} L^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^c \end{pmatrix}$$

The first two columns are a direct consequence of the equations above. The last column arises from the fact $\mathrm{Trace}(VLV^{-1})^n = \mathrm{Trace}\, L^n$.

Now we arrive at a linear system of equations for determining the functions X, Z, Y.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^c \end{pmatrix} VL = L^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^c \end{pmatrix} V$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^c \end{pmatrix} V = \begin{pmatrix} X_a & X_b & X_c \\ Z_a & Z_b & Z_c \\ (e^cY)_a & (e^cY)_b & (e^cY)_c - e^cY \end{pmatrix}$$

e^cY we will denote by Y. A system of 9 equations follows;

$$\begin{pmatrix} \frac{1}{2}(Y_a - X_c) & \frac{1}{2}Y_b - X_a - bX_c & \frac{1}{2}(Y_c - Y) - X_b - \alpha X_c \\ X_a + bY_a - \frac{1}{2}Z_c & X_b + bY_b - Z_a - bZ_c & X_c + b(Y_c - Y) - Z_b - \alpha Z_c \\ Z_a + \alpha Y_a - \frac{1}{2}(Y_c - Y) & Z_b + \alpha Y_b - Y_a - b(Y_c - Y) & Z_c - Y_b \end{pmatrix} = 0$$

Elements $M_{1,1}$ and $M_{3,3}$ lead to a parametrization $X = R_a$, $Y = R_c$, $Z = R_b + f(a,b)$. Elements $M_{2,1}$ and $M_{1,2}$ both lead to equation $(R_a + bR_c)_a = \frac{1}{2}R_{b,c}$. Element $M_{2,2}$ allows us to conclude that the function f depends only from one argument b. Elements $M_{3,1}$ and $M_{1,3}$ are the sane and lead to equation $(R_b + R_c)_a = \frac{1}{2}R_{c,c}$. And finally elements $M_{3,2}$ and

$M_{2,3}$ pass to a third equation in the form $(R_b + f(b) + R_c)_b = (R_a + bR_c)_c$. Thus we have three equations which it is necessary solve

$$(R_a + bR_c)_a = \frac{1}{2}R_{b,c}, \quad (R_b + \alpha R_c)_a = \frac{1}{2}R_{c,c}, \quad (R_b + \alpha R_c)_b = (R_a + bR_c)_c$$

For further calculations the variables b, $\alpha = \frac{1}{2}(a + b^2)$, c will be more suitable. In these variables the system equations above appears as

$$R_{\alpha,\alpha} = 2R_{b,c}, \quad (R_b + bR_\alpha + \alpha R_c)_\alpha = R_{c,c}, \quad (R_b + bR_\alpha + \alpha R_c)_b = \frac{1}{2}R_{\alpha,c}$$

These equations can be interpreted as the vanishing of the curl of some vector which means that vector by itself is gradient of some scalar function, or and that the same system of equations of the second order above can be rewritten as the system of equation of the first order for two unknown functions R, Q

$$R_c = Q_\alpha, \quad R_\alpha = 2Q_b, \quad R_b + bR_\alpha + \alpha R_c = Q_c$$

Eliminating Q we come back to equations for R. Eliminating R, we pass to a system of equations for Q in the form

$$(Q_c+Q)_c = (Q_b+bQ_\alpha+\alpha Q_c)_\alpha, \quad (Q_c+Q)_\alpha = 2(Q_b+bQ_\alpha+\alpha Q_c)_b, \quad Q_{\alpha,\alpha} = 2Q_{b,c} \tag{9}$$

The last is the usual Laplace equation in three dimensions, which is invariant with respect to transformations of the five dimensional rotation group. One is operators of which is exactly $D=\frac{\partial}{\partial b} + b\frac{\partial}{\partial \alpha} + \alpha\frac{\partial}{\partial c}$. The last equation is a consequence of the two first ones. From them it follows that $(DQ)_{\alpha,\alpha} = 2(DQ)_{b,c}$ and operator D commutes with the Laplace operator. Thus the last equation above is consequence of two first ones. Let us seek a solution the last system of equations in the form of Laplace-Fourier transform.

$$Q = \int dk_c dk_b dk_\alpha e^{\alpha k_\alpha + bk_b + ck_c} q(k_\alpha, k_b, k_c)$$

After substituting this into the system of equations above under the sign of integral we obtain two equations

$$(k_c^2+k_c-k_bk_\alpha)q = -k_\alpha(k_\alpha q_{k_b}+k_c q_{k_\alpha}), \quad \frac{1}{2}k_\alpha(k_c+1)-k_b^2 = -k_b(k_\alpha q_{k_b}+k_c q_{k_\alpha}) \tag{10}$$

Eliminating terms with derivatives we arrive at a condition of self consistency

$$(k_c + 1)(k_\alpha^2 - 2k_bk_c) = 0$$

The first possibility $k_c + 1 = 0$ we call the degenerate solution the second one non degenerate one.

6.1. Degenerate case

Let us seek solution Q of Laplace equation under additional condition DQ = 0. In this case simultaneously $Q_c + Q = f(b)$ or $Q = f(b) + e^{-c}\emptyset(b,\alpha)$. Now

$DQ=f_b+e^{-c}(\emptyset_b+b\emptyset_\alpha-\alpha\emptyset)=0$. The solution of the last equation is $\emptyset=e^{(\frac{-3}{b^3}+b\alpha)}F(\frac{-b^2}{2}+\alpha)$ and finally the degenerate solution of the problem is $Q=e^{-c}e^{(\frac{-b^3}{3}+b\alpha)}F(\frac{-b^2}{2}+\alpha)$. But in this case the third equation is no more a consequence of two first ones and it must be satisfied independently. We have

$$F_{\alpha,\alpha} + 2bF_\alpha + b^2F = -2(-b^2+\alpha)F - bF_\alpha, \quad F_{x,x} = -2xF, \quad x \equiv \frac{-b^2}{2} + \alpha$$

This ordinary differential equation may be solved in terms of Bessel functions. Coming back to the function R we obtain

$$-R = e^{-c}e^{(\frac{b^3}{6}+\frac{ba}{2})}(bF(\frac{a}{2}) + \dot{F})$$

By direct calculations it is simple to check that all equations for R above are satisfied. A solution of the continuous Toda chain $u = e^{-c}$ in implicit form is determined from the equations

$$y = R_c = ue^{(\frac{b^3}{6}+\frac{ba}{2})}bF(\frac{a}{2}+\dot{F}), \quad x = -ue^{(\frac{b^3}{6}+\frac{ba}{2})}(\frac{b^2-a}{2})F(\frac{a}{2})+b\dot{F}),$$

$$z = R_b = ue^{(\frac{b^3}{6}+\frac{ba}{2})}(\frac{b^3+ab}{4}F(\frac{a}{2}) + \frac{1}{2}F(\frac{a}{2}) + \frac{b^2+a}{4}\dot{F})$$

6.2. Non degenerate case

In this case $k_\alpha^2 = 2k_bk_c$ or $k_c = \frac{k_\alpha^2}{2k_b}$. The first equation of (10) is rewritten in the form

$$k_c^2 + k_c - k_bk_\alpha = -k_c(2k_b(\ln q)_{k_b} + k_\alpha(\ln q)_{k_\alpha})$$

The solution the last equation of the first order is

$$q = (k_\alpha)^{k_c+1}e^{\frac{(k_\alpha)^3}{6k_c^2}}F(k_c)$$

where F is a scalar function of its argument. Finally we obtain

$$Q = \int k_b dk_b \int dk_c (2k_bk_c)^{\frac{k_c}{2}} (e^{(2k_b)^{\frac{\frac{3}{2}}{3k_c^{\frac{1}{2}}}}} F_1(k_c) + e^{-\frac{(2k_b)^{\frac{3}{2}}}{3k_c^{\frac{1}{2}}}} F_2(k_c)) \tag{11}$$

The first equation gives $R = Q_\alpha$, $W = Q_c$. Substituting into both other equations we pass to a system of two equations

$$Q_{\alpha,\alpha} = 2Q_{b,c}, \quad Q_{\alpha,b} + bQ_{\alpha,\alpha} + \alpha Q_{\alpha,c} = Q_{c,c}$$

Let us seek a solution of this linear system above by a Laplace-Fourier trans-form

$$Q = \int dkdpe^{k\alpha+pb+\frac{k^2}{2p}c}f(p,k)$$

The first equation is satisfied automatically. The second one leads to a differential equation of the first order in partial derivatives for the determination of the function f(k, p) under the integral sign, which for the function $F \equiv k^3 f$ is

$$2p\frac{\partial F}{\partial p} + k\frac{\partial F}{\partial k} = \left(\frac{2p^2}{k} - \frac{k^2}{2p}\right)F$$

with the obvious solution

$$F = k^3 f = e^{\frac{p^2}{3k}+\frac{1}{2}\ln k\frac{k^2}{p}}\phi\left(\frac{k^2}{p}\right) \tag{12}$$

where $\emptyset$ is an arbitrary function of the argument $\frac{k^2}{p}$.

7. Conclusions

We have presented a new idea, unknown up to now to the best of our knowledge in the theory of integrable systems connected with the symmetry equation of the initial system. We have also presented some non-trivial solutions of the 2 + 1 continuous Toda chain. These solutions are often only given in implicit form. An explicit solution is however obtained as a specific example. It may be that this is the best that can be hoped for, i.e. that in the general case the solution is only obtainable in implicit form. This is the case for many non-linear systems, including that of the simple non-linear wave (Monge) equation. The method of solution presented here is far from obvious or straightforward. We hope that a more direct method of finding a solution can be found; now we know that a solution is possible

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