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The Fuzzy Prenucleolus II. Direct Existence Proofs, the Weakly Coalitional Monotonicity and a Characterization

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Abstract We give direct proofs of existence for the prenucleolus of fuzzy game for three different classes: strongly bounded; with lattice politopic sets of preimputations; and for the case when the game to solutions mapping possesses with continuous and monotonic selection functions. Generalized Zhou's result on the weak coalitional monotonicity of fuzzy prenucleolies. In the last part we characterize fuzzy prenucleolies by four properties.

Keywords Fuzzy prenucleolus, Fuzzy prenucleoli, Strongly bounded games, Lattice politopes

1. Introduction

The optimality principle for TU games, which in game theory known as nucleolus has been introduced by D. Schmeidler (1969). Since then it has attracted many researchers working in field of cooperative games. E. Kohlberg (1971) has described new properties of the concept, by Megiddo N. (1974) proved its nonmonotonicity, L. Zhou (1991) had a paper about weak coalitional monotonicity (WCM) of nucleolus. Sobolev A. (1976), Potters J. (1991), Sniders C. (1995), Voorneveld M, Nouveland A. (1998), Orshan and Sudholter P. (2003), have works that characterize the decision rule through various sets of properties.

After Aubin J.-P. (1981) introduced the fuzzy concept in game theory extension of existing in the classical theory decision rules to fuzzy cooperative games has become matter of principal importance. Core has been generalized by Aubin (1981) and Shapley value by Tsurumi et. al. (2001). Maroutian Y. (2017) has extended to fuzzy games classical nucleolus and in Maroutian Y. (2019) in a setting that varies of what described in Tsurumi et. al. (2001) generalized for the fuzzy case Shapley value.

Section 2 devoted to preliminaries that are known from the classical theory and the background material. The latter mostly refers to inductively defined sets [X_p , T_p]. That inductive process after finite number of steps provides with prenucleolus. There are direct existence proofs of prenucleolus in Section 3. For three different classes of games: strongly bounded games, games with lattice polytopic sets of preimputations and for belonging to same class games but possess with certain continuous selection functions we give direct existence proofs for the prenucleolus. In section 4 we extend to fuzzy games result of L. Zhou on weak coalitional monotonicity of prenucleolus (WCM).

Section 5 characterizes prenucleolus through four properties: Nonemptiness, WCM, Consistency and Converse Consistency.

2. Preliminaries and Background

Nucleolus of a classical cooperative game (N, ṽ) is an imputation that is the best in sense of some preference relation $\succ_{\tilde{v}}$. Let for game G= < N, ṽ >

$$Y(\tilde{v})=\{x\in R^N/\ x_i \ge \tilde{v}\,(i),\ \text{for } i\epsilon N,\ \textstyle\sum_{i=1}^{n} x_i = \tilde{v}(N)\},$$

is the set of all imputations for game G.

By magnitude e(s,x) = ṽ(s)-x(s) denotes excess of coalition S from vector x. Vector of excesses θ(x,ṽ) = (θ_1(x,ṽ), …, θ_{2^n} (x,ṽ)) defined with components

$$\theta_m(x,\tilde{v})= max_{\{U\subset 2^N, |U|=m\}}\, min_{s\subset U}\ (\tilde{v}(s) - x(s)).$$

Clear that components of θ(x,ṽ) ordered decreasingly. On set R^N defined quasi-order $\succ_{\tilde{v}}$ following way. For x,yϵR^n x$\preccurlyeq_{\tilde{v}}$y if θ(x,ṽ) $\preccurlyeq_L$ θ(y,ṽ), where $\preccurlyeq_L$ is for lexicographic order. That means exists a number l such that

$$\theta_k(x,\tilde{v}) = \theta_k(y,\tilde{v}),\ \text{for } k \le l - 1 \text{ and } \theta_l(x, \tilde{v}) < \theta_l(y,\tilde{v}).$$

Definition. For Y⊂ $R^{|N|}$ and characteristic function ṽ the set ν(Y)⊂ $R^{|N|}$ is nucleolus for Y if vectors from ν(Y) are minimal in sense of relation $\preccurlyeq_{\tilde{v}}$. i.e.

$$\nu(\tilde{v}) = \{x\epsilon X(\tilde{V})/x \preccurlyeq_{\tilde{v}} y\ for\ every\ y\epsilon X\}$$

Theorem (Schmeidler(1969)) For every nonempty, convex and compact set the nucleolus exists and consists of only one vector.

Theorem (Sobolev(1975)). Let for a game **G**=<N,ṽ> payoff vectors are preimputations from set X(ṽ) = $\{x\epsilon R^{|N|} | \Sigma_{i\epsilon N}\, x_i = \tilde{v}(N)\}$.

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Then game G has a nonempty prenucleolus, which consists of only one vector.

Due to violation of individual rationality for outcomes from X(ṽ), that set no more is compact and hence it is different of the set Y(ṽ). Despite that, the statement on existence and uniqueness of prenucleolus keeps remaining true.

For a fuzzy game (T,ν) where T$\subset [0,1]^n$ is the set of fuzzy coalitions let v: T→ R^N is the characteristic function of that game. Set of preimputations is the set of vectors that satisfy to condition of efficiency:

$$X(\nu) = \{x\epsilon R^N / \sum_{ieN} x_i = \nu(1)\}$$

We inductively define sets X_k, T_k by accepting that X_0= X, T_0=∅ .

For k=0, *l*, …, p we define sets X_{k+1} the following way:

$X_{k+1} = argmin_{x\epsilon X_k} sup_{\tau\epsilon T_k}[(e(\tau,x)-e_0)/ \rho(\tau,T_k)]$ and sets T_k:

$T_k = \{\tau\epsilon T/<x,\tau>=<y,\tau> for\ every\ x,y\epsilon X_k\}$,

where $e(\tau,x) = v(\tau) - <x,\tau>$, $e_0 = min_x\ max_\tau\ e(\tau, x)$ and $\rho(\tau,T_k)$ is the distance between point τ and set T_k:

$$\rho(\tau, T_k) = inf_{\tau' eT}\ \rho(\tau,\tau'), \text{ where}$$

$$\rho(\tau,\tau') = max_i|\tau_i - \tau'_i|.$$

When k increases sets T_k do not decrease: $T_{k+1} \supseteq T_k$.

If for some k_0 it turns out that $T_{k_0+1} = T_{k_0}$, then that in its turn entails stabilization of corresponding set X_{k_0}, or otherwise, with increasing of k X_{k_0} does not decrease any more. That kind of stabilization of sets X_k after finite number of minimization steps can happen in case of games with no piece-wice affine characteristic functions.

Vectors from mentioned kind of sets X_{k_0} we call prenucleolies in different of unique vectors that are prenucleoluses.

3. Direct Existence Proofs

3.1. The Strongly Bounded Games

Definition 3.1.1 A game (T, ν) is strongly bounded if there is a number M>0 such that for every vector xϵX and coalition $\tau\epsilon$T inner product |<x,τ>|<M. It is clear that for a strongly bounded game set of division vectors X$\subset R^n$ is compact and convex.

Definition 3.1.2 (T, ν) is a piece-wice linear characteristic function game if there is a collection of simplexes $\{\Sigma^k\}_{k=1,\dots,q}$, such that for each i, j if i≠j, $\Sigma^i \cap \Sigma^j = \emptyset$, $\cup_{i=1}^q \Sigma^i$= T, as well for $\tau\epsilon\Sigma^k$, where k=1,…,q, $\nu(\tau) = u^k(\tau)$ and $u^k(\tau)$ is a linear function.

The following is a direct proof of existence of fuzzy prenucleolus for strongly bounded piece-wice linear characteristic function games.

Theorem 3.1.1. A strongly bounded piece-wice linear characteristic function game possesses with a unique prenucleolus.

Proof. Let the following equation holds true on some simplex Σ^j:

$$min_{x\epsilon X}\ max_{\tau\epsilon T}\ (\nu(\tau) - <x,\tau>)$$
$$= min_{x\epsilon X}\ max_{\tau\epsilon\Sigma^j}\ (\nu(\tau) - <x,\tau>).$$

Below we deal with minimization problem (MP):

$$\min\ \varepsilon$$
$$u^j(\tau) - <x,\tau> < \varepsilon \text{ where } \tau\epsilon\Sigma^j, x\epsilon X,$$

and X= {x$\epsilon R^n/\sum_1^n x_i = \nu(1)$}.

This problem has a solution because set X is a polytope. Solution of that minimization problem is the set:

$$Y^j = argmin_{x\epsilon X}\ max_{\tau\epsilon\Sigma^j}\ (u^j(\tau) - <x,\tau>)$$

Y^j is a polytope because $Y^j \subset$X. It also is convex and bounded. From there sets Y^k, where k = *l*,…, q are compact and convex. For xϵX and $\tau\epsilon\Sigma^k$ inner product <x,τ> is a linear function of τ. Hence, instead of $u^k(\tau)$ we can deal with inner product <z,τ>, where zϵX is some vector. Expression $max_{\Sigma^k}(u^k(\tau) - <x,\tau>)$ reaches at some coalition $\tau\epsilon\Sigma^k$ for every k. It is well known that $min_X\ max_{\Sigma^k}\ (u^k(\tau) - <x, \tau>)$ reaches only on extX.

From there, as far as in the problem for k = j Y^j is the set of points of minimum for expression $max_{\Sigma^j}(u^j(\tau) - <x, \tau>)$, hence $Y^j \subset$ ext X. Because set of preimputations X is a polytope, so for a bounded game it also bounded, closed and convex. So is set ext X of X. The solution of MP can be represented by a mapping ν: ext X → X. From ν: ext X → $Y^k \subset$ ext X, follows that

ν: ext X → ext X and for each h ϵ ext X ν(h) is a nonempty and convex set. So, according to Kakutani's theorem mapping ν possesses with a fixed point, i.e. with an H*ϵ ext X such that H* ϵ ν(H*) and ν(H*) is a subset of set Y^j: ν(H*) $\subset Y^j = argmin_X\ max_{\Sigma^j}\ e_{u^j}(\tau, x)$. From the written right now it follows that for xϵX $e_{u^j}(\tau,H^*) \leq e_{u^j}(\tau,x)$.

To conclude the proof remains to consider the case, when $x\epsilon Y^k\ and\ k \neq j$.

Started from some number m $T_m \supset \Sigma^j$ and as well $X_{m-1} \subset$ X for arbitrary m. Holds true the inclusion: $argmin_{x\epsilon X_{m-1}} max_{\tau\notin T_{m-1}} e_{u^j}(\tau,x) \subset argmin_{x\epsilon X}\ max_{\tau\epsilon\Sigma^j}\ e_{uj}(\tau,x) = Y^j$. Denote: A= $argmin_{x\epsilon X_k}\ max_{\tau\epsilon\Sigma^k}\ e_{u^k}(\tau, x)$ and B= $argmin_{x\epsilon X}\ max_{\tau\epsilon\Sigma^j}\ e_{u^j}(\tau, x)$, where $k \neq j$ is an arbitrary number.

Accept we have that H*ϵB and HϵA. Due to inclusion above H*$\succ_{(X_m)}$ H. From there, $max_{\tau\epsilon\Sigma^j}\ e_{u^j}(\tau,H^*) < min_{x\epsilon X_k} max_{\tau\epsilon\Sigma^k}\ e_{u^k}(\tau,x)$. The latter one completes the proof that H* is the prenucleolus of game (T,ν).

3.2. Games with Lattice Polytopic sets of Preimputations

In case of strongly bounded games set of preimputations to deal with obtains as a result of excluding from the initial set X big chunks of preimputations. This is a restrictive measure, which can cause fair objections. Dealing with games that have lattice polytopic sets of preimputations is a way for evading of blame.

Definitions. For vectors x, yϵX, x preferable of y in sense of preference $\succ_{(X_k)}$ or x$\succ_{(X_k)}$y, if there is a set X_m such that x, yϵX_m and for some k>m, $x\epsilon X_k$ but $y \notin Y_k$.

All of the followed operations of ordering defined in sense of $\succ_{(X_k)}$- preference.

For A⊂X V_X A ($\wedge_X$ A) denotes least of upper bounds (biggest of lower bounds) of A in X. With respect to preference $\succ_{(X_k)}$ X is a partially ordered set (POS).

A POS X is a join (meet) lattice if for every A⊂X $\vee_X A\epsilon X$ ($\wedge_X A\epsilon X$).

At this point we require that for our polytope X it's set of peaks has been a join lattice in sense of preference $\succ_{(X_k)}$.

Theorem 3.2.1. A piece-wice linear characteristic function game (T, ν) with a lattice polytope set of preimputations possesses with a unique prenucleolus.

Proof. Let for a linear game u^k on Σ^k Y^k is the set of solutions of MP that corresponds to u^k, i.e.

$$Y^k = argmin_{x\epsilon X}\ max_{\tau\epsilon\Sigma^k}\ (u^k(\tau) - < x,\tau >)$$

and ψ^k: $u^k \rightarrow Y^k$ is a mapping.

In difference of strongly bounded games sets Y^k here not bounded and hence they are only closed and convex. Mapping ψ^k: $u^k \rightarrow Y^k$ for games from this class can as well be represented as a mapping v: ext X →X. We have proved in the previous theorem that v: extX $\rightarrow Y^k \subset$ extX, which means that v(extX) is a set of extreme points itself. Farther, we will deal with the least upper bound of set v(extX) in sense of preference $\succ_{(X_k)}$. We need to show that exists $\underline{x}$ = $V_X v$(extX). For the latter recall first what we have written related to mapping $v\ i.e$

v: extX $\rightarrow Y^k \subset$ extX. Then, $\underline{x}$ = $V_X v$(extX) which is l.u.b. of set v(extX) is sense of $\succ_{(X_k)}$ -preference also is l.u.b. for some set Y^k. Because for each Y^k there is only one upper bound that belongs to Y^k, hence due to closedness of Y^k and linearity of excess as a function of x, $\underline{x}\epsilon Y^k \subset X$.

Again for the reason that every $u_k(\tau)$ is a linear function, there is a $z^k\epsilon$X such that $u_k(\tau)$=<z^k, τ>. So for some numbers k_1 ,…, k_l where k_1 < k_2 … < k_l < m in representations of above linear functions for i=1,…,ℓ, each one of z^{k_i} is a l.u.b. for set $\psi^{k_i}(u^{k_i})$. The same $\{z^{k_i}\}_{i=1,\dots,l}$, are as well fixed points for mappings $\psi^{k_i}(u^{k_i})$. Because $\varphi(\nu)=\cup_{i=1}^{n}\ \psi^{k_i}(u^{k_i})$, so from there $\varepsilon(\varphi) = \{z^{k_i}\ /\ z^{k_i}\epsilon\ \varepsilon(\psi^{k_i})\}_{i=1,\dots,l}$. Prenucleolus is the only vector that is biggest in sense of $\succ_{(X_k)}$ preference, or which is the same as it is vector z^j that minimizes expression $min_{z\epsilon X}\ max_{\tau\epsilon T}\ e_\nu(z\ ,\ \tau) = max_{\tau\epsilon\Sigma^j} e_\nu(z^j\ ,\ \tau)$. At the same time for some p $z^j\epsilon X^p$, or $X^p \ni$ max $_{k=1,\dots,m}$ z^k. To resume, vector z^j, which is the prenucleolus of (T, ν) is the least of upper bounds for lattice polytope X and hence minimizes the expression with $min_{z\epsilon X}\ max_{\tau\epsilon T}\ e_\nu(\tau$, z) above.

3.3. Mapping φ Possesses with Continuous Selection Functions

Below we prove existence of prenucleolus in case when mapping φ: X $\rightarrow \cup_{k=1}^{m}\ Y^k$ possesses with continuous selection functions. We assume as well that there are both kinds of them there: monotone increasing and decreasing.

Definition 3.3.1 A mapping $\boldsymbol{\varphi}$: X → X decreases (increases) if for $\boldsymbol{a}$, $\boldsymbol{b\epsilon}$ X and $\boldsymbol{a} \preccurlyeq_{(X_k)}$ b, from x$\boldsymbol{\epsilon\varphi}$(a), y$\boldsymbol{\epsilon\varphi(b)}$ follows that x $\succ_{(X_k)}$y (x $\preccurlyeq_{(X_k)}$y).

Definition 3.3.2. A selection function for a multivalued mapping $\boldsymbol{\varphi}$: X→X is a function

f: X→ $\boldsymbol{\varphi}$(x) such that for every x$\boldsymbol{\epsilon}$X f(x)$\boldsymbol{\epsilon\varphi}$(x).

For the needs of this part we assume that the set of preimputations X endowed with sup norm topology. It is not worth a lot that in the definition of a game inherent a topological assumption regarding the payoff functions. We just need it to ensure the convergence of sequences $\{x_n\}$⊂X and continuity of selection functions for φ.

The existence proof of prenucleolus becomes pretty direct for mapping φ that possesses continuous, monotonic selection functions.

Theorem 3.3.1. Let (T, ν) is a piece-wice linear characteristic function game, with a lattice polytope set of preimputations X, and φ: X →X maps to (T, ν) solutions of it's MP's. Then (T, ν) possesses with a unique prenucleolus if there are monotonic increasing and decreasing continuous selection functions for φ.

Proof. First we prove existence of a fixed point for φ's selection function f and then show not emptiness of φ's set of fixed points: $\varepsilon(\varphi) \neq \emptyset$. After what we show that the obtained this way fixed point, which contained in $\varepsilon(\varphi)$ is greatest in sense of $\succ_{(X_k)}$.

As we have mentioned that earlier, φ(x) = $\cup_{k=1}^{m}\ \ \psi^k$(x), where ψ^k(x) is the set of solutions of MP that corresponds to the game with simplex of fuzzy coalitions Σ^k. Farther for k=1,…,m ψ_k(x) = Y^k and Y^k ⊂extX.

We separate two cases.

Case 1. Let $\underline{x}$ = $\bar{V}_X$ φ(extX) and $\underline{x}\notin\varepsilon(\varphi)$. As least upper bound of X, $\underline{x}\epsilon$X.

The monotone increase of argument x we consider in sense of preference $\succ_{(X_k)}$.

Let for selection function f assume that it is continuous, monotonic increases and $\{x_n\}$ is a sequence of f -iterates from $\underline{x}$. That means x_0 = $\underline{x}$ and x_n= f(x_{n-1}). From there, $\{x_n\}$⊂X is a monotone increasing in sense of $\succ_{(X_k)}$-preference sequence. Since X is a lattice politope, so there is an upper bound of X, which bounds $\{x_n\}$, and from where $\{x_n\}$ converges to a point ē$\epsilon\varphi$(x). Because x_{2n+1} = f(x_{2n}), so due to continuity of f converge both of the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$, which means that ē$\epsilon\varphi$(x) is a fixed point for f, i.e. ē = f(ē). By the other side, because f: x → φ(x), hence f(ē)$\epsilon\varphi(\bar{e})$. That in its turn means, $\bar{e}\epsilon\varphi$(ē), i.e. the same ē as well is a fixed point for φ and set $\varepsilon(\varphi)$ = {x: x∈ φ(x)} is not empty.

Let now show that $\bar{e}$ is the biggest fixed point in sense of preference $\succ_{(X_k)}$. Suppose that $e\epsilon\varepsilon(\varphi)$ and $e \neq \bar{e}$. It is clear that $\underline{x} \geq e$ in the sense of $\succ_{(X_k)}$- preference.

Assume that f' is a monotone decreasing selection function for φ. Sequence of f' - iterates of $\underline{x}$ $\{x'_n\}$ is $\succ_{(x_k)}$-monotone decreasing. From there, for some number n, $x'_n \succ_{(X_k)}$e and for some other number n_k, n_k > n x'_{n_k+1}= f' (x'_{n_k}) $\preccurlyeq_{(X_k)}$ f'(e) = e, which again holds true due to continuity of f '. By induction we obtain that $\bar{e}\ _{(x_k)}\ e$. Hence, in this case (T, ν)'s prenucleolus is fixed point ē.

Case 2. $\underline{x}$ = $\overline{V_X}$ φ(extX) and $\underline{x}\epsilon\varepsilon(\varphi)$. In this case $\underline{x}$ is the biggest point in $\varepsilon(\varphi)$ that can minimize the maximal excess. From there it is the prenucleolus of (T, ν).

An example of a monotonic selection function.

Example 3.3.1 Let φ: X → X is a monotone decreasing mapping and for $a\epsilon$X Z_a be the following set: Z_a= {zϵX / ∃k such that a, zϵX_k and $a \nprec_{(X_k)}$z}

$max_{(\succ_{(X_k)})z\epsilon z_a}$ z, when $a \succ_{(X_k)} min_{(\succ_{(X_k)})x\epsilon\varphi(a)} x$, g(a)$\epsilon\varphi(a)$

g(a) =

$min_{(\succ_{(X_k)})x\epsilon\varphi(a)} x, when\ a \precsim_{(X_k)} min_{(\succ_{(X_k)})x\epsilon\varphi(a)}x$

For $a\epsilon\varphi(a)$ g(a) = $max_{(\succ_{(X_k)})z\epsilon z_a} z$.

Defined this way function g(.) is a decreasing selection function for φ.

Remark 3.3.1. In the example above if to replace the condition of monotonic decreasingness for mapping φ by the same kind of increasingness then function g(.) will become monotone increasing.

4. The Weak Monotonicity Property of Fuzzy Prenucleolus

Let (T, ν) and (T, w) are two fuzzy games and $\boldsymbol{B^z}$ is a set of fuzzy coalitions such that

$$B^z = \{\tau\epsilon T\ /\ w(\tau) - \nu(\tau) = \alpha > 0\},$$
$$\text{and for } \tau'\notin B^z\ w(\tau') = \nu(\tau')\}.$$

Denote by $\nu u(\nu)$ and $\nu u(w)$ the sets of prenucleolies for games (T, ν) and (T, w) respectively.

Definition. The fuzzy prenucleoli is weakly coalitionally monotonic if < τ, x > < < τ, y > for x$\epsilon\nu u(\nu)$, y$\epsilon\nu u(w)$ and $\tau\epsilon B^z$.

The theorem below extends L. Zhu's (1991) property of weak coalitional monotonicity for the classical prenucleolus on prenucleolies for fuzzy cooperative games.

Theorem 4.1 For any fuzzy game its prenucleolies when they exist possess the property of weak coalitional monotonicity.

Proof. Let construct $\{T_k{}^\nu\ ,X_k{}^\nu\}_{k=1,\dots,m}$ for game (T, ν) be such that it results to prenucleoli $\nu u(\nu)$. Then there are sets $\left\{T_{i_j}{}^\nu\right\}_{j=1,\dots,l} \subset \{T_k{}^\nu\}_{k=1,\dots,m}$ such that $B^z \subset T_{i_1}{}^\nu$, ∪ … ∪ $T_{i_l}{}^\nu$.

Depending on if coalitions $\tau\epsilon$T belong to set B^z or not one can describe the containing them sets from these constructs as well their mutual locations in $\{T_k{}^\nu, X_k{}^\nu\}$ and $\{T_i{}^w, X_i{}^w\}$. If

$\tau \notin B^z$ and for a vector xϵX sets, for example $X_p{}^\nu$ and $X_q{}^w$ contain x then they coincide, because the vectors containing in both of the sets minimize equal excesses.

Let now for a coalition $\tau\epsilon B^z$ discuss first it's mutual positions in constructs $\{T_k{}^\nu,\ X_k{}^\nu\}$ and $\{T_i{}^w,\ X_i{}^w\}$. It is clear that there is a number j such that $\tau\epsilon T^\nu{}_j$. Because $w(\tau) = \nu(\tau) + \alpha$, so the position of coalition τ in construct $\{T_i{}^w,\ X_i{}^w\}$ compared with the same in $\{T_k{}^\nu,\ X_k{}^\nu\}$ obtains by moving it up through the column of $\{T_i{}^w,\ X_i{}^w\}$'s to a certain set $T_l{}^w$, where l < j.

Set $X_l{}^w$ that corresponds to $T_l{}^w$ consists of vectors minimizing the excess of coalition τ in game w. Therefore if $\nu u(\nu)$ and $\nu u(w)$ are prenucleolies of games (T, ν) and (T, w) respectively and x$\epsilon\nu u(\nu)$, y$\epsilon\nu u(w)$ then for $\tau\epsilon B^z$ such that $\tau \notin T_{i_j}$

$$\frac{1}{\rho(\tau, T_{i_j} w)} (w(\tau) - < y,\ \tau > -e_0) < \frac{1}{\rho(\tau, T_{i_j} w)}$$
$$(w(\tau) - < x,\ \tau > -\ e_0),$$

From there it follows that < τ, x > < < τ, y >.

Remark 4.1 The proved property remains true as well in the case when a game possesses with prenucleolus.

Remark 4.2 When set B^z contains as well classical coalitions then inequality obtained

above this time refers also to coalitions τ^s = (1, …, 1). For SϵB^z we receive Zhou's classical property of weak coalitional monotonicity i.e.,

$$\sum_{i\epsilon s} \nu_i(w) > \sum_{i\epsilon s} \nu_i(\nu),$$

when $\nu(\nu)$ and $\nu(w)$ instead are prenucleoluses and v,w are piece-wice affine games.

5. A Characterization of Fuzzy Prenucleolies

5.1. Some Preliminaries

For S⊂N we denote by F^s the set of all fuzzy coalitions that belong to cube T^s.

To a S⊂N corresponds fuzzy coalition e^s(N)ϵF^N , which is the vector $e^s{}_i$(N)= 1 if iϵS and $e^s{}_i$(N)= 0 if iϵN \ S. Fuzzy coalition 0^N corresponds to the empty player coalition. We denote by FG^{TU} the class of all fuzzy games (T, ν) that possess with prenucleolies.

For a game (T, ν)ϵFG^{TU} a payoff vector is a function x: N → R^N. x is efficient ($e$ff) if

$\sum_{i\epsilon N} x_i = \nu(e^N)$. The set of preimputations of (T, ν) is: X(ν) = {xϵR^N/ x is $e$ff}.

Restriction of payoff vector x to S is vector $x_s\epsilon R^s$ and for $\tau\epsilon$T < x,τ > denotes inner product of vectors x and τ: < x,τ > = $\sum_{i\epsilon N} x_i\tau_i$.

5.2. The Properties

Below are the properties by what we are going to characterize fuzzy prenucleolies.

P1. Non-emptiness (NEmpt.)

P2. Weak coalitional monotonicity (WCM)

P3. Consistency (Cons.)

P4. Converse Consistency (ConCons)

A solution on FG^{TU} is a mapping σ which associates with each game (T, ν)ϵFG^{TU} a set

$\sigma(\nu)$⊂X(ν) that satisfies to properties P1-P4. The domain for solution σ is the set of fuzzy games FG^{TU}.

As that proved in Maroutian Y. (2017) the minimization process by solving of linear programming problems (LPP) after finite number of steps provides with a single solution, which is the prenucleolus for piece-wice affine characteristic function games.

For not piece-wice affine characteristic function games started from some step k solution of the corresponding MP may not decrease any more, i.e. may be $X_k = X_{k+1}$ for following problem. Vectors that belong to stabilized this way sets X_k we call prenucleoli.

We will use Y - A Hwang's (2007) extension of classical reduced game to games from FG^{TU} that has been introduced by Davis and Mashler (1965).

Definition. Let (T, ν)ϵFG^{TU}, xϵR^N and S⊂N, S ≠ 0. Reduced game with respect to S and x is (S, ν_{s_x}), where

$$\nu_{S,x}(\alpha) = \begin{cases} 0 & \text{if } x = 0^s \\ \nu(e^N) - \sum_{ieN\setminus S} x_i & \text{if } x = e^S(S) \\ \sup\{\ \nu(\alpha,\ \beta) - \sum_{i\epsilon N\setminus S} \beta_i x_i /\ \beta\epsilon F^{N\setminus S}\ \} & \end{cases}$$

Below are definitions of properties P2-P4.

- Consistency (cons.) If (T, ν)ϵFG^{TU}, S⊂N, S ≠∅ and xϵб(ν), then (S, $\nu_{S,x}$)ϵFG^{TU}, $x_S\epsilon$б(S, $\nu_{S,x}$).
- Converse consistency (CCons.) If (T, ν)ϵFG^{TU}, S⊂N, 0 < $|S|$ < N, (S, $\nu_{S,x}$)ϵFG^{TU} and $x_s\epsilon$б(S, $\nu_{S,x}$) then xϵб(ν).
- Weak coalitional monotonicity (WCM).

Let (T, ν) and (T, w)ϵFG^{TU}, B^z is a set of fuzzy coalitions defined in Part 4. If $\nu u(\nu)$ and νu(w) are prenucleolies of (T, ν) and (T, w) respectively, vectors x$\epsilon \nu u(\nu)$ and y$\epsilon \nu u$(w) then fuzzy prenucleolies $\nu u(\nu)$ and νu(w) are weakly coalitionally monotonic if < τ, x > < <τ, y>, when $\tau\epsilon B^z$.

As far as this property of fuzzy prenucleolies we have proved in the theorem of Part 4, so we will just refer to WCM without proving it again here.

5.3 The Results

Lemma 5.3.1 Let (T, ν)ϵFG^{TU}, x$\epsilon \nu u(\nu)$ and S⊂N, S ≠ ∅. For arbitrary yϵX if $\tau\epsilon$T is the coalition of maximal excess then holds true following inequality

$$\nu_{S,y}(\tau^s) - \sum_{i\epsilon s} y_i\tau_i > \nu_{S,x}(\tau^S) - \sum_{i\epsilon s} x_i\tau_i.$$

Proof. Let (T, ν), x$\epsilon \nu u(\nu)$, yϵX and $\tau\epsilon$T are the same as they are in lemma's formulation. Then $\nu(\tau) - \sum_{i\epsilon N} x_i\tau_i < \nu(\tau) - \sum_{i\epsilon N} y_i\tau_i$. We need to show that if S⊂N is an arbitrary coalition then

$$\nu_{S,x}(\tau^S) - \sum_{i\epsilon S} x_i\tau_i < \nu_{S,y}(\tau^S) - \sum_{i\epsilon S} y_i\tau_i.$$

Accept that for some S⊂N in contrary takes place the opposite inequality. By replacing in the inequality above first $\nu_{S,x}(\tau^S)$ by it's expression and then doing the same with $\nu_{S,y}(\tau^S)$, in case if takes place the opposite inequality then that would mean existence of a set X_k such that $x\epsilon X_k$ and at the same time for a k′, k′ > k $y\epsilon X_{k'}$ but x∉ $X_{k'}$. The latter one then would contradict to our assumption about vector x, i.e. x$\epsilon \nu u(\nu)$. With this contradiction our lemma proves.

Lemma 5.3.2. The prenucleolies satisfy consistency.

Proof let (T, ν)ϵFG^{TU}. For x$\epsilon\nu$u(ν) and S⊂N, S ≠ ∅ due to it's definition (S, $\nu_{S,x}$)ϵFG^{TU}. In reduced game (S, $\nu_{S,x}$) x_S is efficient because of xϵX and definition of game (S, $\nu_{S,x}$). From there it remains to show that for all αϵF^S, α ≠ e^S(S) if yϵX is an arbitrary vector then

$$\nu_{S,x}\ (\alpha^S) - \sum_{i\epsilon S} \alpha_i x_i < \nu_{S,y}\ (\alpha^S) - \Sigma_{i\epsilon S}\,\alpha_i y_i.$$

The latter one follows from lemma 5.3.1 and the inequality in its turn means that x_S is a prenucleoli for (S, $\nu_{S,x}$), i.e. $x_S\epsilon$б(S, $\nu_{S,x}$).

Lemma 5.3.3. prenucleolies satisfy to converse consistency.

Proof. Let (T, ν)ϵFG^{TU}, with $|N| \geq 2$, $\nu u(\nu)$ is prenucleoli of game (T,ν) and vector xϵX. We assume that for all coalitions S⊂N, 0 < $|S|$ < $|N|$ and for vector xϵX reduced game

(S, $\nu_{S,x}$)ϵFG^{TU}, $x_S\epsilon\nu u$(S, $\nu_{S,x}$), i.e. x_S is a prenucleoli of (S, $\nu_{S,x}$). We need to prove that x is a prenucleoli for (T, ν) as well, i.e x$\epsilon\nu u(\nu)$.

Let yϵX be an arbitrary vector and for R⊂S τ^R is the coalition of maximal excess in game

(S, $\nu_{S,x}$), Then by Lemma 5.3.1

$$\nu_{S,x}(\tau^R) - \sum_{j\epsilon R} \tau_j y_j > \nu_{S,x}(\tau^R) - \sum_{j\epsilon R} \tau_j x_j.$$

We need to show that for arbitrary $\tau\epsilon$T and y≠x

$$\nu(\tau) - < \text{y},\ \tau > \ < \nu(\tau) - < \text{x},\ \tau >.$$

Otherwice, if there is τ'ϵT such that

$$\nu(\tau') - < \text{y},\ \tau' > \ < \nu(\tau') - < \text{x},\ \tau' > \quad (5.3.1)$$

Then it would mean that for some coalition S⊂N, where S= { i_1, …, i_l}, y_{i_k} > x_{i_k} if k=1, …, l and hence for that coalition S in reduced game (S, $\nu_{S,x}$) x would not be it's prenucleoli. The latter one violates precondition on converse consistency of $\nu u(\nu)$. That means the inequality (5.3.1) is correct and x is a prenucleoli of game (T, ν).

Farther we will prove the uniqueness. It based on Elevator Lemma introduced by Thomson (2005), The variant of this lemma we use says that if a solution б is consistent and on the subdomain of all proper games of (T, ν) it is contained in a solution $\underline{б}$, which is conversely consistent then the inclusion б⊆ $\underline{б}$ always holds.

Theorem 5.1. A solution б on FG^{TU} satisfies NEmpt., WCM, Cons. and CCons. if and only if for all (T, ν)ϵFG^{TU} б(T, ν) = νu(T, ν).

Proof. That νu(T, ν) satisfies to NEmpt. follows from ν belongs to domain FG^{TU}, where every game possesses with a set of prenucleolies. For each one of the rest of properties a corresponding statement we have proved.

Assume that the solution б(T, ν) also in its turn satisfies to WCM, Cons, and CCons. Let (T, ν)ϵFG^{TU}. To prove uniqueness we will use the method of induction on number $|N|$. For $|N|$= 1, б(N, ν) = νu(N, ν). Suppose that б(T, ν) = νu(T, ν) if $|N|$ < k and k ≥ 2.

The case $|N|$ = k. Let xϵб(N, ν). Based on Con. of б, for all S⊂N, with 0 < $|S|$ < $|N|$,

$x_S\epsilon$б(S, $\nu_{S,x}$) = νu(S, $\nu_{S,x}$). By CCon. of the prenucleoli x$\epsilon\nu u$(T, ν). From there, б(T, ν)⊂ νu(T, ν). The inclusion νu(T, ν)⊂б(T, ν) one can show by starting from νu(T, ν) and come to б(T, ν) by applying similar proof. Hence, б(T, ν) = νu(T, ν).

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