

About Competition Between Firms: Equilibrium or Disruption?

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Abstract One resumes the notion of “monopolistic competition” or product differentiation. The model used is Bertrand competition, the demands being deduced from the consumers’ utilities. A consumer is represented by a point u_i in the cube $0 \leq u_i \leq 1$, u_i being the utility of the product i for him. The product differentiation is when the points representing the consumers are in the facets of the cube $u_i = 1$. One studies the mathematical properties of the equilibrium. Some of them correspond to the characteristics of product differentiation, which are: (1) all the consumers make a purchase. When the products are differentiated, the firms are innovative (since the preferred product of a consumer has a utility which is maximal). And innovation has been defined “struggle against no consumption” (Christensen) (2) each firm has its “garden”, consumers in the facet $u_i = 1$ for the firm E_i . A firm keeps the consumers in its “garden”, provided its price is not too high (3) given the existence of “gardens” the profits are sufficient. Product differentiation is symmetrical, and disruption is dissymmetrical: the disruptive firm has a “garden” with a utility higher than the utilities of the “gardens” of its competitors. One demonstrates that the profit of a disruptive firm increases. The goals of the paper are: - To set out a model which describes product differentiation and disruption; - To explore the possibilities of the model used (Bertrand competition, the demands being deduced from the consumers’ utilities). The author has already used this model to study a particular kind of merger: when the merger is profitable, the bought asset being closed down. It is a sign of saturated market. Even, it could be a criterium (for saturated markets) interesting for fintechs. Also, the model allows discriminating non differentiating innovation (when there is product differentiation and all the utilities increase) and differentiating innovation (disruption: only the utility of the product of the disruptive firm increases). One shows that differentiating innovation provides more profit, always, but non differentiating innovation could provide no more profit. Finally, the model is also useful to study the effects of cannibalization. In the paper, a tractable example is set out, allowing to answer this question: is it in the interest of a disruptive firm to buy and close down a competitor before disruption? If the competitor cannibalizes the product of the disruptive firm very much, it is better to buy and close down this competitor.

Keywords Innovation, Disruption, Bertrand competition, Cannibalization

1. Introduction

One starts from the notion of “monopolistic competition”. It was very popular among economists in the 30’s. It is a trade-off between the interests of the firms and consumers. Thanks to differentiated products the firms can choose high enough prices and their profits are sufficient. Concerning the consumers, they benefit from diverse products. More accurately, “monopolistic competition” is between “perfect differentiation of the products” and “price war”.

- “Perfect differentiation of the products” means separate markets. Each firm chooses the monopoly price. There is no effect on the customers of a firm, of the prices chosen by the other firms. It is favorable to firms, which

can choose enough high prices.

- “Price war” means that the products are not enough differentiated, and this makes each firm decrease its price to get customers. Finally, the prices are low and the firms make no profit. It has been described as the Bertrand paradox [1]. Of course, it is favorable to consumers.

“Monopolistic competition” means that the products are partially substitutable and enough differentiated. The important idea is: each firm has its “garden” (or captive customers), customers that the firm is sure to keep (even if there is a condition on the price, which should not be too high). At the equilibrium, each firm has chosen a not too high price, and benefits from its “garden”. Therefore, the profits are sufficient.

Unfortunately, the two models of “monopolistic competition”, the Chamberlain’s one and the Hotelling’s one, were ... erroneous [2]. It has been demonstrated later. When one applies the definition of Nash equilibrium very

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accurately, one demonstrates that the models were erroneous. Tirole, a French economist who was awarded the Nobel prize of economics, demonstrated that there was an error in the Chamberlain's model [2]. And researchers demonstrated that the Hotelling's model was erroneous [3].

In this article one proposes a new definition of "monopolistic competition". The method is Bertrand competition, the demands being deduced from the consumers 'utilities' [4]. The "garden" of a firm E_i is the consumers with a utility $u_i = 1$, if 1 is the maximal price. Each firm keeps the consumers in its garden, provided that its price is the lowest (for instance, if the prices are equal, each firm has for customers exactly the consumers in its garden). To simplify, one considers only three competitors and one supposes that the costs are equal to 0.

The plan of the paper is:

- First, one sets out the method which is used.
- Then one studies the mathematical properties of the "monopolistic competition". One demonstrates the existence of a Nash equilibrium, thanks to the Sperner's lemma. One shows that the characteristics of product differentiation are described by the model.
- One has to define "disruption". The disruption is when one of the firms increases the utility of the consumers in its "garden". For instance, E_1 is disruptive if the utility for the consumers in its garden increases from 1 to 1, 5, while $u_2 = 1$ and $u_3 = 1$. In the general case, when a firm E_i increases the utilities of the consumers u_i from u_i to $u_i + k$ ($k > 0$) its profit P_i increases. One demonstrates it. However, in the particular case of "monopolistic competition" there is a difficulty. One has to set out another demonstration.
- One presents two tractable examples.
- A goal of the paper being to explore the possibilities of the model used, one quotes some results already achieved. They concern the profitability of mergers of some kind, when the bought asset is closed down after the merger. The model can be used also to show that there are two kinds of innovation: non differentiating innovation and differentiating innovation. And it can be used to study the role of cannibalization.
- A tractable example is presented, showing the role of cannibalization. The question which is posed is the best choice for a firm which wants to disrupt: before disruption, it can buy and manage a competitor, or buy and close down it, or make no purchase.
- In the Conclusion one tries to explain why disruption is an attractive choice for entrepreneurs. Also, one considers the possible following of these works.

2. The Method

One uses the model of Bertrand competition the demands being deduced from the utilities of the consumers. This method has already been used by the author in several articles [4].

The utilities for the consumers of the three products sold (u_1, u_2, u_3) are in a cube $0 \leq u_i \leq 1$. For prices (p_1, p_2, p_3) ($0 \leq p_i \leq 1$) the demand D_i corresponds to a "pocket" where are the consumers preferring the product of E_i and buying it. For instance, for E_1 the conditions are:

$$\begin{aligned} - u_1 - p_1 &\geq u_2 - p_2 \\ - u_1 - p_1 &\geq u_3 - p_3 \\ - u_1 - p_1 &\geq 0 \end{aligned} \quad (1)$$

After, the formulas for the profits are obvious:

$$P_i(p_1, p_2, p_3) = p_i D_i(p_1, p_2, p_3)$$

There are several interesting mathematical relationships:

$$- \partial D_i / \partial p_i \leq 0, \partial D_i / \partial p_j \geq 0 \quad (2)$$

$$- \partial D_i / \partial p_j = \partial D_j / \partial p_i \quad (3)$$

It means that to calculate the variation of the consumers 'surplus between two points $M(p_{10}, p_{20}, p_{30})$ and $M'(p'_{10}, p'_{20}, p'_{30})$ one can integrate $dS = -\sum dp_i D_i$ alongside any path from M to M' .

$$- \partial D_1 / \partial p_1 + \partial D_1 / \partial p_2 + \partial D_1 / \partial p_3 \leq 0 \text{ etc.} \quad (4)$$

It is easily demonstrated using (1) and supposing an increase $dp_1 = dp_2 = dp_3 = dp \geq 0$. The demand D_1 decreases.

$$- \partial^2 D_i / \partial p_i \partial p_j + \partial^2 D_j / \partial p_i \partial p_j \leq 0 \quad (5)$$

To prove the existence of a Nash equilibrium one can use the Poincaré Miranda theorem.

The function $(\partial P_1 / \partial p_1, \partial P_2 / \partial p_2, \partial P_3 / \partial p_3)$ has positive values for the coordinate $\partial P_i / \partial p_i$ in the internal facet $u_i = 0$ and negative values in the external facet $u_i = 1$. That $\partial P_i / \partial p_i$ has a negative value if $u_i = 1$ is a consequence of the conditions C_1 and C_2 . It is explained later in the paper. Therefore, there is at least one point where the function is equal to 0: $\partial P_i / \partial p_i = 0$. It is a Nash equilibrium if the second order condition is fulfilled.

One has to make hypotheses. For instance:

C_1 : for any price p_i fixed, the two other prices varying, the prices are strategic complements.

C_2 : there is a unique (stable) equilibrium

C_3 : the second derivatives $\partial^2 P_i / \partial p_i \partial p_j$ are negative (or equal to 0)

The condition C_3 implies C_2 . So, one can choose (C_1, C_2) or (C_1, C_3) . The point where $\partial P_i / \partial p_i = 0$ is a Nash equilibrium, since $\partial^2 P_i / \partial p_i^2 \leq 0$ (condition C_1).

Notice that C_1 has for consequence that $\partial P_i / \partial p_i$ is negative if $u_i = 1$. Indeed, it is the consequence of the existence of a reaction function: when the two other prices are fixed, when p_i varies from 0 to 1, $\partial P_i / \partial p_i$ is positive at the start ($p_i = 0$) then decreases to 0 at the intersection with the reaction function and has a negative value when $p_i = 1$.

Also, the Jacobian which has for coefficients $a_{ij} = \partial^2 P_i / \partial p_i \partial p_j$ has a determinant which is negative. For instance, if the hypotheses are (C_1, C_2) one proves that the determinant is negative by adding the second and the third columns to the first one. One has: $\partial^2 P_1 / \partial p_1^2 + \partial^2 P_1 / \partial p_1 \partial p_2 + \partial^2 P_1 / \partial p_1 \partial p_3 \leq 0$ using (2), (4) and (5) and supposing $\partial^2 D_i / \partial p_i^2 \leq 0$ (a

sufficient condition to have $\partial^2 P_1 / \partial p_i^2 \leq 0$). The coefficients a_{ii} are negative and the a_{ij} are positive (because of C_1). The quantities $a_{ii} a_{jj} - a_{ij} a_{ji}$ are positive (because of C_2). The determinant which is negative is a sufficient condition for the equilibrium being stable (for the small variations of the p_i). All these conditions are fulfilled when the demands are linear. The functions D_i can be any function differentiable twice. One has to suppose that the Nash equilibrium is unique.

3. Definition of the “Monopolistic Competition”

Each firm E_i has its garden, the consumers in the facet $u_i = 1$. All the consumers are in a garden.

One sets out the particular mathematical properties of the “monopolistic competition”. The property P_2 is very interesting from an economic point of view: it states that an equal increase of the prices does not change the demands. It shows why the “monopolistic competition” is advantageous for the firms: if they increase prices all together, no firm loses customers. For instance, in the symmetrical case, the prices can be 1, the maximal price, each firm keeps its “garden” (demand 1/3) and each profit is 1/3.

Also, one demonstrates that there is a Nash equilibrium, thanks to the Sperner’s lemma. It is not necessary, since the Poincaré Miranda theorem could be used (the “monopolistic competition” is a particular case of Bertrand competition). But it is interesting in itself, and allows to handle the mathematical properties of the “monopolistic competition”. These mathematical properties are useful to make some demonstrations, later in the paper.

These mathematical properties are:

P_1 . For any (p_1, p_2, p_3) , all the consumers buy some product. This is obvious. Any consumer belongs to the garden of a firm E_i . Therefore $u_i - p_i = 1 - p_i \geq 0$. This consumer buys the product of E_i , or another, if the net utility is more. Therefore $D_1 + D_2 + D_3$ is constant. If the sum which is constant is 1: $D_1 + D_2 + D_3 = 1$.

P_2 . When (p_1, p_2, p_3) is changed in $(p_1 + k, p_2 + k, p_3 + k)$ the demands remain the same. Take the example of D_1 and consider (1). The two first inequalities remain valid. Now, if a consumer buying the product of E_1 belongs to the garden of E_1 , $u_1 - p_1 = 1 - p_1 \geq 0$ remains valid. This consumer remains in the garden of E_1 . If the consumer buying the product of E_1 belongs to the garden of E_2 $u_2 = 1$ and the first inequality $u_1 - p_1 \geq u_2 - p_2 = 1 - p_2 \geq 0$ shows that $u_1 - p_1 \geq 0$. It is the same if the consumer belongs to the garden of E_3 . Not only the demands but the first and second derivatives are constant when the prices increase of k . The demands are constant along a “ray” that is to say a segment parallel to the bisector $(0, 0, 0)$, $(1, 1, 1)$, linking an internal facet to an external facet. An example of ray is the segment $(\frac{1}{2}, 0, \frac{1}{2})$, $(1, \frac{1}{2}, 1)$ (figure 1). One can choose a “basis” on a ray, some point M_0 and all the points on the ray are represented by a parameter x :

$OM = OM_0 + X$, X being the vector (x, x, x) . For instance, one can choose the middle of the ray described before, as a basis: $M(\frac{3}{4}, \frac{1}{4}, \frac{3}{4})$. The parameter varies from $-\frac{1}{4}$ to $+\frac{1}{4}$. There is this mathematical relationship: $\partial D_1 / \partial p_1 + \partial D_1 / \partial p_2 + \partial D_1 / \partial p_3 = 0$ (and the same for $j = 2$ and $j = 3$).

P_3 . The profit varies in a linear way along a ray. If a “basis” is chosen on the ray the price p_i varies from $p_{i0} + x_1$ to $p_{i0} + x_2$. Therefore, the profit P_i is $P_i = (p_{i0} + x) D_i$. The demand D_i is constant. The profit varies in a linear way.

P_4 There is this mathematical relationship: $\partial^2 D_1 / \partial p_1^2 + \partial^2 D_1 / \partial p_1 \partial p_2 + \partial^2 D_1 / \partial p_1 \partial p_3 = 0$ (the same for $j = 2$ and $j = 3$) (6). Since D_1 is constant along a ray: $\partial D_1 / \partial p_1 + \partial D_1 / \partial p_2 + \partial D_1 / \partial p_3 = 0$. The relationship (6) is obtained by deriving in p_1 . Also, $\partial P_1 / \partial p_1 + \partial P_1 / \partial p_2 + \partial P_1 / \partial p_3 = D_1$, deriving in p_1 : $\partial^2 P_1 / \partial p_1^2 + \partial^2 P_1 / \partial p_1 \partial p_2 + \partial^2 P_1 / \partial p_1 \partial p_3 = \partial D_1 / \partial p_1 < 0$.

Now we can demonstrate the existence of Nash equilibrium using the Sperner’s lemma.

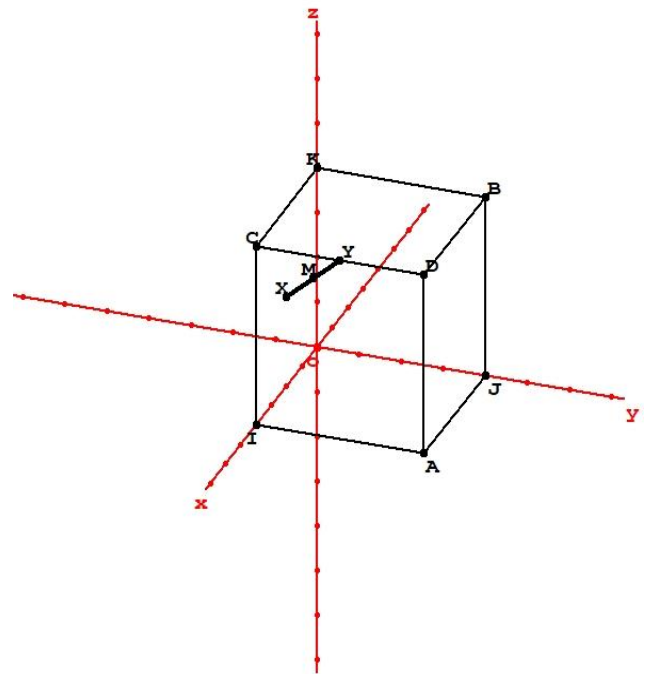


Figure 1. The cube $0 \leq u_i \leq 1$ (or $0 \leq p_i \leq 1$) is shown. A ray, linking X on the internal facet $u_2 = 0$ to Y on the external facet $u_1 = 1$ is shown. The middle M of the segment XY can be chosen as a basis

One starts from this remark: each ray corresponds to (D_1, D_2, D_3) with $D_1 + D_2 + D_3 = 1$ (property P_1). Consider an equilateral triangle of length of edge $2 / \sqrt{3}$: for any point M in the triangle (and on its frontier) the sum of the distances to edges is 1: $d_1 + d_2 + d_3 = 1$ (d_i is the distance to the edge which does not contain the vertex i). One ray is represented by a point M . To color the vertices of some triangulation one chooses colors 1, 2 or 3 corresponding to the vertices 1, 2 or 3: the color i is chosen if $\partial P_i / \partial p_i / \partial D_i / \partial p_i \leq \partial P_j / \partial p_j / \partial D_j / \partial p_j$ for any j different from i in a point M_0 chosen as a basis on the ray. Let us call q_i the quantity $\partial P_i / \partial p_i / \partial D_i / \partial p_i$. One checks:

- At the vertex i the color i is chosen. For instance, the

vertex 1 of the triangle corresponds to the ray / point (0, 1, 1). At this point $q_1 \leq q_2$ and $q_1 \leq q_3$ because $\partial P_1 / \partial p_1$ is positive and $\partial P_2 / \partial p_2$ and $\partial P_3 / \partial p_3$ are negative.

- A point of the edge 1 – 2, for instance, corresponds to a point on the edge of the cube (0, 1, 1) – (0, 0, 1) or the edge (0, 0, 1) – (1, 0, 1). The demand D_3 is equal to 0. On the first edge, for instance, $\partial P_1 / \partial p_1 \geq 0$ and $\partial P_3 / \partial p_3 \leq 0$. Therefore, $q_1 \leq q_3$. The color chosen is 1 or 2: it is 1 if $q_1 \leq q_2$ and it is 2 if $q_2 \leq q_3$. The conditions of the Sperner's lemma are fulfilled. When the triangulation becomes smaller, the center of a triangle with three colors on its vertices tends towards some point M_0 . At this point the q_i are equal.

After, it is straightforward. Along the ray of the point M_0 , in any point the $\partial P_i / \partial p_i$ are equal. One takes the point M_0 as a basis: $p_i = p_{i0} + x$.

$$\partial P_i / \partial p_i = D_i + p_i \partial D_i / \partial p_i$$

$$\partial P_i / \partial p_i (M) = D_i (M_0) + (p_{i0} + x) \partial D_i / \partial p_i (M_0) = \partial P_i / \partial p_i (M_0) + x \partial D_i / \partial p_i (M_0).$$

$$q_i (M) = q_i (M_0) + x.$$

For another value of i (2 or 3) the value is the same.

The quantity q_i (m) passes by the value 0 when M passes from the internal facet to the external facet. Suppose a ray linking the internal facet $u_2 = 0$ to the external facet $u_1 = 1$ (it is the example of the fig 1). On the internal facet $u_2 = 0$ the three equal quantities are negative ($\partial P_2 / \partial p_2 \geq 0$). On the external facet $u_1 = 1$, the three equal quantities are positive ($\partial P_1 / \partial p_1 \leq 0$). One applies the intermediate values theorem. At the point where the three equal quantities are equal to 0, the $\partial P_i / \partial p_i$ are equal to 0. It is a Nash equilibrium, if the condition of second order is fulfilled ($\partial^2 P_i / \partial p_i^2 \leq 0$).

Possibly the Nash equilibrium is not unique. However, it is unique in the symmetrical case. The point M where the $\partial P_i / \partial p_i$ are equal to 0 is necessarily on the diagonal (0, 0, 0), (1, 1, 1) because of symmetry. There is a single ray where the Nash equilibrium can be. And on a ray only one point can be a Nash equilibrium ($\partial P_i / \partial p_i$ varies in a linear way along a ray).

Also, the determinant of the Jacobian should be negative. When adding the second column and the third column to the first one obtains quantities like $\partial^2 P_1 / \partial p_1^2 + \partial^2 P_1 / \partial p_1 \partial p_2 + \partial^2 P_1 / \partial p_1 \partial p_3 = \partial D_1 / \partial p_1$ which are negative (Property P₄).

Some of the mathematical properties correspond to the characteristics of “monopolistic competition”, or product differentiation. First, all the consumers make a purchase (property P₁). Product differentiation means innovation, since the firms are able to trigger the maximal utility of their product, for any consumer. And innovation has been defined “struggle against no consumption” by Christensen¹ [6]. Also, each firm has its “garden”. And the existence of “gardens” guarantees that the profits are sufficient.

4. Defining Disruption

There is disruption when the gardens are not symmetrical. For instance, E_1 is disruptive if $u_1 = 1, 5$ in its garden while $u_2 = 1$ in the garden of E_2 and $u_3 = 1$ in the garden of E_3 .

One defines disruption:

E_1 (the firm which is disruptive) increases of k ($k > 0$) the utilities u_1 of the consumers. Therefore, the new demands are:

$$D'_1 = D_1 (p_1 - k, p_2, p_3)$$

$$D'_2 = D_2 (p_1 - k, p_2, p_3)$$

$$D'_3 = D_3 (p_1 - k, p_2, p_3).$$

And the new profits are:

$$P'_1 = p_1 D'_1$$

$$P'_2 = p_2 D'_2$$

$$P'_3 = p_3 D'_3$$

One demonstrates that (k being small): $dp_2 \leq 0$, $dp_3 \leq 0$, $dP_1 \geq 0$, $dP_2 \leq 0$, $dP_3 \leq 0$. The effect (of E_1 being a disruptive firm) is that the profit of E_1 increases, and the profits of E_2 and E_3 decrease. One needs the hypothesis: the determinant of the Jacobian $a_{ij} = \partial^2 P_i / \partial p_i \partial p_j$ (i : from 1 to 3, j : from 1 to 3) is negative. The conditions (C_1 , C_2) are sufficient but ... there is a hurdle.

If one considers the first tractable example presented in the paper later, one sees that the Jacobian is not defined at the equilibrium point ($p_1 = p_2 = p_3 = 1/2$). Indeed, it could be frequent, when there is symmetry of the utilities. One calculates the demand in different cases such as $p_1 < p_2 < p_3$, or $p_2 < p_1 < p_3$ etc. The formulas for the demand being different is every case, possibly when $p_1 = p_2 = p_3$, there is no continuity of the second derivative. And the Jacobian is not defined. In the tractable example there are different second derivatives for $p_i < 1/2$ and $p_i > 1/2$. But the conditions C_1 and C_2 are fulfilled. Considering that a price is fixed, the two other ones varying, on the bisector the reaction functions have two tangents (one below, one above the bisector, for the reaction R_1). In other words, the reaction functions are made up of two different “strands” (one below, one above the bisector, for R_1).

One proposes a particular demonstration for this case which does not use the Jacobian (since it is not defined).

One needs to demonstrate two lemmas.

There are two equilibriums:

- $E (p_{10}, p_{20}, p_{30})$ is the initial equilibrium. One knows that it exists and is unique (in the case of the tractable example, because the utilities are symmetrical).
- $E' (p'_{10}, p'_{20}, p'_{30})$ is the equilibrium after the increase of the utilities u_1 (from u_1 to $u_1 + k$, $k > 0$). One knows that it exists (thanks to the Poincaré Miranda theorem) and one supposes it is unique.

One calls S a sequence of “steps” $S_1 S_2 S_3 S_1 S_2 S_3 \dots S_i$ being the step when E_i maximizes its profit, the two other prices being constant.

One will demonstrate that $dP_1 \geq 0$ (k being small) using only the conditions C_1 and C_2 .

¹ For Christensen, «disruptive innovation» allows products which are more useful and more accessible. It is the motor of economic growth. For instance, the progress in computer technology allowed mainframe computers, then personal computers, and finally smartphones.

Lemma 1: one passes from E to E' thanks to the sequence S with: $dp_1 > 0$ ($dp_1 < k$), at the step 1, $dp_2 < 0$ at the step 2, $dp_3 < 0$ at the step 3, $dp_1 < 0$ at the step 4 etc. One considers the deformation of the reaction functions. At each step the right strand is to consider since the reaction function is made up of two "strands". One uses inequalities such as: $\partial^2 P_1 / \partial p_1^2 \leq 0$, $\partial^2 P_1 / \partial p_1 \partial p_2 \geq 0$ etc. The decreasing sequences of the p_i being bounded, there is convergence toward a limit, which is $(p'_{10}, p'_{20}, p'_{30})$. By continuity the limit is E', a point where a firm does not need to adjust its choice. One knows $dp_2 < 0$ and $dp_3 < 0$, when one passes from E to E'. One has:

$dp_1 = -kp_1 \partial D_1 / \partial p_1 + p_1 \partial D_1 / \partial p_2 dp_2 + p_1 \partial D_1 / \partial p_3 dp_3$. If one demonstrates $dp_2 > -k$ ($|dp_2| < k$) and $dp_3 < -k$ ($|dp_3| < k$), then $dp_1 < 0$, since $dp_1 > -kp_1 (\partial D_1 / \partial p_1 + \partial D_1 / \partial p_2 + \partial D_1 / \partial p_3) = 0$ (Property P₂).

Lemma 2: When one passes from E to E', $dp_2 < 0$ and $|dp_2| < k$. The same for p_3 .

One starts at the point P ($p_{10}, p_{20} - k, p_{30} - k$) to go to E' thanks to the sequence S. At the point P, the $\partial P'_1 / \partial p_i$ are positive.

$$P'_1 = p_1 D_1 (p_1 - k, p_2, p_3)$$

$$\partial P'_1 / \partial p_1 = D_1 (p_1 - k, p_2, p_3) + p_1 \partial D_1 (p_1 - k, p_2, p_3) / \partial p_1.$$

$$\text{At the point P, } \partial P'_1 / \partial p_1 = 0.$$

$$P'_2 = p_2 D_2 (p_1 - k, p_2, p_3)$$

$$\partial P'_2 / \partial p_2 = D_2 (p_1 - k, p_2, p_3) + p_2 \partial D_2 (p_1 - k, p_2, p_3) / \partial p_2.$$

$$\text{At the point P, } \partial P'_2 / \partial p_2 = -k \partial D_2 / \partial p_2 > 0.$$

$$\text{Also, } \partial P'_3 / \partial p_3 > 0.$$

Therefore, the sequences of p_i are increasing. And they are bounded. They converge to $(p'_{10}, p'_{20}, p'_{30})$. One knows: $p_{20} - p'_{20} > 0$ and $p_{30} - p'_{30} > 0$. Therefore, $dp_2 > -k$ and $dp_3 > -k$.

Notice that one has no proof that the equilibrium is stable (for small variations of the prices). One has not the condition of the determinant of the Jacobian which is negative, since the Jacobian is not defined. And the convergence of S when one starts from E or P does not prove the stability. The definition of the stability of the equilibrium is: for any sequence of steps (infinite, with each firm making an infinity of steps) and starting from any point, there is convergence.

5. Tractable Examples

The first tractable example is interesting because ... the demands are not linear. One supposes a density of utility which is areal, homogeneously distributed on the external facets of the cube $u_i = 1$ (this density is equal to 1/3) (figure 2). The formulas for the demands depend of inequalities such as $p_1 < p_2 < p_3$ etc.

For instance, if $p_1 \leq p_2 \leq p_3$:

$$D_1 = 1/3 [1 + p_1^2/2 - p_3^2/2 - p_1 p_2 + p_2 p_3 - 2p_1 + p_2 + p_3]$$

$$D_2 = 1/3 [1 - p_1^2/2 - p_3^2/2 + p_1 p_3 + p_1 - 2p_2 + p_3]$$

$$D_3 = 1/3 [1 + p_3^2 + p_1 p_2 - p_1 p_3 - p_2 p_3 + p_1 + p_2 - 2p_3]$$

Of course, the Nash equilibrium is on the diagonal (0, 0, 0) (1, 1, 1) and there are different formulas for $p_1 < 1/2$ and $p_1 > 1/2$, for instance. The Nash equilibrium is at $(1/2, 1/2, 1/2)$ and the profits are equal to 1/6. But there is a discontinuity of

the tangent to the reaction function, at the equilibrium point. One cannot define the determinant of the Jacobian. However, the particular demonstration of the previous paragraph allows to state that if the firm E₁ adds k to the utilities u_i ($u_i \rightarrow u_i + k, k > 0$), its profit P_1 increases.

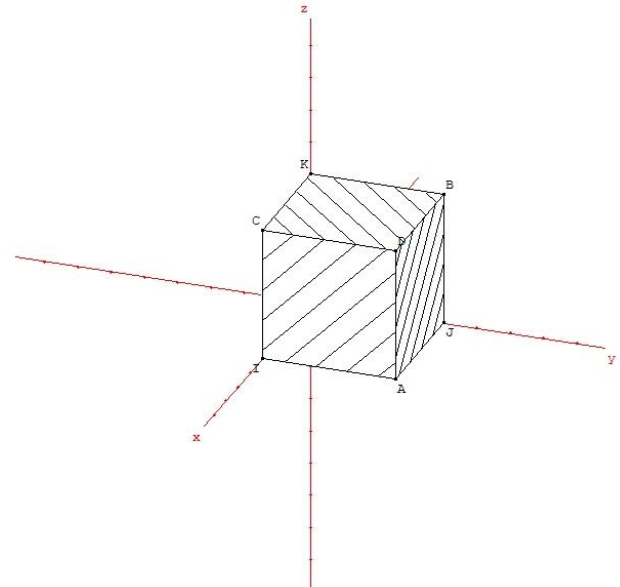


Figure 2. In the first tractable example the density is areal, homogeneous and distributed on the external facets of the cube. These facets are stripped on the figure

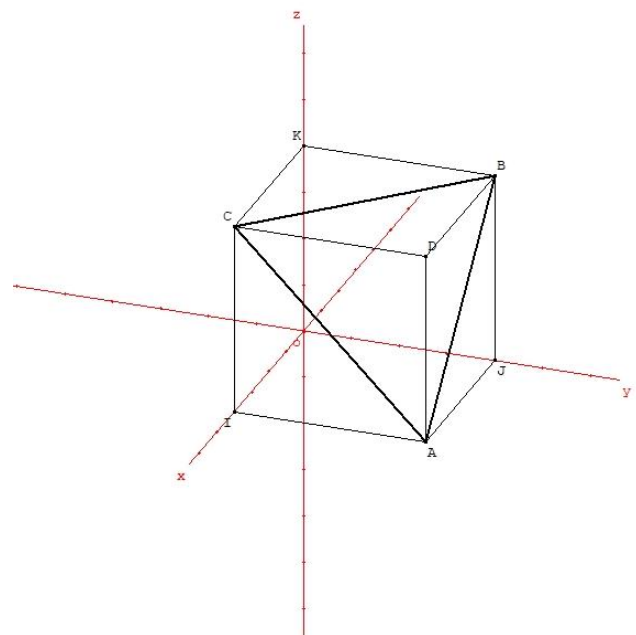


Figure 3. The density in the case of the second tractable example is shown. It is linear, homogeneous and distributed on the second bisectors in the external facets of the cube

The second tractable example is simpler, since the demands are linear. The Jacobian is defined and is negative (the equilibrium is stable for small variations of the prices). The second derivatives of the demands are equal to 0. One supposes that the density is linear, homogeneously

distributed and on the second bisector in each external facet $u_i = 1$ (density equal to $1/\sqrt{2}$). The density is shown on the figure 3.

The demands are:

$$D_1 = 1/3 [1 - 2p_1 + p_2 + p_3]$$

$$D_2 = 1/3 [1 + p_1 - 2p_2 + p_3]$$

$$D_3 = 1/3 [1 + p_1 + p_2 - 2p_3]$$

The Nash equilibrium corresponds to $p_1 = p_2 = p_3 = 1/2$ and $P_1 = P_2 = P_3 = 1/6$. To study disruption in this case, one can suppose that E_1 increases all the utilities of 1. The demands become:

$$D'_1 = 1/3 [3 - 2p_1 + p_2 + p_3]$$

$$D'_2 = 1/3 [p_1 - 2p_2 + p_3]$$

$$D'_3 = 1/3 [p_1 + p_2 - 2p_3]$$

(provided that for p_1, p_2, p_3 all the D_i are positive).

One checks that for $(1, 0, 0)$ all the demands are $1/3$. It is extremely advantageous for E_1 , since its price is high (compared to the two other prices) and it keeps the consumers in its “garden” (that is to say the consumers preferring the product sold by E_1).

One easily calculates the profits at the Nash equilibrium $p'_{10} = 9/10, p'_{20} = p'_{30} = 3/10$ (in accordance with the model, p_2 and p_3 decrease): $P'_1 = 54/100$ (which is more than $1/6$) and $P'_2 = P'_3 = 3/50$ (which is less than $1/6$). In accordance with the model, the profit P_1 increases and the profits P_2 and P_3 decrease.

One has to check that the Nash equilibrium is “inside the formulas” ($D_i \geq 0$). It is not necessary in the case of that example because the demands are linear. Since $\partial P_i / \partial p_i = D_i + p_i \partial D_i / \partial p_i = 0$, $\partial D_i / \partial p_i$ being a negative coefficient (the demand D_i is linear), $D_i \geq 0$.

The second tractable example is resumed later in the paper. One considers a value of k which is $1/2$, not 1, to be “inside the formulas”. The game is with two players, since one considers a “buy and manage” and a “buy and close down”.

6. About the Possibilities of the Model Used

A goal of this paper is to show the possibilities of the model used (Bertrand competition, the demands being deduced from the consumers ‘utilities’).

The model can be used to study the profitability of a merger, when the asset which is bought is closed, to discriminate between non differentiating innovation and differentiating innovation, and study the effects of cannibalization:

Profitability of buy and close down.

This kind of operation, to buy an asset and to close down it, which is supposed to be profitable (otherwise the managers would not have decided it) seems rare. Indeed, it is discrete, because the Opinion disapproves any destruction of means of production. In France, several years ago, the firm UPM (United Paper Mills) decided to close a site where paper was

produced, in a village (Docelles). During a night, members of the staff destroyed the machine. It was made by night, not because it was illegal, but because it was scandalous. The destroyed machine could not be resold, and the firm lost money. But another unit of the same firm should benefit from higher prices. This compensates and besides the loss of money. It is the same calculation than in the case of “buy and close down”.

The topic is interesting for several reasons:

- The question of the regulator’s reaction is posed. He could disapprove this operation, which shrinks competition. But his opposition is easy to circumvent. It is easy to not invest in a branch, or to not invest enough, and some day one has to close it.
- The profitability of an operation “buy and close down” is obviously a sign of saturated market. In case of Bertrand competition, a merger is always profitable (given some conditions). The profitability of a merger cannot be the sign of a saturated market. But the profitability of an operation “buy and close down” implies a strong strategic effect, because of less competition. It is explained by the saturation of a market. Two causes (when a market is saturated) are possible: (1) a competitor creates low utility for consumers, is obliged to choose a low price and “steals” customers from the other competitors (2) the products are not enough differentiated, that is to say the Bertrand paradox applies. The model used allows to present tractable models of the two kinds [4].
- When the “buy and close down” is profitable, it is a reason more to choose the merger (“buy and manage”), because there is an insurance. Often to “integrate” the bought firm in the buying firm is difficult [7]. Quarrels between the two teams of managers can raise. If the “buy and close down” is profitable, it is easy to make pressure on the team of managers of the bought firm (“if you do not cooperate, we can close down the asset, it will remain profitable”). Therefore, the profitability of the “buy and close down”, if it exists, incites to merge.

Non differentiating innovation and differentiating innovation.

There is non differentiating innovation when an innovation is beneficial to all the competitors. In the model, all the utilities u_i become $u_i + k$ ($k > 0$). If at the start there is product differentiation, the “gardens” correspond to the utility $1 + k$. The profits increase. But there is a drawback: when the increase of all the utilities is the same, there is less product differentiation. An easy calculation which is not set out in this paper, shows this: non differentiating innovation can trigger a kind of saturated market (at the start, a “buy and close down” is not profitable, then after non differentiating innovation it becomes profitable).

Let us give an “example of non-differentiating innovation. The dressmaker Karl Lagerfeld (1939 – 2019) made his career in Paris and was notorious. He worked in many firms

which were in competition: in high fashion (Chanel, Fendi and his own firm), in ready to wear (Chloe, H&M) and even in sale on line (NetaPorter). Thanks to his talent, he brought innovation in several firms in competition. Therefore, the consumers' utilities increased. The consequence was that the profits of the firms in which he worked increased (and himself won more money). This seems to be a digression but perhaps is not. The author has stated in a paper that there is a production function of the modern artist [8]. An artist has to be visible (participating in galas, events, having a humanitarian activity, appearing in medias ... to have an "image") and to be creative (in his works). The artist's gain is an increasing function of the two factors (visibility and creativity). In this sense, Karl Lagerfeld was really a modern artist. He was an entrepreneur in visibility and creativity. His prolific activity is accurately depicted in the book "Karl Lagerfeld de A à Z" [9].

At the opposite, differentiating innovation (disruption) is beneficial to a single firm, the disruptive firm. But a disruptive firm should not "buy and close down" a competitor. It is to shrink competition thanks to a means that the regulator could refuse.

In his book "The innovator's dilemma", Christensen has explained that a firm the market of which is threatened by innovating competitors will react by protecting its market, aiming at keeping its customers thanks to an upgrade of its product. It is risky because some day the new products could replace its own [10]. Of course, there is the temptation to buy and close down the innovating firm which is a threat. The threatening firm could also be bought then its development is slowed. A famous example is Instagram bought by Facebook. It has been said that years later, Instagram could have become a real leader (as Facebook), but it has been bought by Facebook. This is made possible by the fact that it is difficult to value a start-up (in the fields of Information Technologies or bio-engineering, for instance). Examples are given in the article "Antitrust and innovation: welcoming and protecting disruption" [11].

Recently fears have appeared that the regulation in the USA, could slacken. This fear is expressed in the Philippon's book "The great reversal: how America has given up on free markets" [12].

All this shows that the profitability of operations of "buy and close down" is an interesting topic. Since disruption is wished and welcomed, one fears an incumbent which "buys and closes down" a small, innovating firm which threatens its market or its product.

Effects of cannibalization.

The model can be used to study the effects of cannibalization. One gives a tractable example in the following chapter.

7. A Tractable Example on Disruption

There are often portfolios of brands in many fields: technology, fashion (clothes, leather goods, jewelry ...) or

medias. These brands are often in competition. These portfolios of brands in competition pose puzzling questions. Is it in the interest of the owner to choose a disruptive strategy for one of his firms, given that it will harm the other very much? Is it not in his interest to choose disruption for one of his firms and close down another? These questions could be interesting for the regulator, too. Possibly, the regulator could disagree, if the owner of several firms in competition buys a firm with the intention of not developing it, perhaps of closing down it (to develop another firm he owns).

One can use the second tractable example set out in the paper to illustrate the effect of cannibalization.

The disruption can be achieved in three ways:

- At the start, there is equilibrium with three firms, "monopolistic competition". One calls this game E (as equilibrium).
- At the start, the disruptive firm E_1 has bought E_3 and will keep it. One calls this game BM (as Buy and Manage).
- At the start, the disruptive firm E_1 has bought E_3 and will close down it. One calls this game BC (as Buy and Close down).

The calculations show that E is less profitable than BC, which is less profitable than BM. How to explain that BC is less profitable than BM? In case of BC, the disruptive firm does not benefit from the profit of E_3 , but there is no cannibalization. At the opposite, in case of BM, the disruptive firm E_1 benefits from the profit of E_3 , but there is cannibalization (that is to say, E_3 "steals" customers from E_1 , and the margin is less, because $p_3 < p_1$).

The calculations are needed to know what is more profitable (BM or BC).

One can put it in other words. Consider BM, after disruption. One passes to BC by closing E_3 ($p_3 = 1$), then the two prices p_1 and p_2 change and the two profits P_1 and P_2 change, also. What happens when p_3 becomes equal to 1? There are three kinds of customers of E_3 : (1) those choosing E_1 . It is the effect "end of cannibalization", since these customers will allow the margin p_1 instead of p_3 (and $p_1 > p_3$) (2) those choosing E_2 who are customers lost for E_1 and (3) those choosing no purchase, who are also customers lost for E_1 . The demand for E_1 has decreased (D_1 is less after $p_3 \rightarrow 1$ than $D_1 + D_3$ before). After, there are changes of p_1 and p_2 , and one does not know the change of D_1 . The calculations show that the demand D_1 is less, after the close down. This explains that the profit is less. And this, notwithstanding the cannibalization which no more exists. The outcome is "reassuring" for the regulator. A disruptive firm should not buy a competitor then close down it. It should buy it and keep it. Notice that one has only examined a particular case.

This question is interesting, since the disruptive firm benefits from high margins during a first period, only. There are three periods. During the first one the disruptive firm has a behavior described by speed and brand. The goal is growth. It has to make its product well known and attractive for

customers. The high margins do not last a long time because during the second and third period there are entries and imitators [10]. The two last periods are necessary to make the prices of the new products lower, which is a condition of disruption according to Christensen (the products are “affordable and accessible”). During the first period, there are priorities and other goals like reduction of the costs or upgrading the distribution of the product, are neglected. And in this first period, the disruptive firm could be interested in buying a competitor, if it allows more profit.

8. Conclusions

One has modelled “monopolistic competition”, or product differentiation, thanks to a method which is presented in the paper: Bertrand competition, the demands being deduced from the consumers’ utilities. The mathematical properties of the equilibrium, the existence of which is demonstrated, correspond to the characteristics of product differentiation: all the consumers buy a product, the competitors benefit from a “garden” (a set of customers easily kept) and the profits are sufficient. Disruption is defined: the “garden” of the disruptive firm has a utility which is more than the utilities of the “gardens” of the other competitors. One demonstrates that the profit of a firm which disrupts, increases.

A goal of the paper is to explore the possibilities of the method presented. This method allows to study these topics: the profitability of mergers when the asset which is bought is closed down, the difference between non differentiating innovation and differentiating innovation (disruption) and the effects of cannibalization.

Tractable examples are set out, which allow to illustrate how the method can be used. For instance, a tractable example brings an answer to the question: What is the most profitable for a disruptive firm: to “buy and manage” a competitor, to “buy and close down” it or to make no purchase? This question implies to study the effects of cannibalization.

Could this method be upgraded? One can deal with this question considering three aspects:

- *The choice of three players.* It is chosen to simplify. It would be easy to generalize the game to n players.
- *Neglecting the costs.* Let us recall the difference between “disruptive innovation” and “efficient innovation” according to Christensen. We have already quoted the definition of “disruptive innovation”. The efficient innovation is an innovation which does not concern the product, but triggers a decrease of the costs [13]. Since the goal of this paper is to study disruptive innovation, to neglect the costs is without consequences.
- *The problem of moral obsolescence.* It can be understood thanks to a cognitive bias, sometimes called the “dolphin bias”. It has been described, among other cognitive biases, by Nassim Taleb [14]. It concerns an

inquiry on these questions: “How much money do you want to give to save dolphins?” and “How much money do you want to give to save kids from some illness?”. The results are not the same if the questions are posed separately or simultaneously. If the questions are separately posed the answers could be 50 dollars (first question) and 30 dollars (second question). If the questions are simultaneously posed, the order is reverse: 30 dollars (first question) and 50 dollars (second question). In other words, evaluation of something depends on information. When the consumers having chosen the old product are informed on the new product, their evaluation of the utility of the old product decreases (in particular when the usage implies networks). In the model used on the paper, in case of disruption one could change u_1 into $u_1 + k$ ($k > 0$), E_1 being the disruptive firm, u_2 into $u_2 - k_2$ ($k_2 > 0$) and u_3 into $u_3 - k_3$ ($k_3 > 0$), E_2 and E_3 being the other competitors. This means that the profit P_1 would increase more, and the profits P_2 and P_3 decrease more.

The model used seems interesting, in particular, because it provides a criterium for saturated markets (the profitability of “buy and close down”). Perhaps practical applications should be possible. Market studies provide a cloud of points representing the utilities of the consumers. It corresponds approximately to an analytical function, or not. If not, a computer could be used to find the Nash equilibrium (provided that it exists and is unique). After, to calculate if the “buy and close down” is profitable, or not, should be feasible. If the “buy and close down” is profitable, it is a criterium indicating that the market is saturated.

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