Journal of Game Theory 2019, 8(1): 16-24
DOI: 10.5923/j.jgt.20190801.03

The Shapley Pre-value for Fuzzy Cooperative Games

Yeremia Maroutian

11806 Miles Ave Apt. 112, Cleveland Oh. 44105, USA

Abstract In this paper for fuzzy cooperative games we generalize the classical Shapley value through using one of the known approaches for its redefinition. For fuzzy cooperative games formalized as well the Shapley axioms. For the fuzzy pre-value of Shapley we research certain of known properties that hold true for classical Shapley value.

Keywords Fuzzy cooperative games, Fuzzy coalition, Fuzzy Shapley pre-value

1. Introduction

The cooperative fuzzy games reflect type of situations in which for players allowed to take part in coalitions with participation levels that can vary from non-cooperation to full cooperation. Fuzzy coalitions describe levels of participation at which players involved in cooperation. The reward for players in this type of games defines depending on their level of cooperation.

An important topic of research for fuzzy cooperative theory is the extension of existing in classical theory decision concepts on fuzzy games. It is known, that not every classical concept has its natural counterpart for fuzzy games. At the same time some results in classical theory allow to be transformed on fuzzy case, with of course significant differences.

The concept of fuzzy sets as a continuum of grades of membership for the members of sets has been introduced by Zadeh (1965). Let N= {1, 2… n} be the set of all players. We will call $uzzy\ coalition$ an n-dimensional vector $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ with $0 \leq \tau_i \leq 1$ for each $i \in N$. A cooperative fuzzy game with the players set N is a pair (T, v), where $T \subset [0,1]^n$, is the set of fuzzy coalitions and v is the characteristic function of that game that maps a real number to each fuzzy coalition.

The cooperative fuzzy games reflect type of situations in which for players allowed to take part in coalitions with participation levels that can vary from non-cooperation to full cooperation. Fuzzy coalitions reflect levels of participation at which players involved in cooperation. The reward for a player in this type of games defines depending on level of cooperation.

* Corresponding author:
yermars0131@gmail.com (Yeremia Maroutian)
Published online at http://journal.sapub.org/jgt

An important topic of research for fuzzy cooperative theory is the extension of existing in classical theory decision concepts on fuzzy games. It is known, that not every classical concept has its natural counterpart for fuzzy games. At the same time some results in classical theory allow to be transformed on fuzzy case, with of course significant differences.

Shapley value is the most studied concept in the classical cooperative theory. It's generalization for the fuzzy games considered as one of the important topics for the fuzzy theory of games.

There are known different approaches for extension of classical Shapley value on fuzzy cooperative games. Have been obtained different fuzzy values that depend on applied approaches. It pays to separate the Aubin's (1981) diagonal value, which represents itself an integral formula for the Shapley value. For this and some other values it is specific, that they do not use all of the information that is in the square of fuzzy coalitions (in our case the latter one is an n-dimensional unit cube). The mentioned fact separates types of games to what the fuzzy Shapley value suitable. The value proposed by Tsurumi et al (2001) represents of itself a payoff vector for each fuzzy game that depends on the final formed coalition. Hwang and Liao (2009) gave a concept of value that results to a fuzzy Shapley value, where a payoff obtained for each level of participation. Jimenez-Losada et al. (2019) for the fuzzy cooperative games propose a new extension of the Shapley value that improves the known classical Shapley value for fuzzy cooperative games.

Among the known axiomatizations for the fuzzy Shapley values can be separated Shapley (1953), Myerson (1980), Young (1985), Weber (1988), Hart and Mas-Colell ((1989), Casajus (2014). Each one of them demonstrates certain interesting properties of the Shapley value.

This research in its entirety contains novel results. New is the offered approach on generalizing the classical Shapley value for the fuzzy coalition game. Among the aspects what are new in fuzzy theory should be separated the constructed functional, point of minimum for what is the fuzzy Shapley value. In classical cooperative games a payment for each player in coalition is a component of certain traditional payoff vector. In contrast to that for the discussed fuzzy model the payoff is a function $\mathrm{F}(v)$, which satisfies to modified analogues of Shapley axioms. These axioms again have been newly formulated here. In different of values that obtained according to listed approaches, the Shapley value we propose defined for games on the unit cube that satisfy to the condition of integrability. That is much weaker condition than the requirement of being differentiable which is in case of Shapley's some other fuzzy extensions. It makes essentially wider the class of fuzzy games to what the defined our way fuzzy Shapley value can be applied. For the fuzzy pre-value we research certain of known properties that hold true for classical Shapley value.

Results included in paper devoted to generalizing of the classical Shapley value for fuzzy cooperative games, through one of the known approaches. According to that approach the classical Shapley value is the point of minimum for a certain functional on the set of all pre-imputations. Included in Section 2 preliminary facts and definitions that refer to cooperative games, the classical functional, fuzzy coalitions, and a simpler form for functional in case of fuzzy games, formulated fuzzy analogues of Shapley axioms. Has been defined the functional $Q(x,v)$ for the fuzzy case. Section 3 devoted to finding of a function $\mathrm{F}(v)$ that minimizes $Q(x,v)$. For functions $g_{k,l}^{n}(\tau^{L})$ participating in the expression of functional $Q(x,v)$, formulated the requirements at what for the function $\mathrm{F}(v)$ hold true the Shapley axioms. In section 4 has been discussed the properties of function $\mathrm{F}(v)$. It has been proven, that for a class of fuzzy games obtained from the classical cooperative games through the Owen's multilinear extension, the pre-value $\mathrm{F}(v)$ coincides with the classical Shapley pre-value.

Further, has been obtained necessary and sufficient conditions at what the function $\mathrm{F}(v)$ satisfies to the strengthened dummy's axiom. Has been proven a preposition that allows constructing functions $g_{k,l}^{n}(\tau^{L})$, which satisfy to the strengthen dummy's axiom with fewer players.

2. Preliminary Facts and Definitions

In SOBOLEV A. (1978) for classical cooperative games has been proved that the Shapley vector is a point of the set

$$X=\{x/\,x(N)=v(N)\},$$

on what the functional

$$\bar{Q}(x,v)=\sum_{S\neq\emptyset,\ N}(|S|-1)!\,(|N|-|S|-1)!\,(v(S)-x(S))^{2}$$

gets its minimum.

That property of Shapley vector allows using it as an alternative definition. The defined that way Shapley vector in its turn by a quite natural way can be extended on fuzzy cooperative games.

Let T be the set of all fuzzy coalitions, i.e. $T\subset[0,1]^{n}$. A coalition in its classical sense is a peak of the cube T, or more precisely, that peak for what

$$\tau_{i}^{S}=\begin{cases}1 & \text{if} \quad i\in S\\ 0 & \text{if} \quad i\notin S\end{cases}$$

For a measure $\bar{\mu}$ that concentrated on peaks τ^{S} of the cube $[0,1]^{n}$ and accepts values $(|S|-1)!\,(|N|-|S|-1)$ the expression for $\bar{Q}(x,v)$ can be rewritten in a simpler form:

$$Q\,(x,v)=\int_{T}\,(v(\tau)-x(\tau))^{2}d\bar{\mu}$$

On the cube T instead of $\bar{\mu}$ one can take an arbitrary measure and define the value $\mathrm{F}(v)$ for a game $v(\tau)$ as a point on what the newly obtained functional $Q\,(x,v)$ will reach its minimum.

Farther we will choose a measure such that the function $\mathrm{F}(v)$ would possess with properties defined by axioms, similar to Shapley axioms for classical cooperative games.

Below formulated analogs of Shapley axioms for fuzzy cooperative games.

A1 (Symmetry) Let π is an injection from the set N^{v} on $N^{v'}$ such, that for each coalition $\tau\in\mathrm{T}$

$$v'(\pi^{*}\tau)=v(\tau),$$

where $\pi^{*}:T\to T$ is a map, for what $(\pi^{*}\tau)_{i}=\tau_{\pi^{-1}i}$.

Then, for all $i\in N^{v}$

$$F_{\pi\,i}(v')=F_{i}\,(v).$$

A.2 (Pareto optimality)

$$\sum_{i\in N}F_{i}\,(v)=v(1)-v(0).$$

The unusual form of this axiom compared with its classical analogue caused by the fact that the possibility of $v(0)\neq 0$ not excluded.

A.3 (Inefficiency of 'dummy') If in the game (T,v) i_{0} is a player such, that $v(\tau)=v(\tau||\tau_{i_0}{=}0)$, for arbitrary $\tau\in\mathrm{T}$, then $F_{i_0}(v)=0$. The expression $v(\tau)=v(\tau||\tau_{i_0}{=}0)$ means, that the value of the coalition τ in the game v equals to the same value for the coalition with $\tau_{i_0}{=}0$.

A.4 (Aggregation) If u and v are games with the same set of coalitions T and $w=u+v$, then

$$F(w)=F(u)+F(v).$$

Besides A.3 we will also deal with its strengthened form. There will be a need for the next definition.

Definition 1. We will say that the game (T^{n+1},u) derived from the game (T^{n},v) by adding to it a 'dummy' player $(n+1)$, if for every coalition $\tau\in T^{n}$ and every $\tau_{n+1}\in[0,1]$ takes place:

$$v(\tau)=u(\tau,\tau_{n+1}).$$

A3'. (The strengthened axiom of 'dummy') Holds true

$$F_{j}(v)=F_{j}\,(u),$$

for arbitrary $j\neq n+1$.

The A3' otherwise, means that including a 'dummy' in a game does not change gains of other players.

It is obvious, that if A.2 remains true, then the A.3 follows from A.3'.

In general the choice of measure μ is not unique. Partially as such a measure can be also $\bar{\mu}$. The axiom A.1 holds true, if the measure has been defined the same way on all of the sides of the cube.

Let now consider the following sets. For arbitrary $K \subset N$ and $L \subset N \setminus K$, denote by <K, L> the following set: $< K, L > = \{\tau \in \mathrm{T} / \ 0 < \tau_i < 1, \ i \in \mathrm{L}; \ \tau_i = 1, \ i \in \mathrm{K}; \ \tau_i = 0, \ i \in \mathrm{N} \setminus (\mathrm{K} \cup \mathrm{L})\}$.

$$k = |K|, \ l = |L|, \ \tau^L = (\tau_i)_{i \in \mathrm{L}}.$$

Sets <K, L> actually are the faces of the cube $[0,1]^n$. We will deal with the case, when on each one of its faces measure defined by a density function $g^n_{k,l}(\tau^L)$. To satisfy axiom A.1 will be accepted that $g^n_{k,l}(\tau^L)$ are symmetric, none negative functions of their arguments. As a result, we will have the following functional:

$$Q(x, v) = \sum_{K \subset N} \sum_{L \subset N \setminus K} \int_{<K,L>} (v(\tau) - x(\tau))^2 \, g^n_{k,l}(\tau^L) d\tau^L,$$

where $k = |K|, \ l = |L|, \ \tau^L = (\tau_i)_{i \in \mathrm{L}}$.

Our goal is finding relations between the functions $g^n_{k,l}(\tau^L)$ so that the function $\mathrm{F}(v)$ which minimizes functional $Q(x)$, has satisfied to axioms A.3 and A.3'.

3. Pre-value for Fuzzy Cooperative Games

Below we will find explicit formula for the function $F(v)$, which minimizes functional $Q(x, v)$. At the same time for functions $g^n_{k,l}(\tau^L)$ participating in the expression of functional $Q(x, v)$, we will formulate requirements so that with them hold true the Shapley axioms for $F(v)$.

Farther we will use the defined below magnitudes:

$$A^n_i(v) = \sum_{\{K: K \ni i, K \subset N\}} \sum_{\{L \subset N \setminus K\}} \int_{<K,L>} (v(\tau) \, g^n_{k,l}(\tau^L) d\tau^L + \sum_{\{K: K \subset N\}} \sum_{\{i: i \in \mathrm{L} \subset N \setminus K\}} \int_{<K,L>} (v(\tau) \ \tau_i \, g^n_{k,l}(\tau^L) d\tau^L;$$

$$\alpha^n_{k,l} = \int_{<L>} \tau_i^2 \, g^n_{k,l}(\tau^L) d\tau^L, \ where \ i \in \mathrm{L}, \quad l \geq 1$$

$$\beta^n_{k,l} = \int_{<L>} \tau_i \, \tau_j \, g^n_{k,l}(\tau^L) d\tau^L, \ where \ i, j \in \mathrm{L}, i \neq j \quad l \geq 2$$

$$\gamma^n_{k,l} = \int_{<L>} \tau_i \, g^n_{k,l}(\tau^L) d\tau^L, \ where \ i \in \mathrm{L}, \quad l \geq 1$$

$$\delta^n_{k,l} = \begin{cases} g^n_{k,0} & if \quad \mathrm{l} = 0 \\ \int_{<L>} g^n_{k,l}(\tau^L) d\tau^L & if \ l \geq 1, \ \mathrm{k} \geq 1 \end{cases}$$

Let extend definitions of $\alpha^n_{k,l}$, $\gamma^n_{k,l}$ for $l = 0$ and $\beta^n_{k,l}$ for $l = 0, 1$, by accepting that

$\alpha^n_{k,0} = \beta^n_{k,0} = \gamma^n_{k,0} = \beta^n_{k,1} = \beta^n_{k,0} = 0$. Besides, let also accept that $\delta^n_{0,l} = 0$.

Further we will need to deal with numbers:

$$c_n(k, l, n-k-l) = \frac{n!}{k! \, l! \, (n-k-l)!}$$

We denote:

$$\Delta^n_{k,l} = c_{n-1}(k, l-1, n-k-l)\alpha^n_{k,l} - c_{n-2}(k, l-2, n-k-l)\beta^n_{k,l} - 2 \, c_{n-2}(k-1, l, n-k-2) \, \delta^n_{k,l},$$

and

$$\Delta^n = \sum_{l=0}^{n} \sum_{k=0}^{n-l} \Delta^n_{k,l}.$$

Preposition 1. If $\Delta^n \neq 0$, then the functional Q(x, v) on the set

$$X = \{x / \ x(N) = v(1) - v(0)\}$$

its minimum by x accepts on the vector $F(v)$ with following components:

$$F_i(v) = \frac{1}{n} \{ (1) - v(0) + \frac{1}{\Delta^n} \sum_{\{<K,L>, \ i \notin K,L, \ K \cap L = \emptyset\}} \int_{<K \cup i, \ L>} v(\tau)(g^n_{k+1, \ l}(\tau^L)(n - k - 1 - \textstyle\sum_{j \in L} \tau_j) d\tau^L + \int_{<K, \ L \cup i>} v(\tau)(g^n_{k, \ l+1}(\tau^{L \cup i})((n-1)\tau_i - k - \textstyle\sum_{j \in L} \tau_j) \, d\tau^{L \cup i} - \int_{<K, \ L>} v(\tau)(g^n_{k, \ l}(\tau^L)\left(k + \textstyle\sum_{j \in L} \tau_j\right) d\tau^L]\}. \quad (1)$$

Proof. The vector on what functional $Q(x, v)$ reaches its minimum, according to the method of Lagrange's multipliers should satisfy to the following system of linear equations:

$$\begin{cases} \frac{\partial Q}{\partial x_i} = \lambda^{(n)} \quad for \ \forall \, i \in N & (2) \\ \sum_{i \in N} x_i = v(1) - v(0) & (3) \end{cases}$$

To write these equations in explicit form we need to find the partial differentials of $Q(x, v)$:

$$-\frac{1}{2} \frac{\partial Q}{\partial x_i} = \sum_{K \ni i} \sum_{L \subset N \setminus K} \int_{<K, \ L>} (v(\tau) - < x, \tau >) g^n_{k,l}(\tau^L) d\tau^L + \sum_{K \subset N} \sum_{\{L: i \in L \subset N \setminus K} \int_{<K,L>} (v(\tau) - < x, \tau >) \tau_i g^n_{k, \ l}(\tau^L) \, d\tau^L \ ]\}.$$

There will be need as well for the following notations:

$$p^n = \sum_{\{K: \, i \in K \subset N\}} \sum_{\{L: \, L \subset N \setminus K\}} \int_{<K, \ L>} g^n_{k, \ l}(\tau^L) \, d\tau^L + \sum_{\{K: \, i \in K \subset N\}} \sum_{\{i \in \ L \subset N \setminus K\}} \int_{<K, \ L>} \tau_i^2 \, g^n_{k, \ l}(\tau^L) d\tau^L$$

$$q^n = \sum_{\{K: \, K \subset N\}} \sum_{\{L: i, j \in \ L \subset N \setminus K\}} \int_{<L>} \tau_i g^n_{k, \ l}(\tau^L) \, d\tau^L + \sum_{\{K: \, i, j \in K \subset N\}} \sum_{\{L: \ L \subset N \setminus K\}} \int_{<K, \ L>} g^n_{k, \ l}(\tau^L) d\tau^L + 2 \sum_{\{K: \, i \in K \subset N \setminus j\}} \sum_{\{L: \, j \in \ L \subset N \setminus K\}} \int_{<K, \ L>} \tau_j \, g^n_{k, \ l}(\tau^L) d\tau^L.$$

We need to prove that $\Delta^n = p^n - q^n$. So, let rewrite the expressions for p^n and q^n in different forms:

$$p^n = \sum_{k=1}^{n} \sum_{l=0}^{n-k} \binom{n-1}{k-1} \binom{n-k}{l} \delta^n_{k,l} + \sum_{k=0}^{n-1} \sum_{l=1}^{n-k} \binom{n-1}{k} \binom{n-k-1}{l-1} \alpha^n_{k,l}$$

$$= \sum_{k=1}^{n} \sum_{l=0}^{n-k} c_{n-1}(k-1, l, n-k-l) \, \delta^n_{k,l} + \sum_{k=0}^{n-1} \sum_{l=1}^{n-k} \binom{n-1}{k} c_{n-1}(k, l-1, n-k-l) \alpha^n_{k,l}.$$

$$q^n = \sum_{k=0}^{n-2} \sum_{l=2}^{n-k} \binom{n-2}{k-1} \binom{n-k-2}{l-2} \beta^n_{k,l} + \sum_{k=1}^{n} \sum_{l=0}^{n-k} \binom{n-2}{k-2} \binom{n-k}{l} \delta^n_{k,l} +$$

$2\ \sum_{k=2}^{n-2}\sum_{l=1}^{n-k}\binom{n-2}{k-1}\binom{n-k-1}{l-1}\gamma_{k,l}^{n}\ =$
$\sum_{k=0}^{n-2}\ \sum_{l=2}^{n-k} c_{n-2}\ (k,l-2,n-k-l)\ \beta_{k,l}^{n}\ +$
$\frac{k-1}{n-k}\sum_{k=1}^{n}\sum_{l=0}^{n-k} c_{n-2}\ (k-1,l,n-k-l)\delta_{k,l}^{n}\ +$
$2\sum_{k=2}^{n-2}\sum_{l=1}^{n-k} c_{n-2}\ (k-1,l-1,n-k-l)\ \gamma_{k,l}^{n}.$

As far as above we redefined the magnitudes $\alpha_{k,0}^{n} = \beta_{k,0}^{n} = \beta_{k,1}^{n} = \gamma_{k,0}^{n} = \delta_{0,l}^{n} = 0$, so, while figuring out the difference $p^n - q^n$, we can formally extend summing by the index k from 0 to n and by the index l from 0 to $n-k$. That will result the required equality instantly.

Further, by using the defined above notations we can rewrite (2) - (3) in an equivalent form:

$$\begin{cases} \sum_{i\in N} x_i = v(1) - v\ (0). & (3') \\ -\frac{1}{2}\frac{\partial Q}{\partial x_i} = A_i^n(v) - q^n \sum_{j\neq i} x_j - p^n\, x_i = \lambda^{(n)} & \\ for\ arbitrary\ i \in N. & (2') \end{cases}$$

Summing equations (2') by i and taking in account the (3'), we will obtain, that

$\lambda^{(n)} = \frac{1}{n}\ [\sum_{i\in N} A_i^n(v) - (q^n\ (n-1) +$
$p^n\ (v(1) - v\ (0)].$

That expression for $\lambda^{(n)}$ together with the system (2') – (3') gives:

$F_i\ \ (v) = \frac{1}{\Delta^n}\ [-\lambda^{(n)} + A_i^n(v)\text{-}\ q^n\ (v(1) - v\ (0)]=$
$\frac{1}{n}[v(1) - v\ (0) + \frac{1}{\Delta^n}\,[\ (n-1)A_i^n(v) - \sum_{j\neq i} A_j^n(v)\ ].$

Submitting in this formula the expression for $A_i^n(v)$ and uniting the integrals with coinciding domains of integration, we will have the following expression for $F_i\ \ (v)$:

$F_i\ \ (v) = \frac{1}{n}[v(1) - v\ (0)$
$+\frac{1}{\Delta^n}\,[(n-1)$
$(\sum_{\{K:\ K\ni i,\ \ K\subset N\}}\sum_{\{L:\, L\subset\ N\setminus K\ \}}\int_{<K,L>} v(\tau)\ g_{k,l}^{n}(\ \tau^L)d\ \tau^L$
$+\ \sum_{\{K:\ K\subset N\}}\sum_{\{L:\ i\ \epsilon\, L\subset\ N\setminus K\ \}}\int_{<K,L>} v(\tau)\ \tau_i g_{k,l}^{n}(\ \tau^L)d\ \tau^L\ )$
$-\sum_{j\neq i}(\sum_{\{K:\, j\epsilon K\subset N\}}\sum_{\{L:\ \mathrm{L}\subset\ N\setminus K\ \}}\int_{<K,L>} (v(\tau)\ g_{k,l}^{n}(\ \tau^L)d\ \tau^L\ +$
$\sum_{\{K:\ K\subset N\}}\sum_{\{\ i:\ j\ \epsilon\, L\subset\ N\setminus K\ \}}\int_{<K,L>} v(\tau)\ \tau_j\, g_{k,l}^{n}(\ \tau^L)d\ \tau^L\ )]\}$
$=\ \frac{1}{n}[v(1) - v\ (0) + \frac{1}{\Delta^n}$
$\sum_{\{<K,L>,\ \ K\cap L=\emptyset\ K,l\not\ni i\}}\int_{<K\cup i\ ,L>} v(\tau)$
$(n-k-1-\sum_{\{\,j\ \epsilon\, L\ \}}\tau_j)\quad g_{k+1,\ \ l}^{n}(\ \tau^L)d\ \tau^L\ +$
$\int_{<K,\ \ L\cup i>} v(\tau)\ ((n-1)\ \tau_i\ - k - \sum_{\{\,j\ \epsilon\, L\ \}}\tau_j))\quad g_{k,\ \ l+1}^{n}(\ \tau^{L\cup i})d\ \tau^L$
$-\int_{<K,\ \ L>} v(\tau)\ (k + \sum_{\{\,j\ \epsilon\, L\ \}}\tau_j)\quad g_{k,\ \ l}^{n}(\ \tau^L)d\ \tau^L\ \ ]\}.$

As a result we have obtained the required formula (2.1).

Remark.1. In general the solution $F\ (v)$ for system (2) - (3) is unique for arbitrary (and not only for not negative) functions $g_{k,l}^{n}(\ \tau^L)$, which participate in (x,v) 's expression. The obtained that way solution for system (2) - (3) still can be accepted as some analogue for the Shapley function, despite of that it already may not minimize the functional $Q(x,v)$.

Remark.2. For a functional $Q(x,v)$ with arbitrary measure μ the $F_i\ (v)$'s expression becomes much simpler. So let

$$Q\ (x,v)\ =\ \int_T\ (v(\tau) - x(\tau))^2 d\mu$$

By using the following notations,

$$T^i = \{\ \tau\ \epsilon\ \mathrm{T}/\ \ 0 < \tau_i {<} 1\ \},\qquad T_1^i = \{\ \tau\ \epsilon\ \mathrm{T}/\ \ \tau_i\ = 1\}.$$

We can rewrite the expression for $A_i^n(v)$ the following way:

$$A_i^n\ (v) = \int_{T_1^i} v(\tau)\quad d\mu\ \ + \int_{T^i} v(\tau)\,\tau_i\quad d\mu$$

By submitting that expression for $A_i^n\ (v)$ in the formula for $F_i\ \ (v)$, we will have that

$F_i\ \ (v) =$
$\frac{1}{n}\{v(1) - v\ (0) + \frac{1}{\Delta^n}\,[(n-1)A_i^n(v)\ \text{-}\ \sum_{j\neq i} A_j^n(v)]\}$
$=\ \frac{1}{n}\{v(1) - v\ (0) + \frac{1}{\Delta^n}\,[(n-1)(\int_{T_1^i} v(\tau)\quad d\mu\ \ +$
$\int_{T^i} v(\tau)\,\tau_i\quad d\mu\,)\ \text{-}\ \sum_{j\neq i}(\int_{T_1^j} v(\tau)\quad d\mu\ \ + \int_{T^j} v(\tau)\,\tau_j\quad d\mu\,).$ (4)

Proposition 2. Let $\Delta^n\ \neq 0$. For the function $F(v)$ to satisfy axiom $A3$ it is necessary and sufficient, that hold true the following equations:

$g_{k,\ \ l}^{n}(\ \tau^L)(k + \sum_{i\epsilon L}\tau_i) + g_{k+1,\ \ l}^{n}(\ \tau^L)(\sum_{i\epsilon L}\tau_i\ - (n-k-1))$
$+\int_0^1 g_{k,l+1}^{n}(\ \tau^{L\cup n}\ )\ (k + \sum_{\{\ i\epsilon\ L\ \}}\tau_i) - (n-1)\ \tau_n)\ d\tau_n\ = 0,$ (5)

For such pairs (k,l) that $k \geq 0,\ l \geq 0,\ k+l \leq n-1$, besides $(0,0)$ *and* $(n-1,\ 0)$. In addition will also take place the following equations:

$$\Delta^n = (\mathrm{n}-1)\ (g_{n-1,\ \ 0}^{n} +\ \int_0^1 g_{n-1,1}^{n}(\ \tau_n)\ (1-\ \tau_n)\ d\tau_n$$
$$= (\mathrm{n}-1)\ (g_{1,\ \ 0}^{n} +\ \int_0^1\ \tau_n g_{1,0}^{n}(\ \tau_n)\ \ d\tau_n\ ) \qquad (6)$$

Proof. We will derive the conditions (5) based on axiom $A3$. That way we will prove the necessity of our statement. While doing so, due to the reversibility of applied judgments we will also prove the sufficiency of this proposition.

We will accept, that the player n is a 'dummy', i.e. $v(\tau) = v(\tau || \tau_n{=}0)$, for all $\tau\ \epsilon\ \mathrm{T}$.

The integral equations below take place due to the 'dummies' property:

$$\int_{<K\cup n,\ \ L>} v(\tau)\quad g_{k+1,\ \ l}^{n}(\ \tau^L)d\ \tau^L$$
$$=\ \int_{<K,\ \ L>} v(\tau)\quad g_{k+1,\ \ l}^{n}(\ \tau^L)d \qquad (7)$$
$$\int_{<K,\ \ L\cup n>} v(\tau)\quad g_{k,\ \ l+1}^{n}(\ \tau^{L\cup n})d\ \tau^L$$
$$= \int_{<K,\ \ L>} v(\tau)\quad [\int_0^1\ g_{k,\ \ l+1}^{n}(\ \tau^{L\cup n})d\tau_n\ ]d\ \tau^L \qquad (8)$$

From formula (1) for the component $F_n\ (v)$ by using equalities (7) and (8), we will obtain the following expression:

$F_n\ (v) = \frac{1}{n\Delta^n}\{v(1)[\Delta^n - (\mathrm{n}-1)\,(g^n_{n-1,\ 0} + \int_0^1 g^n_{n-1,\ 1}(\tau_n)\ d\tau_n)]$-$v\,(0)[\Delta^n - (\mathrm{n}-1)\,(g^n_{1,0} + \int_0^1 \tau_n\, g^n_{0,\ 1}(\tau_n)d\tau_n)] +$
$\sum_{<K,L>\in:=\{<K,\ L>:K,\ L\ \subset N\setminus n\ ,<K,L>\neq<\emptyset,\ \emptyset>,\ <N\setminus n,\ \emptyset>\}}$
$\int_{<K,\ L>} v(\tau)\ g^n_{k+1,\ l}(\tau^L)(n-k-1-\sum_{i\epsilon L}\tau_i)$
$-g^n_{k,\ l}(\tau^L)(k+\sum_{i\epsilon L}\tau_i)$
$+\ \int_0^1 g^n_{k,l+1}(\tau^{L\cup n})\,((n-1)\,\tau_n - k$
$-\sum_{\{\,i\epsilon\, L\,\}}\tau_i))\,d\tau_n]\ \ d\,\tau^L\ \}.$

$F_n\ \ (v) = 0$ based on the axiom $A3$. At the same time $v\,(0) \neq 0$, $v(1) \neq 0$ and $v(\tau)$ is for arbitrary coalition τ. From there, the expressions in square brackets should be equal to 0. Hence, as a result we will obtain the equations (5) and (6).

To conclude it is remaining to show, that the equations (6) can be derived from (5), to what will be devoted the rest of the proof.

Let consider the following magnitudes:

$u_{k,l} = \dfrac{1}{n-1}c_{n-1}\,(k,l,n-k-l-1)$
$[l\alpha^n_{k,l} + l(l-1)\beta^n_{k,l} + 2kl\gamma^n_{k,l} + k^2\delta^n_{k,l}]$

$v_{k,l} = \dfrac{1}{n-1}c_{n-1}\,(k-1,l,n-k-l)$
$[l\alpha^n_{k,l} + l(l-1)\beta^n_{k,l}$
$+(2k-n-1)l\gamma^n_{k,l} + (k-n)(k-1)\delta^n_{k,l}]$

$f_{k,l} = \Delta^n_{k,l} - u_{k,l} - v_{k,l}.$ (9)

That statement we will prove trough the next four lemmas.

Lemma 1. Takes place the following equation:

$$f_{k,l} = -u_{k,l} - v_{k+1,\ l-1}. \quad (10)$$

Proof. Let $L\subset\ N$, and $j\,\epsilon\,\mathrm{L}$. We will consider the integral below:

$h_{k,l} = \dfrac{c_{n-1}(k,l-1,n-k-l)}{n-1}$
$\int_{<L>}(k+\sum_{\{\,i\,\epsilon\,L\setminus j\}}\tau_i)\begin{pmatrix}k+\sum_{i\epsilon L\setminus j}\tau_i\\ -(n-1)\tau_j\end{pmatrix}g^n_{k,\ l}(\tau^L)d\,\tau^L$.

Let transform the sub integral expression by using the symmetry of functions $g^n_{k,l}(\tau^L)$ and invented in the beginning notations:

$h_{k,l} = \frac{c_{n-1}(k,l-1,n-k-l)}{n-1}\int_{<L>} g^n_{k,\ l}(\tau^L)(k^2 + (l-1)\tau_i{}^2$
$+ (l-1)(l-n-1)\tau_i\,\tau_j + k(2l-n-1)\tau_i\ d\tau^L =$
$\frac{c_{n-1}(k,l-1,n-k-l)}{n-1}\ [(l-1)\alpha^n_{k,l} + (l-1)].$

The expression below we will obtain by applying the condition (5) to the integral $h_{k,l}$:

$h_{k,l} = \dfrac{c_{n-1}(k,l-1,n-k-l)}{n-1}$
$[\int_{<L\setminus j>}(k+\sum_{i\epsilon LJay}\tau_i)^2\,g^n_{k,\ l-1}(\tau^{L\setminus j})d\,\tau^{L\setminus j}\ -$
$\int_{<L>}(k+\sum_{\{\,i\,\epsilon\,L\setminus j\}}\tau_i)$
$\left(k+\sum_{i\epsilon L\setminus j}\tau_i - (n-1)\right)g^n_{k,\ l}(\tau^{L\setminus j})d\,\tau^{L\setminus j}$
$= -u_{k,l-1} - v_{k+1,\ l-1}.$

That is what has been required to prove.

Lemma 2. For every $m \geq 1$, takes place the following presentation:

$\Delta^n = \sum_{l=0}^{n}\sum_{k=0}^{n-l}\Delta^n_{k,l} = \sum_{l=0}^{m-1}\sum_{k=0}^{n-l}\Delta^n_{k,l} + \sum_{k=0}^{n-m}f_{k,m}$.(12)

Proof. This statement we will prove by using the method of mathematical induction.

1. Let prove first, that the statement is correct for $m = n$. Really, as far as

$u_{0,\ n} = v_{0\,,n} = 0$, so
$\Delta^n = \sum_{l=0}^{n}\sum_{k=0}^{n-l}\Delta^n_{k,l}$
$= \sum_{l=0}^{n-1}\sum_{k=0}^{n-l}\Delta^n_{k,l} + \Delta^n_{0,\ n} - u_{0,\ n} - v_{0\,,n}$
$= \sum_{l=0}^{n-1}\sum_{k=0}^{n-l}\Delta^n_{k,l} + f_{0,\ l}$.

2. Accept that the expression (12) is correct for arbitrary m. We will use the formula (11) of the *Lemma 1*, as well as the equalities:

$$u_{n-m+1,\ m-1} = v_{0\ ,\ m-1} = 0.$$

Below we have performed some quite natural transformations:

$\Delta^n = \sum_{l=0}^{m-2}\sum_{k=0}^{n-l}\Delta^n_{k,l} + \sum_{k=0}^{n-m+1}\Delta^n_{k,\ m-1} + \sum_{k=0}^{n-m}f_{k,m}$
$= \sum_{l=0}^{m-2}\sum_{k=0}^{n-l}\Delta^n_{k,l} + \sum_{k=0}^{n-m+1}\Delta^n_{k,\ m-1} -$
$\sum_{k=0}^{n-m}u_{k,\ m-1} - \sum_{k=0}^{n-m}v_{k+1,\ m-1}$
$= \sum_{l=0}^{m-2}\sum_{k=0}^{n-l}\Delta^n_{k,l} + \sum_{k=0}^{n-m+1}\Delta^n_{k,\ m-1} -$
$\sum_{k=0}^{n-m+1}u_{k,\ m-1} - \sum_{k=0}^{n-m+1}v_{k,\ m-1}$
$= \sum_{l=0}^{m-2}\sum_{k=0}^{n-l}\Delta^n_{k,l} +$
$\sum_{k=0}^{n-m+1}(\Delta^n_{k,\ m-1} - u_{k,\ m-1} - v_{k,\ m-1})$
$= \sum_{l=0}^{n-2}\sum_{k=0}^{n-l}\Delta^n_{k,l} + \sum_{k=0}^{n-m+1}f_{k,m-1}$

The lemma has been proved.

Lemma 3. $\Delta^n = (\mathrm{n}-1)\,(g^n_{n-1,\ 0} + \int_0^1(1-\tau_n)g^n_{n-1,\ 1}(\tau_n)\ d\tau_n)$.

Proof. For $m = 1$, according to presentation from the *Lemma 2*,

$\Delta^n = \sum_{k=0}^{n-l}\Delta^n_{k,0} + \sum_{k=0}^{n-1}f_{k,1} = \Delta^n_{1,0} + \Delta^n_{n-1,0} +$
$\sum_{k=2}^{n-2}\Delta^n_{k,0} + f_{0,1} + f_{n-1,1} + \sum_{k=1}^{n-2}f_{k,1}.$

Based on formulas (11) and (10),

$\Delta^n = \Delta^n_{1,0} + \Delta^n_{n-1,0} + f_{0,1} + f_{n-1,1} +$
$\sum_{k=2}^{n-2}\Delta^n_{k,0} - \sum_{k=1}^{n-2}u_{k,0} - \sum_{k=1}^{n-2}v_{k+1,\ 0} =$
$\Delta^n_{1,0} + \Delta^n_{n-1,0} + f_{0,1} + f_{n-1,1} - u_{1,0} - v_{n-1,\ 0} +$
$\sum_{k=2}^{n-2}(\Delta^n_{k,0} - u_{k,0} - v_{k,\ 0}) = \Delta^n_{1,0} + \Delta^n_{n-1,0}$

$+ f_{0,1} + f_{n-1,1} - u_{1,0} - v_{n-1,\ 0} - \sum_{k=2}^{n-2} f_{k,0}$.

Let submit in the obtained expression submit values for the following magnitudes:

$\sum_{k=2}^{n-2} f_{k,0} = 0\ \Delta_{1,0}^{n} = u_{1,0} = g_{1,\ 0}^{n}$,

$\Delta_{n-1,0}^{n} - v_{n-1,\ 0} = (\mathrm{n}-1)\ g_{n-1,\ 0}^{n}$.

$f_{n-1,1} = (\mathrm{n}-1)\int_0^1 (1-\tau_n) g_{n-1,\ 1}^{n}(\tau_n)\ d\tau_n$,

We will have the final expression

$$\Delta^n = (\mathrm{n}-1)\left(g_{n-1,\ 0}^{n} + \int_0^1 (1-\tau_n) g_{n-1,\ 1}^{n}(\tau_n)\ d\tau_n\right).$$

Lemma 4. $g_{n-1,\ 0}^{n} + \int_0^1 (1-\tau_n) g_{n-1,\ 1}^{n}(\tau_n)\ d\tau_n$

$= g_{1,\ 0}^{n} + \int_0^1 \tau_n g_{0,\ 1}^{n}(\tau_n)\ d\tau_n$.

Proof. Below we are integrating the equality (5) by the domain $< K, L >$ and using the notations above:

$$0 = \int_{<L>} g_{k,\ l}^{n}(\tau^L)\left(k + \sum_{i\epsilon L}\tau_i\right)^n d\tau^L +$$

$$\int_{<L>} g_{k+1,\ l}^{n}(\tau^L)\left(\sum_{i\epsilon L}\tau_i - (n-k-1)\right) d\tau^L +$$

$$\int_{<L>}\left(\int_0^1 g_{k,l+1}^{n}(\tau^{L\cup n})\left(k + \sum_{\{i\epsilon L\}}\tau_i - (n-1)\tau_n\right) d\tau_n\right) d\tau^L$$

$$= k\int_{<L>} g_{k,\ l}^{n}(\tau^L)\ d\tau^L$$

$$+ k\int_{<L>} g_{k,\ l+1}^{n}(\tau^{L\cup n})\ d\tau^L$$

$$+ l\int_{<L>} \tau_i g_{k,\ l}^{n}(\tau^L)\ d\tau^L +$$

$$(k+1-n)\int_{<L>} g_{k+1,\ l}^{n}(\tau^L)\ d\tau^L +$$

$$\int_{<L>} \tau_i g_{k+1,\ l}^{n}(\tau^L)\ d\tau^L +$$

$$(l+1-n)\int_{<L\cup n>} \tau_n g_{k,\ l+1}^{n}(\tau^{L\cup n})\ d\tau^{L\cup n}$$

$$= (l+1-n)\gamma_{k,l+1}^{n} + l\gamma_{k,l}^{n} + l\gamma_{k+1,l}^{n}$$

$$+(k+1-n)\delta_{k+1,l}^{n} + k\delta_{k,l+1}^{n} + k\delta_{k,l}^{n}.$$

If to multiply the equalities above by $c_{n-1}(k, l, n-k-l-1)$ and sum them by all pairs (k, l), for what takes place the (5), then we will obtain:

$$\zeta = \sum_{l=0}^{n-1}\sum_{k=0}^{n-l-1}[(l+1-n)\gamma_{k,l+1}^{n} + l\gamma_{k,l}^{n} + l\gamma_{k+1,l}^{n}$$

$$+(k+1-n)\delta_{k+1,l}^{n} + k\delta_{k,l+1}^{n}$$

$$+ k\delta_{k,l}^{n}]c_{n-1}(k, l, n-k-l-1).$$

As far as $\gamma_{k,0}^{n} = \delta_{0,l}^{n} = 0$, so we will have, that

$$\zeta = \sum_{l=1}^{n-1}\sum_{k=0}^{n-l-1}\{\gamma_{k,l}^{n}[c_{n-1}(k, l, n-k-l-1) +$$

$$(l-n)c_{n-1}(k,\ l-1,\ n-k-l) +$$

$$l c_{n-1}(k,\ l-1,\ n-k-l)] +$$

$$\sum_{l=0}^{n-l-1}\sum_{k=0}^{n-l-1}\delta_{k,l}^{n}[(k-n)c_{n-1}(k-1,\ l,\ n-k-l-1)$$

$$+k c_{n-1}(k,\ l-1,\ n-k-l)$$

$$+k\ c_{n-1}(k,\ l,\ n-k-l-1)]\} =$$

$$\sum_{l=1}^{n-1}\sum_{k=0}^{n-l-1}\{\gamma_{k,l}^{n}[\tfrac{l(n-1)!}{k!l!(n-k-l-1)!} + \tfrac{(l-n)(n-1)!}{k!(l-1)!(n-k-l)!} +$$

$$\tfrac{l\,(n-1)!}{(k-1)!l!(n-k-l)!}] + \sum_{l=0}^{n-l-1}\sum_{k=0}^{n-l-1}\delta_{k,l}^{n}[\tfrac{(k-n)(n-1)!}{(k-1)!l!(n-k-l)!} +$$

$$\tfrac{k(n-1)!}{k!(l-1)!(n-k-l)!}].$$

Expressions in the square brackets are equal to each other:

$$\frac{(n-1)!}{(k-1)!(l-1)!(n-k-l-1)!}\left[\frac{1}{k} + \frac{l-n}{k(n-k-l)} + \frac{1}{n-k-l}\right] = 0$$

$$\frac{(n-1)!}{(k-1)!(l-1)!(n-k-l-1)!}\left[\frac{1}{l} + \frac{k-n}{l(n-k-l)} + \frac{1}{n-k-l}\right] = 0.$$

As a result $\zeta = 0$.

Now recall that (5) takes place for pairs (k, l), where that $k \geq 0,\ l \geq 0,\ k + l \leq n-1$, besides the pairs $(0,0)$ and $(n-1,\ 0)$.

If in the written above expression to separate from the sum members that correspond to pairs $(0,0)$ and $(n-1,\ 0)$, then based on (5) the remaining will be equal to 0.

The separated members will give us the following expression:

$$(n-1)\gamma_{n-1,1}^{n} + (n-1)\delta_{n-1,1}^{n} + (n-1)\delta_{n-1,0}^{n}$$

$$-(n-1)\gamma_{0,1}^{n} - (n-1)\delta_{1,0}^{n} = 0$$

By substituting the values for participating magnitudes, we will obtain what has been required:

$$g_{n-1,\ 0}^{n} + \int_0^1 (1-\tau_n) g_{n-1,\ 1}^{n}(\tau_n)\ d\tau_n = g_{1,\ 0}^{n} + \int_0^1 \tau_n g_{0,\ 1}^{n}(\tau_n)\ d\tau_n.$$

The last two lemmas conclude the proof of preposition 2.

4. Properties of the Function $F(v)$

Let $F(v)$ be the pre-value for the game v. That means, it is the point on what the functional $Q(x,v)$ reaches its minimum, and participating in $Q(x,v)$ functions $g_{k,l}^{n}(\tau^L)$ satisfy to the specified for $\boldsymbol{F(v)'s}$ Shapley axioms conditions.

In this paragraph we will show, that for a class of fuzzy games obtained from the classical cooperative games through the Owen's multilinear extension, the prevalue $\boldsymbol{F(v)}$ coincides with the Shapley prevalue for the classical game. Further, we will obtain necessary and sufficient conditions at what the function $\boldsymbol{F(v)}$ satisfies to the strengthened dummies axiom. We will also prove a preposition allowing constructing functions $\overline{g_{k,l}^{n}}(\tau^L)$ that satisfy to the axiom $A3'$ for fewer players.

4.1. Fuzzy games we are going to deal with defined through certain n dimensional distribution functions $\psi_n(x_1, \ldots, x_n)$, that are continuous by each one of their arguments and satisfy to following condition:

$$\psi_n(x_1, \ldots, x_{i-1}, x_i, \ldots, x_n)$$

$$+ \psi_n(x_1, \ldots, x_{i-1}, 1 - x_i, \ldots, x_n)$$

$$= \bar{\psi}_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\ \ (a)$$

where $\bar{\psi}_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is some $(n-1)$ dimensional distribution function of variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ and possesses with the same kind of properties as functions $\psi_n(x_1, \ldots, x_n)$.

Preposition 3. Let (T,v) is a fuzzy game that obtained from a classical game $(N,\overline{v})$ by the following way:

$$v(\tau)=\sum_{S\subset N}\ \psi_n(\hat{\tau}^S)\bar{v}(S), \qquad (13)$$

where

$$\hat{\tau}_i^S=\begin{cases}\tau_i, & if\ i\in S\\ 1-\tau_i & if\ i\notin S\end{cases}$$

and $\psi_n(\hat{\tau}^S)$ is a function that satisfies to the condition (a). If $\Phi(\overline{v})$ is the Shapley prevalue for game $(N,\overline{v})$ and $F(v)$ the prevalue for fuzzy game (T,v) then $\Phi(\overline{v})=F(v)$.

Proof. Let $F(v)=(F_1(v),\dots,F_n(v))$ is the solution for system $(2)-(3)$ of game (T,v). By using the explicit expression (4) for $F_i(v)$ and formula (13) we can prove that $F_i(v)$ allows some linear representation that depends on $\bar{v}(S)$:

$$F_i(v)=\sum_{S\subset N} b_{S,i}\,\overline{v}(S)=x_i(\bar{v}).$$

The axioms $A1,\ A2,\ A4$ for $x(\overline{v})$ hold true due to properties of $F(v)$. So to prove our statement, we need to show, that for $x_i(\bar{v})$ takes place the 'dummies' axiom. So, we need to prove that if a player $i_0\in N$ is a dummy in the game $(N,\overline{v})$, then i_0 is so also in (T,v). Let for all $S\subset N$, $\overline{v}(S\cup i_0)=\overline{v}(S)$. Then

$$v(\tau)=\sum_{S\subset N}\psi(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_n)\,\overline{v}(S)=$$
$$\sum_{S\not\ni i_0}[\psi(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_{i_0},\dots,\hat{\tau}^S{}_n)\overline{v}(S)+$$
$$\psi(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_{i_0},\dots,\hat{\tau}^S{}_n)\overline{v}(S)$$
$$+\psi(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_{i_0-1},1-\hat{\tau}^S{}_{i_0},\hat{\tau}^S{}_{i_0+1}\dots,\hat{\tau}^S{}_n)\overline{v}(S\cup i_0)$$
$$=\sum_{S\not\ni i_0}[\hat{\psi}(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_{i_0-1},\hat{\tau}^S{}_{i_0+1}\dots,\hat{\tau}^S{}_n)\overline{v}(S)$$

By the other side:

$$v(\tau\|\tau_{i_0}=0)=$$
$$\sum_{\{S\subset N,\ S\not\ni i_0\}}[\psi(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_{i_0-1},0,\hat{\tau}^S{}_{i_0+1}\dots,\hat{\tau}^S{}_n)$$
$$++\psi(\hat{\tau}^S{}_1,\dots,\widehat{\tau^S{}_{i_0-1}},1,\hat{\tau}^S{}_{i_0+1},\dots,\tau^S{}_n)]\,\overline{v}(S)=$$
$$=\sum_{\{S\subset N,\ S\not\ni i_0\}}\psi(\hat{\tau}^S{}_1,\dots,\hat{\tau}^S{}_{i_0-1},\hat{\tau}^S{}_{i_0+1},\dots,\hat{\tau}^S{}_n)\overline{v}(S).$$

From there $v(\tau)=v(\tau\|\tau_{i_0}=0)$, and hence, $x_{i_0}(\overline{v})=F_{i_0}(v)=0$.

As a result we have obtained that the prevalue $x(\overline{v})$ satisfies to all of the Shapley axioms, which means, that $x(\overline{v})=\Phi(\overline{v})$.

Remark. In a special case, when players participate in coalition independently of each other, i.e., when

$$\psi(\tau_1,\dots,\tau_n)=\prod_{\{i\epsilon S\}}\tau_i\prod_{\{j\notin S\}}(1-\tau_j),$$

then $v(\tau)$ is a multilinear extension of Owen. One can check it easily, that then for functions $\psi(\tau_1,\dots,\tau_n)$ takes place the condition (a).

Below we will derive relations that concern to the strengthened axiom of 'dummy'.

Preposition 4. Let $\Delta^n\neq 0$, $\Delta^{n+1}\neq 0$. The function $F(v)$ in an $n-$person game satisfies to the strengthened 'dummies' axiom if and only if, when for all such pairs (k,l) that $k\geq 0,\ l\geq 0, k+l\leq n$, besides the $(0,0)$ and $(n,\ 0)$, takes place the following equation:

$$\frac{1}{\Delta^{n+1}}[g^{n+1}_{k,\ l}(\tau^L)+g^{n+1}_{k+1,\ l}(\tau^L)+\int_0^1 g^{n+1}_{k,\ l+1}(\tau^{L\cup\{n+1\}})d\tau_{n+1}]=\frac{1}{\Delta^n}g^n_{k,\ l} \qquad (14)$$

and the equation (5) for the $(n+1)-$players game:

$$g^{n+1}_{k,\ l}(\tau^L)\big(k+\sum_{j\epsilon L}\tau_j\big)+$$
$$g^{n+1}_{k+1,\ l}(\tau^L)\big(k-n-\sum_{j\epsilon L}\tau_j\big)+$$
$$\int_0^1 g^{n+1}_{k,l+1}\big(\tau^{L\cup\{n+1\}}\big)(k+\sum_{\{j\,\epsilon\,L\}}\tau_j-n\,\tau_{n+1})\,d\tau_n=0 \quad (5')$$

Proof. Necessity. The equation (5') follows from the *Proposition 2*, because, if the axiom *A2.* holds true, then *A3* follows from the *A3'*.

The same way, as we did it in case of formula (2.1), let again write the expression for $F_i\ (v)$ by separating the player $n+1$, while summing by the side $<K,L>$ of the cube and replacing $u(\tau\ ,\tau_{n+1})$ by $v(\tau)$, based on the equation $u(\tau\ ,\tau_{n+1})=v(\tau)$:

$$F_i\ (v)=\frac{1}{n+1}\ \{v(1)-v(0)+\frac{1}{\Delta^{n+1}}$$
$$\sum_{\{<K,L>,\ i,\{n+1\}\notin K,L,\ K\cap L=\emptyset\}}$$
$$[\int_{<K\cup i,\ L>}v(\tau)(g^{n+1}_{k+1,\ l}(\tau^L)(n-k-\sum_{j\epsilon L}\tau_j)d\tau^L+$$
$$\int_{<K,\ L\cup i>}v(\tau)g^n_{k,\ l+1}(\tau^{L\cup i})(n\tau_i-k-\sum_{j\epsilon L}\tau_j)\,d\tau^{L\cup i}-$$
$$\int_{<K,\ L>}v(\tau)(g^{n+1}_{k,\ l}(\tau^L)\big(k+\sum_{j\epsilon L}\tau_j\big)d\tau^L+$$
$$\int_{<K\cup i\cup\{n+1\},\ L>}v(\tau)(g^{n+1}_{k+2,\ l}(\tau^L)(n-k-1-\sum_{j\epsilon L}\tau_j)d\tau^L+$$
$$\int_{<K\cup\{n+1\},\ L>}v(\tau)(g^{n+1}_{k+1,\ l+1}(\tau^{L\cup i})\big(n-k-1-\sum_{j\epsilon L}\tau_j\big)d\tau^L+$$
$$\int_{<K\cup\{n+1\},\ L>}v(\tau)(g^{n+1}_{k+1,\ l}(\tau^L)\big(k+1+\sum_{j\epsilon L}\tau_j\big)d\tau^L+$$
$$\int_{<K\cup i,\ L\cup\{n+1\}>}v(\tau)(g^{n+1}_{k+1,\ l+1}\big(\tau^{L\cup\{n+1\}}\big)\big(n\tau_i-k-\sum_{j\epsilon L}\tau_j\big)d\tau^L+$$
$$\int_{<K,\ L\cup\{n+1\}>}v(\tau)(g^{n+1}_{k+1,\ l}\big(\tau^{L\cup\{n+1\}}\big)\big(k+\sum_{j\epsilon L\cup\{n+1\}}\tau_j\big)d\tau^L]\} \qquad (15)$$

The formula (1) also gives as an expression for $F_i\ (u)$. According to the strengthened axiom of 'dummy', gains for the same player $i\in$ N in both of the games (T^n,v) and (T^{n+1},u) are the same. That is the reason because of what the obtained for $F_i\ (v)$ and $F_i\ (u)$ expressions are equal. Let write that equation:

$$\frac{1}{n}\ \{\ v(1)-v(0)+\frac{1}{\Delta^n}$$
$$\sum_{\{<K,L>,\ i\notin K,L,\ K\cap L=\emptyset\}}$$
$$\int_{<K\cup i,\ L>}v(\tau)(g^n_{k+1,\ l}(\tau^L)(n-k-1-\sum_{j\epsilon L}\tau_j)d\tau^L+$$
$$\int_{<K,\ L\cup i>}v(\tau)(g^n_{k,\ l+1}(\tau^{L\cup i})((n-1)\tau_i-k-\sum_{j\epsilon L}\tau_j)\,d\tau^{L\cup i}-$$
$$\int_{<K,\ L>}v(\tau)(g^n_{k,\ l}(\tau^L)\big(k+\sum_{j\epsilon L}\tau_j\big)d\tau^L]\}=\frac{1}{n+1}$$
$$\{(1)-v(0)+\frac{1}{\Delta^{n+1}}$$
$$\sum_{\{<K,L>,\ i,\{n+1\}\notin K,L,\ K\cap L=\emptyset\}}$$
$$[\int_{<K\cup i,\ L>}v(\tau)(g^{n+1}_{k+1,\ l}(\tau^L)(n-k-\sum_{j\epsilon L}\tau_j)d\tau^L+$$

$\int_{<K,\ L\cup i>} v(\tau)(g^n_{k,\ l+1}(\tau^{L\cup i})(n\tau_i - k - \sum_{j\epsilon L}\tau_j) +$
$g^{n+1}_{k+1,\ l+1}(\tau^{L\cup i})(n\tau_i - k - \sum_{j\epsilon L}\tau_j) +$
$\int_0^1 g^{n+1}_{k+1,\ l+2}(\tau^{L\cup i\cup\{n+1\}})(n\tau_i - k - \sum_{j\epsilon L\cup\{n+1\}}\tau_j)\, d\tau^{L\cup i} -$
$\int_{<K,\ L>} v(\tau)(g^{n+1}_{k,\ l}(\tau^L)(k + \sum_{j\epsilon L}\tau_j)$
$+g^{n+1}_{k,\ l+1}(\tau^L)(k + 1 + \sum_{j\epsilon L}\tau_j)$
$+ \int_0^1 g^{n+1}_{k,\ l+1}(\tau^{L\cup\{n+1\}})(k + \sum_{j\epsilon L\cup\{n+1\}}\tau_j)\, d\,\tau_{n+1})d\tau^L\,]\}.$

By equalizing coefficients at $v(\tau)$ for similar coalitions in both of the sides of equation we will obtain relations that connect functions $g^n_{k,l}(\tau^L)$ and $g^{n+1}_{k,l}(\tau^L)$. Forms of the mentioned coefficients depend on the cubes sides to what coalition τ belongs. More precisely, to which one of the sets K, L or $N\backslash(K\cup L)$ belongs the player i. Besides that special views have the coefficients at $v(1)$ and $v(0)$. As a result we will have, that the equality $F_i\ (v) = F_i\ (u)$ (i.e. the axiom $A3$') is equivalent to the following five relations.

The equality of coefficients at $v(\tau)$ for $\tau\,\epsilon < K, L >$, where $i \notin K, L$ (besides the side $<\emptyset,\emptyset>$) gives as the relation:

$$\frac{1}{n\Delta^n} g^n_{k,\ l}(\tau^L)(k + \sum_{j\epsilon L}\tau_j)$$
$$= \frac{1}{(n+1)\Delta^{n+1}} g^{n+1}_{k,\ l}(\tau^L)(k + \sum_{j\epsilon L}\tau_j)$$
$$+g^{n+1}_{k+1,\ l}(\tau^L)(k + 1 + \sum_{j\epsilon L}\tau_j) +$$
$$\int_0^1 g^{n+1}_{k,\ l+1}(\tau^{L\cup\{n+1\}})(k + \sum_{j\epsilon L\cup\{n+1\}}\tau_j)\, d\tau_{n+1}\}. \quad (16)$$

for all pairs (k,l) that $k \geq 0,\ l \geq 0,\ k + l \leq n - 1$, besides the pair $(0,0)$.

For the sides $< K, L >$, where K∋ i, besides the side $< N,\emptyset >$, we will obtain the following relation:

$$\frac{1}{(n+1)\Delta^{n+1}} g^{n+1}_{k,\ l}(\tau^L)[(n - k + 1 - \sum_{j\epsilon L}\tau_j)$$
$$+g^{n+1}_{k+1,\ l}(\tau^L)(n - k - \sum_{j\epsilon L}\tau_j) +$$
$$\int_0^1 g^{n+1}_{k,\ l+1}(\tau^{L\cup\{n+1\}})(n - k + 1 - \sum_{j\epsilon L\cup\{n+1\}}\tau_j)\, d\tau_{n+1}] \quad (17)$$

which takes place for all pairs (k,l) that
$k \geq 0,\ l \geq 0, k + l \leq n$, besides the pair $(n,0)$.

For the sides $< K, L >$, where $L \ni i$, we will obtain the following relation:

$$\frac{1}{n\Delta^n} g^{n+1}_{k,\ l}(\tau^L)(n\tau_i - k - \sum_{j\epsilon L}\tau_j) =$$
$$\frac{1}{(n+1)\Delta^{n+1}}\{g^{n+1}_{k,\ l}(\tau^L)((n+1)\tau_i - k - \sum_{j\epsilon L}\tau_j) +$$
$$g^{n+1}_{k+1,\ l}(\tau^L)\left((n+1)\tau_i - k - 1 - \sum_{j\epsilon L}\tau_j\right) +$$
$$\int_0^1 g^{n+1}_{k,\ l+1}(\tau^{L\cup\{n+1\}})\left((n+1)\tau_i - k - \sum_{j\epsilon L\cup\{n+1\}}\tau_j\right) d\tau_{n+1}\} \quad (18)$$

which is correct for all pairs (k,l) that $k \geq 0,\ l \geq 1$,
$k + l \leq n$.

Finally, by equalizing the coefficients at $v(1)$ and $v(0)$ we will obtain

$$\frac{1}{(n+1)\Delta^{n+1}}[\Delta^{n+1} + g^{n+1}_{n,\ 0} +$$
$$\int_0^1 (1 - \tau_{n+1}) g^{n+1}_{n,\ 1}(\tau_{n+1})\ d\tau_{n+1} = \frac{1}{n} \quad (19)$$
$$\frac{1}{(n+1)\Delta^{n+1}}[\Delta^{n+1} + g^{n+1}_{1,\ 0} +$$
$$\int_0^1 \tau_{n+1} g^{n+1}_{0,\ 1}(\tau_{n+1})\ d\tau_{n+1} = \frac{1}{n} \quad (20)$$

Adding to each other the equalities (16) and (17) we will obtain that (14) takes place for pairs

(k,l) where $k \geq 1,\ l \geq 0,\ k + l \leq n - 1$.

Adding to each other equalities (16) and (18) and dividing the sum by τ_i we will obtain that (14) takes place for pairs (k,l) where $k \geq 0,\ l \geq 1,\ k + l \leq n - 1$.

Subtracting from each other the equalities (16)and (18) and dividing the result by $(1 - \tau_i)$ we will have that (14) takes place for pairs (k,l), where $k \geq 1,\ l \geq 1$, $k + l \leq n$, besides the pair $(n,0)$.

So, we have proved that the equality (14) takes place for all pairs (k,l) that $k \geq 0,\ l \geq 0,\ k + l \leq n$, besides the pairs $(0,0)$ and $(n,0)$.

The Proposition (14) has been proved fully.

Below proved one more proposition that allows constructing functions $\overline{g^n_{k,l}}(\tau^L)$ *such* that for $n < n_0$ satisfy equations (5) if they do the same for n_0.

Proposition 5. Let for some $n_0\ \Delta^{n_0} \neq 0$ and for functions $\overline{g^{n-1}_{k,l}}(\tau^L)$,

$$\overline{g^{n-1}_{k,l}}(\tau^L) = \frac{c}{\Delta^n}\ [g^n_{k,\ l}(\tau^L) + g^n_{k+1,\ l}(\tau^L) +$$
$$\int_0^1 g^{n+1}_{k,\ l+1}(\tau^{L\cup n})\, d\tau_n] \quad (21)$$

take place equations (5). Then $\Delta^{n-1}(\overline{g})$ =c and functions $\overline{g^{n-1}_{k,l}}(\tau^L)$ satisfy to the equation (5) too.

Proof. The equation $\Delta^{n-1}(\overline{g})$ =c can be proved by using equations (21) and the known earlier formula, but this time for $n - 1$ players, i.e.

$$\Delta^{n-1} = \sum_{l=0}^{n}\sum_{k=0}^{n-l-1}\Delta^{n-1}_{k,l}$$

To check that functions $\overline{g^{n-1}_{k,l}}(\tau^L)$ satisfy (5) it is enough using the formula (21) to figure out taking part in (5) magnitudes that written for $(n-1)$, and after that summing them by taking in account that the (5) takes place for the $n-$game.

3. Below are examples for the sets of none negative functions that satisfy to 'dummies' axiom in case of some fixed number of players n.

Example 1. Simplest is the set of functions $g^n_{k,l}$ defined the following way:

$$g^n_{k,\ l} = \begin{cases} (k-1)!\,(n-k-1)! & \text{if} \quad l = 0 \\ 0 \quad \text{if} & l \geq 1 \end{cases}$$

Gain of a player in case of defined functions depends only on game's value on the peaks of cube T^n, which are the coalitions in the classical sense. The functional $Q\ (x,\ v)$ in its turn coincides with the functional $\bar{Q}$(x, v) that defined for classical cooperative games.

Example 2. Let for some n define functions $g_{k,l}^{n}$ the following way:

$g_{k,\ 1}^{n}(\tau)=\frac{1}{(n-1)!}\prod_{j=1}^{n-k-1}(j-\tau)\prod_{j=0}^{k-1}(j+\tau)$

$g_{k,\ l}^{n}(\tau)=0, for\ l\geq 2$,

and by using the formula below recurrently define $g_{k,\ 0}^{n}$:

$g_{k+1,\ l}^{n}(\tau)=\frac{1}{n-k-1}\{ kg_{k,\ 0}^{n}-\int_0^1 g_{k,\ 0}^{n}(\tau)\big((n-1)\tau_i-k\big)d\tau \}$,

where one can choose $g_{1,\ 0}^{n}$ arbitrary, but big enough to get $g_{k,\ 0}^{n}$ none negative.

For l=0 equations (5) immediately follow from the formula that recurrently defines the

functions $g_{k+1,\ 0}^{n}$.

For $l=1$ we have:

$g_{k,1}^{n}(\tau)(k+\tau)+g_{k+1,\ 1}^{n}(\tau)\big(\tau-(n-k-1)\big)$

$+\int_0^1 g_{k,\ 2}^{n}(k+\tau-(n-1)\tau)d\tau=$

$\frac{(k+\tau)}{(n-1)!}\prod_{j=1}^{n-k-1}(j-\tau)\prod_{j=0}^{k}(j+\tau)\ +$

$\frac{(\tau-(n-k-1))}{(n-1)!}\prod_{j=1}^{n-k-1}(j-\tau)\prod_{j=0}^{k}(j+\tau)\ =$

$\frac{1}{(n-1)!}\prod_{j=1}^{n-k-1}(j-\tau)\prod_{j=0}^{k}(j+\tau)$

$-\ \frac{1}{(n-1)!}\prod_{j=1}^{n-k-1}(j-\tau)\prod_{j=0}^{k}(j+\tau)\ =0.$

Further we will apply the *Proposition* 5, by accepting that Δ^{n}= c. Let check, that for $m\leq n$ the functions $g_{k,\ 1}^{m}$ can be defined by the formula (21):

$g_{k,1}^{n+1}(\tau)=g_{k,\ 1}^{n}(\tau)+g_{k+1,\ 1}^{n}(\tau)+$

$\int_0^1 g_{k,\ 2}^{n}(\tau,\ \tau_n)\,d\tau_n$=

$\frac{1}{(n-1)!}\ \prod_{j=1}^{n-k-1}(j-\tau)\prod_{j=0}^{k-1}(j+\tau)$

$+\ \frac{1}{(n-1)!}\ \prod_{j=1}^{n-k-2}(j-\tau)\prod_{j=0}^{k}(j+\tau)=$

$\frac{1}{(n-1)!}\ \prod_{j=0}^{n-k-2}(j-\tau)\prod_{j=0}^{k-1}(j+\tau)(n-k-1-\tau+k+\tau)=$

$\frac{1}{(n-1)!}\ \prod_{j=0}^{n-k-2}(j-\tau)\prod_{j=0}^{k-1}(j+\tau).$

That way we have constructed example of functions $g_{k,l}^{n}(\tau^{L})$ that satisfy to the axiom $A3$' for some number of players n that does not succeed to certain n_0. However, doing the same for arbitrary n this method unfortunately does not allow.

We should mention, that in this example essential are only those coalitions, which get described by peaks and the line segments that connect adjacent peaks of cube T^{n}.

Example 3. For $n=2$ to axiom $A3$ satisfy the following functions that defined on square:

$g_{0,1}^{2}(\tau_1)=\ (1-\tau_1)^2$.

$g_{1,1}^{2}(\tau_1)=\tau_1{}^2$.

$g_{0,2}^{2}(\tau_1\,,\tau_2)=\ 12\tau_1\tau_2\,(1-\tau_1)\,(1-\tau_2).$

Remark 4. All of the functions $g_{k,l}^{n}(\tau^{L})$ that constructed in examples above satisfy to the following condition:

$$g_{k,\ l}^{n}(\tau_1,\dots,\tau_l)\ =g_{k,\ l}^{n}(1-\tau_1,\dots,1-\ \tau_l),$$

which can be considered as another property for the distribution functions $g_{k,l}^{n}(\tau^{L})$.

As a conclusion the author would like to appreciate any information about copyright breach of these results.

ACKNOWLEDGEMENTS

The author thanks the editor and two anonymous referees for their helpful suggestions.

REFERENCES

[1] AUBIN JP (1981). Cooperative fuzzy games. Math Oper.res 6: 1-13.

[2] CASAJUS A. (2013). The Shapley value without efficiency and additivity. Mathematical Social Sciences 68(1): 1-4.

[3] HART S. and MAS-COLELL A. (1989). Potential, value and consistency. Econometrica 57(3): 589-614.

[4] HWANG Y.A. and LIAO Y.H. (2009). The consistent value of fuzzy games. Fuzzy Sets and Systems 160: 644-656.

[5] JIMENENEZ-LOSADA A. et al. (2019). A new Shapley value for games with fuzzy coalitions. Fuzzy Sets and Systems. January 20019.

[6] MYERSON R.B. (1980). Conferences structures and fair allocation rules. International Journal of Game Theory 9: 169-182.

[7] SHAPLEY L.S. (1953). A value for n-person games, in: The Shapley Value, A.W. Tucker and H. Kuhn (Eds.), Annals of Mathematics Studies, Vol. 28. Princeton University Press: 307-317.

[8] SOBOLEV A. (1978). Nonlinear analogues of Shapley function. – Modern Directions in Game Theory - Vilnius, pp.119-126. (In Russian).

[9] TSURUMI M., TANINO T. and INUIGUCHI, M. (2001). A Shapley function on a class of cooperative fuzzy games. European Journal of Operational Research 129: 596-618.

[10] WEBER R.J. (1988). Probabilistic values for games. In: The Shapley Value. Roth A.R. (eds). Cambridge University Press.

[11] YOUNG H.P. (1985). Monotonic solutions of cooperative games. International Journal of Game Theory 14: 65-72.

[12] ZADEH L. A. (1965). Fuzzy sets. Information Control 8: 338-353.