Cyclicity in Cooperative Games – A Theoretical Model for an Infinite Order of Stable Sets in Pillage Games

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Abstract We consider the question whether a pillage game can have infinitely many stable sets under certain regularity and dominance conditions. We establish a rigidity result in continuous pillage games. This property implies that in any continuous pillage game, an allocation's dominance structure is essentially determined by local regularity behaviours of the power function. Using this property, we derive a weak convergence theorem on the power set of the n-players near any point in the allocation space by passing any sequence of allocations through specially constructed filters subject to geometric, parity, and unidirectionality conditions. Combining these results, a sufficient condition for the theoretical existence of an infinite set of stable sets is proposed. We also discuss the extension of our results in the wider scope where some of the conditions are lifted.

Keywords Cooperative game theory, Pillage games, Stable set

1. Introduction

Cooperative games require players to examine their individual strategies as well as to coordinate with other players to maximize the payoff of a subset of related players. These subsets give rise to the concept of coalitions. In pillage games which are a type of cooperative games, a coalition's strength is measured by its current possession of a given set of goods.

As with other cooperative games, stable sets are a key interest. Conceptually, stable sets are locus of allocations for a coalition where once all the players in the coalition attain one such allocation of the goods, they suffer a loss in valuation determined by the power function against players outside of the coalition if one player unilaterally moves away from such allocation, i.e. a dynamic equilibrium is reached, similar to an attractor in a dynamical system. Jordan proved that the cardinality of any given stable set in continuous pillage games is finite [1]. McKenzie, Kerber and Rowat showed that the cardinality of an internally stable set is bounded above by $R_c(4) - 1$ where R_c is the diagonal multicolor Ramsey number and c is the number of players in the game [2]. Examining strictly monotonic sequences in dimensions greater than 1, Saxton showed that $|S|$, the cardinality of a stable set in an n-player pillage game satisfies

$$|S| < 3^{\frac{1}{2}\binom{2d}{d}} \tag{1}$$

where $d = \lfloor (n-1)/2 \rfloor$ is the floor function [3]. Using measure theory, Beardon and Rowat was able to represent any internally stable set as a finite union of sets satisfying certain pairwise comparison conditions, thereby proving that a stable set's topological dimension in an n-player, m-good pillage game is bounded above by $m(n-1) - 1$ [4].

These properties on stable sets prompted the question: can there be a pillage game with infinitely many stable sets, namely is there a pillage game where the collection of stable set is of order $\aleph_0$ or greater? This question is significant in cooperative games. Historically, Shapley found examples of cooperative games that have infinitely many stable sets by using characteristic functions [5]. In pillage games, McKenzie, Kerber, and Rowat constructed an example of a pillage game with multiple independent stable sets [2]. Thus the above question deserves some serious consideration. In this paper, we give a partial answer to this question. We consider the case of a single good continuous pillage game with non-empty core under a forward dominance structure. We examine a theoretical model of a continuous pillage game satisfying the forward dominance condition. The creation of a concrete model depends on a function satisfying the numerical constraints that permits these conditions. Examples will be given in this paper, we defer the task of finding a concrete example satisfying the many numerical properties set forth in his paper since such a concrete example requires the careful coding by a computer language specialist who encodes these properties in an algorithm and tests it on a work station with sufficient computing power to effect such concrete examples. As such, the primary

contribution of this paper is theoretical.

The format of this paper is as follows: section 2 of the paper gives the basic background of pillage games. Section 3 introduces a basic property of continuous pillage games - a rigidity theorem. This theorem gives the local structure of dominance in an infinite sequence of allocations as it approaches its limit point. In section 4, using the rigidity theorem, we establish a weak convergence theorem on the dominance structure of any sequence of allocations by using ultrafilters that select subsequences based on geometric properties, unidirectionality conditions, and parity comparison conditions. In section 5, we combine the above results to show a theoretical model in which a set of sufficient conditions for an infinite set of stable sets exists under a cyclic forward dominance structure.

In the discussions of section 6, we provide a possible future approach to the study of pillage games and indeed cooperative games in general by highlighting certain fibre bundle properties embedded in pillage games and the possibility of a representation theorem for power functions. The fibre bundle properties are a signature property satisfied by pillage games without the requirement of continuity. Fibre bundles considerations are connected to measure-theoretic integrations in the broader sense. Integration over the base in a fibre bundle points to a representation theorem for pillage games similar to the Riesz representation theorem in classical functional analysis. As a result, fundamental questions in these games such as dominance, stability, and power dynamics among coalitions can be examined through questions about their respective fibre-theoretic analogues such as minimax plateau studies, fibre neighbourhood investigations, and local-global duality studies on function spaces over the fibre bases.

2. Fundamentals of Pillage Games

A single good n-player pillage game is a game among the set I of the n players $I = \{1,2,3,.....n\}$ dividing a single good/commodity where the players' shares of the good are represented as vectors in the convex hull X with vertex at unity on each of the n axes in $\mathbb{R}^n$:

$$X \equiv \{(x_1, x_2, x_3, \dots x_n) |\ x_1 \geq 0 \text{ and } \sum x_i = 1\} \quad (2)$$

Each element $x \in X$ is called an allocation of the good. Given an allocation x, its utility with respect to a subset of players $C \subset I$ is measured by a real-valued function called the power function $\pi: 2^I \times X \to \mathbb{R}$ satisfying the three order axioms:

- (WC): if $C \subset C'$, then $\pi(C', x) \geq \pi(C, x)\ \forall\ x \in X$.
- (WR): if $y_i \geq x_i\ \forall\ i \in C \subset \{1,2,3,.....n\}$, then $\pi(C, y) \geq \pi(C, x)$.
- (SR): if $\emptyset \neq C$ and $y_i > x_i\ \forall\ i \in C \subset \{1,2,3,.....n\}$, then $\pi(C, y) > \pi(C, x)$.

The notation 2^I denotes the power set of the n players. Thus a pillage game is a triple (n, π, X) with the above properties. If there is the additional requirement that for each

$C \subset \{1,2,3,.....n\}$, π is continuous on the allocation variable x, then we call (n, π, X) a continuous pillage game.

An allocation a dominates another allocation b, denoted as $a \succ b$ if $\pi(W, b) > \pi(L, b)$ where $W = \{i|a_i > b_i\}$ and $L = \{i|b_i > a_i\}$. We call W the win set and L the lose set. The idea of dominance measured by the power function is that given an allocation, a more dominant player can move toward another allocation that confers higher utility. The power function therefore both informs and is informed by the allocation structures. Thus its properties are of critical interest in pillage games.

Given $A \subset X$, the dominion of A, denoted as $D(A)$ is the set of allocations that are dominated by some element of A: $\{x \in X|\ \exists a \in A \text{ such that } a \succ x\}$. Thus $D(X)$ is the set of allocations that are dominated by at least one other allocation. The core, denoted as K, is the set of un-dominated allocations, i.e. $K = X/D(X)$.

A set A is said to satisfy internal stability if dominance does not occur between allocations of A: $A \cap D(A) = \emptyset$. Clearly the core K is both internally stable and contained in any internally stable set. A is said to satisfy external stability if $A \cup D(A) = X$. A set A is called a stable set if it satisfies both internal and external stability. Jordan showed that an internally stable set of a pillage game can contain only finitely many allocations [1].

3. A Rigidity Theorem

Let (n, π, X) be a continuous pillage game. Let $x \in X$ be an allocation. If A and B are two non-empty and non-intersecting subsets of 2^I, then the power function can assume any of the following 3 all-inclusive but mutually exclusive relations on A and B at x:

$$\pi(A, x) > \pi(B, x), \pi(A, x) = \pi(B, x), \text{ or}$$
$$\pi(A, x) < \pi(B, x)$$

Definition 1. Let x be an allocation in a continuous pillage game (n, π, X).

(1) Let $R_{x,>}$ denote the set of all non-empty and non-intersecting pairs of sets $(A, B) \in 2^I \times 2^I$ such that $\pi(A, x) > \pi(B, x)$.

(2) Similarly, let $R_{x,>,=}$ denote the set of all non-empty and non-intersecting pairs of sets $(A, B) \in 2^I \times 2^I$ such that $\pi(A, x) \geq \pi(B, x)$.

Theorem 1. Given a continuous pillage game (n, π, X), let $x \in X, \exists \delta_x > 0$ such that $\forall \check{x}$ with $|\check{x} - x| < \delta_x$, we have $R_{x,>} \subset R_{\check{x},>}$. Here $|x|$ is the Euclidean norm on $\mathbb{R}^n$.

Proof. We examine the disjoints sets $R_{x,>}$, $R_{x,<}$, and $R_{x,>,=}$. Let $(A, B) \in R_{x,>}$, by definition, $\pi(A, x) > \pi(B, x)$. Denote $\pi(A, x) - \pi(B, x) = \sigma_{A,B,x} > 0$. Notice $R_{x,>}$ is a finite set because the power set $|2^n| < \infty$. In fact

$$R_{x,>} \leq \binom{2^n}{2} \quad (3)$$

Assume without the loss of generality that $R_{x,>}$ has cardinality m: $R_{x,>} = (A_1, B_1), (A_2, B_2), \dots, (A_m, B_m)$. We

also have the corresponding $\sigma_{A_i,B_i,x}$'s, one for each pair of $(A_i, B_i) \in R_{x,>}$. The continuity of (n, π, X) means that the power function $\pi(C, x)$ is continuous on the allocation variable x given a fixed set $C \in I$. This implies for any $\varepsilon > 0$, there is a $\delta_{C,x} > 0$ such that $\forall \check{x}$ with $|\check{x} - x| < \delta_{C,x}$, we have $|\pi(C, x) - \pi(C, \check{x})| < \varepsilon$. One needs take caution that given an x and an ε, unless uniform continuity is assumed, different subsets of $I = \{1,2,3,\dots\dots n\}$ will require different δ's, one for each subset $C \in I$.

Let $(A_i, B_i) \in R_{x,>}$, as before, we have $\pi(A_1, x) - \pi(B_1, x) = \sigma_{A_1,B_1,x} > 0$. Thus we have

$$\frac{\sigma_{A_i,B_i,x}}{3} > 0 \tag{4}$$

By the continuity of the power function π, we choose two neighborhoods: a $\delta_{i,A}$-neighbourhood for the set A_i and a $\delta_{i,B}$-neighbourhood for the set B_i such that $\forall \check{x}$ with $|\check{x} - x| < \delta_{i,A}$ and $|\check{x} - x| < \delta_{i,B}$ we have

$$|\pi(A_i, x) - \pi(A_i, \check{x})| < \frac{\sigma_{A_i,B_i,x}}{3} \text{ and}$$

$$|\pi(B_i, x) - \pi(B_i, \check{x})| < \frac{\sigma_{A_i,B_i,x}}{3}.$$

Let $\delta_i = \min(\delta_{i,A}, \delta_{i,B})$, then for all $\check{x}$ with $|\check{x} - x| < \delta_i$, we have

$$0 < \pi(A_i, \check{x}) - \pi(B_i, \check{x}) < \frac{2\sigma_{A_i,B_i,x}}{3} \tag{5}$$

This means for any $\check{x}$ within a small neighborhood of x, if $(A_i, B_i) \in R_{x,>}$, then $(A_i, B_i) \in R_{\check{x},>}$. There are only finitely many pairs of sets (A_j, B_j) in $R_{x,>}$. Each pair (A_j, B_j) has a corresponding

$$\frac{\sigma_{A_j,B_j,x}}{3}$$

and a corresponding δ_j. To extend the above inclusion to all pairs of sets (A_j, B_j) in $R_{x,>}$, we define

$$\delta_x = \min(\delta_1, \delta_2, \dots, \delta_j, \dots \delta_m).$$

Then for all $\check{x}$ with $|\check{x} - x| < \delta_x$ and for all j with $1 \le j \le m$, we have

$$0 < \pi(A_j, \check{x}) - \pi(B_j, \check{x}) < \frac{2\sigma_{A_j,B_j,x}}{3} \tag{6}$$

Thus within a sufficiently small neighbourhood, we have the set inclusion $R_{x,>} \subset R_{\check{x},>}$. This completes the proof.

This theorem implies that the dominance structure at a point is informed or "bounded" by a larger set of dominance structures of the points in a small neighbourhood near that point. In the next section, we will examine the case of a set of infinitely many allocations under a forward dominance structure. The theorem above will create certain useful properties in a special subset of these allocations.

4. Weak Convergence of Dominance

In this section, we will first examine a simple geometric property of the space X of allocations based on compactness. With this property, we will then examine its implications in the case of an infinite set of allocations sets $\{S_i\}_{i=1}^{\infty}$ and

derive a convergent theorem on the dominance structures.

Jordan proved that the cardinality of a stable set in a pillage game is finite [1]. Thus if an infinite set of stable sets is possible in a pillage game with non-empty core K, each stable set must have finite cardinality and each S_i must contain at least one non-core allocation x_i.

Since x_i is not in the core, it is dominated by some other allocations. To achieve external stability, each stable set S_j with $j \ne i$ must contain at least one allocation that dominates x_i. Conversely, S_i must also contain elements that dominate $x_j \in S_j$. To maintain internal stability, none of the elements that dominate x_i can be in S_i. In particular, x_i cannot be dominated by any core element. The Jordan finiteness theorem means that each stable set S_i must achieve external stability $S_i \cup D(S_i) = X$ with only finitely many elements. Denote this set of non-core allocations in S_i by C_i. Together these two conditions imply that if a continuous pillage game with non-empty core has infinitely many stable sets, each stable set S_i must contain a finite set $C_i = S_i \backslash K$ with $x_i \in C_i$ and C_i must contain allocations that dominate x_j for all $i \ne j$. With the above observations, we prove an immediate property of these pillage games.

Theorem 2. If a continuous pillage game (n, π, X) with infinitely many stable sets $\{S_i\}_{i=1}^{\infty}$ described above exists, then each C_i contains at least two allocations.

Proof. Given two distinct stable sets S_i and S_j in this setup, to satisfy external stability, S_j must contain a non-core element in $x_j \in C_j$ to dominate x_i. In turn, S_i must contain a second element $a_{i,1}$ that dominates x_j to achieve external stability. Furthermore, $a_{i,1}$ cannot be a core element because S_j contains the same core K as S_i and $a_{i,1} > x_j$ violates internal stability for the stable set S_j. So there are at least two allocations $a_{i,1}$ and x_i in C_i. A symmetric argument shows that S_j must also contain at least two elements. This completes the proof.

Notice since $a_{i,1}$ is not in the core, it must also be dominated by some non-core allocations in $S_j \forall j \ne i$ for S_j to achieve external stability. Therefore when the non-core elements of S_i and S_j are compared, there is a chain of dominance between these allocations. External stability of each stable set implies that any chain of dominance between allocations from two stable sets, starting with a non-core allocation in S_i must also end with a non-core allocation in S_i. A symmetric argument shows that this is also true if we start with a non-core allocation in S_j against S_i. The finiteness condition along with the above analysis implies a structural necessity for S_i: since card $(S_i) < \infty$, each non-core allocation in C_i must complete a dominance loop of finite length which also ends with an allocation in C_i to achieve external stability. Therefore, there is a cyclicity in the dominance structure. This cyclicity allows closure in the sense that a chain starting with a non-core element in one stable set must return upon the same stable set when compared with the non-core elements in anther stable set. Thus a model of pillage game with infinite stable sets must have this cyclic closure property. Once this is achieved, the logical impossibility of a dominance loop never returning to

itself is removed. The remaining work is to find a specific function with the numerical characteristics that satisfies the above mentioned cyclic dominance structure conditions.

Consider a continuous pillage game (n, π, X) with non-empty core K. Let $\{C_i\}_{i=1}^{\infty}$ be a countably infinite set of allocations sets, each with finite cardinality. We can find a non-degenerate infinite sequence of allocations $\{x_i\}_{i=1}^{\infty}$ with $x_i \in C_i$ for each i. Since the total space of allocations X is compact, any non-degenerate infinite sequence in X contains a non-degenerate convergent subsequence. By an abuse of notation, we still denote this convergent subsequence by $\{x_i\}_{i=1}^{\infty}$ Denote the limit point of this convergent subsequence by x: $\lim_n x_n = x$.

Since $|K| < \infty$, elements in the core can be discarded from a non-degenerate infinite set of allocations. Thus we can assume $\{x_i\}_{i=1}^{\infty} \cap K = \emptyset$. We can filter this convergent subsequence further to give a non-degenerate subsequence monotone in each of the n coordinate indices $\{1,2,3, \ldots \ldots n\}$ using a geometric filter as follows. In this context, monotonicity means for each of the index $k \in \{1,2,3, \ldots \ldots n\}$, the sequence of real numbers $\{x_i^k\}_{i=1}^{\infty}$ is either an increasing sequence of real numbers, or a decreasing sequence of real numbers, or a constant non-changing sequence consisting of a single real number. To apply the geometric filter, notice that for an arbitrary non-degenerate sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ in $\mathbb{R}$, there is a monotone convergent subsequence. Thus for the convergent non-degenerate subsequence of allocations $\{x_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$, we can find a monotone subsequence of $\{x_i\}_{i=1}^{\infty}$ by passing to a monotone subsequence of real numbers in each one of the n coordinate indices starting with the first one. For example, if we denote the sequence of real numbers formed by the kth coordinate of all the allocations in the convergent sequence of allocations in its full multi-index expansion notation by $\{x_1^k, x_2^k, x_3^k, \ldots x_n^k\}$, then we can find a strictly monotone subsequence for this sequence of real numbers. We form a new convergent subsequence in the original convergent sequence of allocations $\{x_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$ by including only those allocations for which the monotone subsequence is positioned. So for example, if every fourth element in the sequence of real numbers $\{x_1^k, x_2^k, x_3^k, \ldots x_n^k\}$ is discarded by this geometric filter, then every corresponding fourth element in the original sequence of convergent allocations $\{x_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$ is discarded. An example of the geometric filter is given below.

Example: consider the following non-degenerate infinite sequence of allocations in a 1-good 3-player pillage game:

$$\{(0.4, 0.3 + (1/4)^i, 0.3 + (-1)^{i+1}(1/4)^i)\}_{i=1}^{\infty}$$

If we pass the sequence formed by the second coordinate through the geometric filter, then a monotone increasing convergent subsequence is achieved by selecting all the even terms $\{0.3 + (1/4)^{2i}\}_{i=1}^{\infty}$. The reader is cautioned that in doing so, not only are the odd terms in the sequence of real numbers formed by the second coordinate discarded, odd terms in the allocations are also discarded. Therefore, the geometric filter gives the following convergent subsequence, monotone in each of the three indices:

$$\{(0.4, 0.3 + (1/4)^{2i}, 0.3 - (1/4)^{2i})\}_{i=1}^{\infty}$$

The reader is further cautioned that in the above example the second coordinate and the third coordinate are algebraically coupled.

It is possible that in the original sequence of allocations $\{x_i\}_{i=1}^{\infty}$, one or more coordinates is already degenerate before we apply the geometric filter. That is, for a particular index k with $1 \leq k \leq n$, the sequence $\{x_1^k, x_2^k, x_3^k, \ldots x_n^k\}$ consists of only finitely many real numbers. In that case, we choose one real number for this index and retain all the terms in which this real number appears. Since the original sequence of allocations is non-degenerate, there is at least one index where the sequence of real numbers must consist of infinitely many non-equal real numbers. We start with the first index $1 \in \{1,2,3, \ldots \ldots n\}$ and repeat the process up to the nth index. Since there are only finitely many indices, the geometric filter will result in a convergent non-degenerate sequence monotone in each of the n indices. Due to numerous upper indices, lower indices, and the many filters, we still denote the resulting sequence of allocations as $\{x_i\}_{i=1}^{\infty}$. Notice the filtered sequence has the property that in each coordinate $1 \leq k \leq n$, each one of the following conditions holds:

$$x_1^k < x_2^k < \cdots < x_n^k < \cdots$$
$$x_1^k = x_2^k = \cdots = x_n^k = \cdots$$
$$x_1^k > x_2^k > \cdots > x_n^k > \cdots$$

With this filtered sequence, we now consider a forward dominant structure where the more dominant allocations are the ones with higher indices:

$$\cdots \succ x_n \succ x_{n-1} \succ \cdots \succ x_1 \qquad (7)$$

Forward dominance relations have appeared in previous studies of stable sets. For example, McKenzie, Kerber, and Rowat constructed a non-transitive forward dominance loop in a 1-good 4-player pillage game

$$a \succ d \succ b \succ c \succ a$$

to show the non-uniqueness of stable sets [2]. Each of the above four elements is an "exceptional allocation" in the set $E = \{a, b, c, d\}$ that dominates all other allocations in X/E except for the elements in the core K. In their example, the power function has the special numerical property that it uses two different sets of double linear weights with the sum of 1, placed at three different coordinates to equalize the valuations for a and b and likewise for c and d. With these numerical constraints satisfied, the stable sets in this 1-good 4-player pillage game are

$$\{\{K\}, a, b\} \text{ and } \{\{K\}, c, d\}$$

Hence a non-transitive forward dominance structure is carefully constructed using numerical analysis. This construction reveals certain intricate complexities that the simple order axioms set forth in the definition of pillage games allow. As such, special numerical analytic constructions can be achieved to allow for these "exotic" features of pillage games satisfying a set of rational and minimal expectations similar to the philosophical impasse in Arrow's impossibility theorem. In contrast, our approach is a

classic algebra-geometric approach where analytic concepts such as compactness and bounded convergence are used to derive the existence of sequences of allocations with special properties. Recall that the current convergent sequence of allocations is monotone in each of the n indices. With the above forward dominance structure, we have

Lemma 1. There exists a subsequence of $\{x_i\}_{i=1}^{\infty}$ in the above setup such that for any subset C of $\{1,2,3,\ldots.n\}$, only one of the following three mutually exclusive relations holds:

$$\pi(C,x_1) < \pi(C,x_2) < \pi(C,x_3) < \cdots$$
$$\pi(C,x_1) = \pi(C,x_2) = \pi(C,x_3) = \cdots$$
$$\pi(C,x_1) > \pi(C,x_2) > \pi(C,x_3) > \cdots$$

Proof. Since $\{x_i\}_{i=1}^{\infty}$ is strictly monotone in each index, if A is a subset of $\{1,2,3,\ldots.n\}$ where the x_i's are increasing in their indices $j \in A \subset \{1,2,3,\ldots.n\}$, then by the SR axiom, we have

$$\pi(A,x_{i+k}) > \pi(A,x_i), \forall k \in \mathbb{z}^+ \qquad (8)$$

Conversely, if B is a subset of $\{1,2,3,\ldots.n\}$ where the x_i's are decreasing in their indices $j \in B \subset \{1,2,3,\ldots.n\}$, then by the SR axiom, we have

$$\pi(B,x_{i+k}) < \pi(A,x_i), \forall k \in \mathbb{z}^+ \qquad (9)$$

If C is a subset of $\{1,2,3,\ldots.n\}$ where the x_i's contain both increasing and decreasing indices, we use a unidirectionality filter to achieve monotonicity in $\{\pi(C,x_i)\}_{i=1}^{\infty}$ as follows. As a sequence of real numbers, $\{\pi(C,x_i)\}_{i=1}^{\infty}$ either consists of an infinitely many real numbers or it consists of only finitely many real numbers.

If it is infinite, then $\{\pi(C,x_i)\}_{i=1}^{\infty}$ either contains a non-degenerate monotone increasing subsequence or a non-degenerate monotone decreasing subsequence. Assume without the loss of generality that $\{\pi(C,x_i)\}_{i=1}^{\infty}$ contains a monotone increasing subsequence. Then a new subsequence can be achieved by discarding the terms in $\{\pi(C,x_i)\}_{i=1}^{\infty}$ with indices j where the jth term in $\{\pi(C,x_i)\}_{i=1}^{\infty}$ is not monotone increasing. We are then left with a non-degenerate subsequence of allocations $\{x_i\}_{i=1}^{\infty}$ where for the subset $C \subset \{1,2,3,\ldots.n\}$, the sequence of real numbers $\{\pi(C,x_i)\}_{i=1}^{\infty}$ is a monotone increasing sequence. We still denote the newly filtered subsequence of allocations by $\{x_i\}_{i=1}^{\infty}$.

If the sequence of real numbers $\{\pi(C,x_i)\}_{i=1}^{\infty}$ is a finite set, then we take the one of the valuations assumed by the power function restricted to the subset C. Notice because the geometrically filtered allocation sequence $\{x_i\}_{i=1}^{\infty}$ is monotone in each of the n indices, the first case where $\{\pi(C,x_i)\}_{i=1}^{\infty}$ is an infinite set will occur by the SR axiom. Since there are only finitely many subsets of $\{1,2,3,\ldots.n\}$, after finitely many filtering steps, we arrive at a non-degenerate sequence of allocations, monotone in each index with the property that for each subset C of $\{1,2,3,\ldots.n\}$, $\{\pi(C,x_i)\}_{i=1}^{\infty}$ is a monotone sequence of real numbers. This completes the proof.

The unidirectionality filter establishes a uniformity in the valuation by the power function with respect to any subset of $\{1,2,3,\ldots.n\}$. We now consider the comparison of valuations by the power function with respect to any pair of non-empty and disjoint subsets of $\{1,2,3,\ldots.n\}$. For any pair of disjoint and non-empty sets $A \times B$ of $2^I \times 2^I$ with $A \cap B \neq \emptyset$, we examine the sequence of equalities and inequalities that exists between $\pi(A,x_i)$ and $\pi(B,x_i)$ for the filtered allocation sequence $\{x_i\}_{i=1}^{\infty}$. We have

Lemma 2. There exists a subsequence of $\{x_i\}_{i=1}^{\infty}$ such that for each pair of disjoint and non-empty subsets $A \times B$ of $2^I \times 2^I$, only one of the following three mutually exclusive relations holds for all allocations in the sequence:

$$\pi(A,x_i) > \pi(B,x_i)$$
$$\pi(A,x_i) = \pi(B,x_i)$$
$$\pi(A,x_i) < \pi(B,x_i)$$

Proof. For each pair of disjoint non-empty subsets $A \times B$, at least one of the parity relations $>$, $=$, or $<$ exists for infinitely many terms. It is certainly possible that two or all three parity relations exist for infinitely many terms in $\{x_i\}_{i=1}^{\infty}$ for a particular pair $A \times B$. We select a further subsequence of $\{x_i\}_{i=1}^{\infty}$ by the following parity comparison filter.

If only one of the three above parity relations exists for infinitely many allocations in $\{x_i\}_{i=1}^{\infty}$, then there are at most only finitely many allocations in the sequence where the other two parity relations exist. In this case, we filter $\{x_i\}_{i=1}^{\infty}$ by discarding the finitely many terms for which the other two parity relations hold. For example, if for the pair $A \times B$, all but finitely many terms in $\{x_i\}_{i=1}^{\infty}$ have $\pi(A,x_i) > \pi(B,x_i)$, then we discard the finitely terms where $\pi(A,x_i) = \pi(B,x_i)$ and $\pi(A,x_i) < \pi(B,x_i)$.

If two or more of the above parity relations exist for infinitely many allocations in $\{x_i\}_{i=1}^{\infty}$, then we select precisely one relation and discard the other allocations for which the chosen relation does not hold. Since there are only finitely many proper subsets of $\{1,2,3,\ldots.n\}$, there are only finitely many disjoint non-empty pairs of $A \times B$. Therefore the parity comparison filter will stop after all finitely many such pairs $A \times B$ have been filtered, each resulting in an infinite non-degenerate subsequence of allocations. For the convenience of notations, we still denote the final filtered sequence of allocations as $\{x_i\}_{i=1}^{\infty}$. This completes the proof.

Recall that we denoted by x the limit point of the convergent allocation sequence $\{x_i\}_{i=1}^{\infty}$. We now prove a local regularity property of the filtered sequence.

Theorem 3. There exists an $\delta > 0$ such that in an δ-neighborhood of x, the following equivalence of dominance structure holds:

$$R_{x_i,>,=} = R_{x_k,>,=} \qquad (10)$$

Proof. With the final convergent sequence thus filtered, we have that for every pair of disjoint non-empty subsets $A \times B$, the entire filtered sequence satisfies the same parity relations: $<$, $=$, or $>$. But these pairs of disjoint non-empty subsets A and B are precisely the pair of subsets that define the dominance relations for each allocation x_i. That is, these pairs are precisely the pairs of subsets that define the win sets and lose sets for x_i against any other allocation. For example, if $\pi(A,x_i) > \pi(B,x_i)$, then we have

$$(A, B) \subset R_{x_i,>,=} \qquad (11)$$

Therefore, if $\pi(A, x_i) > \pi(B, x_i)$ for all indices in the ultrafiltered sequence of allocations $\{x_i\}_{i=1}^{\infty}$ for each pair of disjoint non-empty subsets, then we have

$$R_{x_i,>,=} = R_{x_k,>,=}$$

Notice the above equality relation may not hold true if we only have a convergent sequence of allocations approaching x. The three filters added a further rigidity to the original sequence of allocations such that in the end with the ultra-filtered sequence, these desired geometric and parity comparison properties hold. This completes the proof.

We now summarize the three different filters from the above discussions in their prescribed order into an algorithm. The reader is reminded that because of the limit of mathematical notations, we still denote the filtered sequence by $\{x_i\}_{i=1}^{\infty}$ after each of the three filters. Given an infinite set of allocations $\{x_i\}_{i=1}^{\infty}$, the first filter is the geometric filter where we pass the original sequence through to achieve a subsequence monotone in each of the n coordinates. The second filter is the unidirectionality filter where for each proper subset $C \subset \{1,2,3,\ldots\ldots n\}$, an increasing, decreasing, or constant infinite subsequence of real numbers in $\{\pi(C, x_i)\}_{i=1}^{\infty}$ is selected. The third filter is the parity comparison filter where for each pair of disjoint non-empty subsets $A \times B$, we select an infinite subsequence in which only one of the three parity relations $<, =,$ or $>$ holds true all allocations in the chosen subsequence. This triple step algorithm gives an infinite sequence of allocations $\{x_i\}_{i=1}^{\infty}$ with the desired dominance relation properties.

Notice in the forward dominance relation, we did not assume transitivity. That is, we did not assume $x_n > x_k \; \forall n > k$. But we will show that with our filtered sequence of allocations satisfying the geometric, the unidirectionality, and parity comparison relations, the forward dominance structure is in fact transitive.

Theorem 4. The filtered sequence of allocations $\{x_i\}_{i=1}^{\infty}$ is transitively dominant: $\forall n > k, x_n > x_k$.

Proof. Since $\{x_i\}_{i=1}^{\infty}$ is strictly monotone in each coordinate, if i is an index where $x_{n+1}^i > x_n^i$, then $x_{n+k}^i > x_n^i \; \forall n \in \mathbb{z}^+$. Then by monotonicity, if W_n is the win set for x_{n+1} over x_n and L_n is the lose set for x_{n+1} over x_n, then W_n is also the win set for x_{n+k} over x_n and L_n is also the lose set for x_{n+k} over x_n. Therefore we have $\pi(W_n, x_n) > \pi(L_n, x_n)$ when x_{n+k} is compared to x_n. But this is $x_{n+k} > x_n$. This completes the proof.

5. The Theoretical Model

In the previous sections, we started with a highly redundant infinite sequence of allocations $\{x_i\}_{i=1}^{\infty}$. Then different filters were used to reduce these allocations in order to achieve a convergent sequence with desired properties. In this section, we put these results together and examine a theoretical model that satisfies a set of conditions necessary for the existence of an infinite set of stable set.

Since any stable set must have finite cardinality, the challenge in finding a model that makes an infinite set of stable sets possible is closing the dominance loop for any non-core allocations in a given allocation set S_i against the non-core allocations in another set S_j for a countably infinite family of such sets. This implies when comparing two distinct allocation sets S_i and S_j, any chain of dominance starting with a non-core allocation in S_i must end after finitely many allocations with a non-core allocation in the same set S_i in order for S_i to satisfy external stability against S_j:

$$z_i > z_j > \cdots > x_j > x_i$$

with a_i and z_i in S_i and a_j and z_j in S_j. Since this relation must also hold true for S_j against S_i, a necessary condition is the existence of two distinct sets of allocations. Since this second set of allocations consists of non-core elements, dominance structures also exist within these allocations.

Let (n, π, X) be a continuous pillage game with a non-empty core K. Suppose we have an infinite sequence of forward dominant allocations $\{x_i\}_{i=1}^{\infty}$ satisfying the monotonicity, unidirectionality, and parity comparison conditions in section 4. This implies that two distinct allocations x_i and x_j can belong to two distinct sets S_i and S_j to satisfy internal stability.

Let K and x_i be in S_i. By theorem 4, $\{x_i\}_{i=1}^{\infty}$ is transitively dominant. External stability necessitates allocations $\{a_{1,i}\}_{i=1}^{\infty}$ that dominate x_j's down the chain by theorem 2. In the notation $a_{1,i}$, the first right index 1 indicates that this sequence of allocations $\{a_{1,i}\}_{i=1}^{\infty}$ is the first sequence needed after the original sequence $\{x_i\}_{i=1}^{\infty}$. The second right index i indicates the position of the element $a_{1,i}$ in the set of allocations. Namely, $a_{1,i} \in S_i$. Thus we have for $i, k \in \mathbb{z}^+$

$$a_{1,i} > x_{i+k} > x_i$$

Lemma 3. $\{a_{1,i}\}_{i=1}^{\infty}$ is a sequence of pairwise distinct allocations. That is, $a_{1,i} \neq a_{1,j}$ for $i \neq j$.

Proof. Suppose for $i \neq j$, $a_{1,i} = a_{1,j}$. Assume without the loss of generality that $i < j$. Since x_i and $a_{1,i}$ are in S_i and x_j and $a_{1,j}$ are in S_j, forward dominance of $\{x_i\}_{i=1}^{\infty}$ implies that we have

$$x_j > x_i$$

By external stability of S_i, we also have

$$a_{1,i} > x_j > x_i$$

Since $a_{1,j} \in S_j$, if $a_{1,j} = a_{1,i}$, the dominance sequence above implies that

$$a_{1,j} > x_j > x_i$$

But this contradicts the internal stability of S_j. Thus we must have $a_{1,i} \neq a_{1,j}$ for $i \neq j$. This completes the proof.

The above lemma implies that an infinite sequence of allocations $\{a_{1,i}\}_{i=1}^{\infty}$ where $a_{1,i}$ dominates an infinite subset of $\{x_j\}_{j=i+1}^{\infty}$ is needed to ensure external stability. Thus we have that K, x_i and $a_{1,i}$ now belong to S_i. Since

we have $a_{1,i} \succ x_j \succ x_i$ and $a_{1,i}$'s are non-core allocations, there is an allocation in S_j that dominates $a_{1,i}$. This necessitates the existence of a second sequence of allocations $\{a_{2,i}\}_{i=1}^{\infty}$ with $a_{2,i} \in S_i$. We now consider a set of conditions sufficient for the existence of an $\aleph_0$-order set of stable sets in a continuous pillage game:

1. (n, π, X) contains pairwise distinct forward dominant sequences $\{x_i\}_{i=1}^{\infty}$, $\{a_{1,i}\}_{i=1}^{\infty}$, and $\{a_{2,i}\}_{i=1}^{\infty}$ in the interior of X such that each sequence satisfies the conditions of monotonicity, unidirectionality, and parity comparison.

2. There exists a finite set of non-core allocations $C = \{c_1, c_2, c_3, \cdots, c_n,\}$ with $c_i \notin \{x_i\}_{i=1}^{\infty} \cup \{a_{1,i}\}_{i=1}^{\infty} \cup \{a_{2,i}\}_{i=1}^{\infty}$; $x_i, a_{1,i}, a_{2,i}, c_i$, and elements in K do not dominate each other; and c_i does not dominate c_j for $i \neq j$.

3. $\forall\, b \in X / \{K, C, \{x_i\}_{i=1}^{\infty}, \{a_{1,i}\}_{i=1}^{\infty}, \{a_{2,i}\}_{i=1}^{\infty}\}$, b is either dominated by a subset of the core, or by allocations in C, or by an allocation in $\{x_i\}_{i=1}^{\infty}$, $\{a_{1,i}\}_{i=1}^{\infty}$, or $\{a_{2,i}\}_{i=1}^{\infty}$ for each i.

4. The following inter-sequence dominance structure holds $\forall\, j, k \in \mathbb{z}^+$:

$$x_{j+k} \succ x_j \succ a_{2,j+k} \succ a_{2,j} \succ a_{1,j+k} \succ a_{1,j} \succ x_{j+k}$$

Theorem 6. A continuous pillage game with a power function $f: 2^I \times X \to \mathbb{R}$ satisfying the three order axioms and conditions (1) through (4) contains an $\aleph_0$-order of stable sets $\{S_i\}_{i=1}^{\infty}$ with $S_i = \{K, C, x_i, a_{1,i}, a_{2,i}\}$.

Proof. Let $S_i = \{K, C, x_i, a_{1,i}, a_{2,i}\}$. Condition (1) gives the infinite sequences where valuation comparisons can be made by the power function. These sequences satisfy the conditions of monotonicity, unidirectionality, and parity comparison. Forward dominant means that $\forall\, j, k \in \mathbb{z}^+$, we have the following relations:

$$x_{j+k} \succ x_j,\ a_{1,j+k} \succ a_{1,j},\ \text{and}\ a_{2,j+k} \succ a_{2,j} \quad (12)$$

Conditions (1) and (4) taken together give the closed loop of finite length as follows. $\forall\, j, k \in \mathbb{z}^+$, any non-core allocation in S_j gives a unique chain of dominance that ends also in S_j. For $x_j \in S_j$, conditions (1) and (4) give the following finite chain of dominance:

$$x_j \succ a_{2,j+k} \succ a_{2,j} \succ a_{1,j+k} \succ a_{1,j} \succ x_{j+k} \succ x_j \quad (13)$$

For $a_{1,j} \in S_j$, conditions (1) and (4) also give the following finite chain of dominance:

$$a_{1,j} \succ x_{j+k} \succ x_j \succ a_{2,j+k} \succ a_{2,j} \succ a_{1,j+k} \succ a_{1,j} \quad (14)$$

Similarly for $a_{2,j} \in S_j$, conditions (1) and (4) give the following finite chain of dominance:

$$a_{2,j} \succ a_{1,j+k} \succ a_{1,j} \succ x_{j+k} \succ x_j \succ a_{2,j+k} \succ a_{2,j} \quad (15)$$

Therefore when comparing S_j with S_{j+k}, if we start with any non-core allocation in S_j, the chain of dominance structure is always a cyclic loop of finite length that ends in a non-core allocation in S_j to ensure external stability. Starting with any non-core allocation in S_{j+k}, conditions (1) and (4) also give a chain of finite length against non-core allocations in S_{j+k}. For $x_{j+k} \in S_{j+k}$, conditions (1) and (4)

give the following finite chain of dominance:

$$x_{j+k} \succ x_j \succ a_{2,j+k} \succ a_{2,j} \succ a_{1,j+k} \succ a_{1,j} \succ x_{j+k} \quad (16)$$

For $a_{1,j+k} \in S_{j+k}$, conditions (1) and (4) also give the following finite chain of dominance:

$$a_{1,j+k} \succ a_{1,j} \succ x_{j+k} \succ x_j \succ a_{2,j+k} \succ a_{2,j} \succ a_{1,j+k} \quad (17)$$

And similarly for $a_{2,j+k} \in S_{j+k}$, conditions (1) and (4) give the following finite chain of dominance:

$$a_{2,j+k} \succ a_{2,j} \succ a_{1,j+k} \succ a_{1,j} \succ x_{j+k} \succ x_j \succ a_{2,j+k} \quad (18)$$

Thus, when comparing the non-core allocations in two stable sets S_m and S_n, starting with any non-core allocation in the first stable set S_m, a dominance loop of finite length ending also with an allocation in S_m exists both for $n < m$ and $m < n$. Condition (2) gives internal stability in that for any pair of allocations a and b in $S_i = \{K, C, x_i, a_{1,i}, a_{2,i}\}$, if we denote $W = \{i | a_i > b_i\}$ and $L = \{i | b_i > a_i\}$, then we have $\pi(W, a_i) = \pi(L, a_i)$ and $\pi(W, b_i) = \pi(L, b_i)$. Condition (3) gives external stability for S_i. The reader is cautioned that the theorem only gives a special set of sufficient conditions – these conditions may not be necessary. Indeed there can be numerous examples of pillage games in which conditions other than conditions (1) through (4) are satisfied but still have infinitely many stable sets.

6. Conclusions

Continuity gives many advantages in analysing pillage games. For example, since $C \times X$ is a compact set for any $C \subset \{1,2,3,\ldots..n\}$, its image $\pi(C \times X)$ is also a compact set in $\mathbb{R}$. Hence $\pi(C \times X)$ is a bounded set. In addition, since a continuous function on a compact set is also uniformly continuous, π is uniformly continuous on $C \times X$. These rich structural properties lead to a simpler path of analysis of continuous pillage games. We now propose two possible future approaches to the study of pillage games, one for the continuous case and the other for the general case without the assumption of continuity.

If a pillage game is continuous, the boundedness property and the subsequent uniform continuity property stated above allow for a profile of (n, π, X) to be constructed as follows. Given any $C \subset \{1,2,3,\ldots..n\}$ and $\varepsilon > 0$, $\exists\, \delta_{C,\varepsilon} > 0$ depending on C and ε such that $\forall\, x_1, x_2 \in X$ with $\|x_1 - x_2\| < \delta_{C,\varepsilon}$, we have $|\pi(C, x_1) - \pi(C, x_2)| < \varepsilon$. Since there are only finitely many subsets of $\{1,2,3,\ldots..n\}$, a lower bound on all of the $\delta_{C,\varepsilon}$'s with C spanning $\{1,2,3,\ldots..n\}$ can be found. As a result, the power function's growth rate can be controlled. Thus starting at any allocation x, one can ask: what is the maximum rate a coalition C's collective power can grow by moving to a nearby allocation x'.

In the general case without the assumption of continuity, notice that pillage games have certain measure-theoretic properties that give insights and characterize classes of games beyond pillage games. Given an allocation x, the WC condition introduces a property similar to that of a measure μ: if $C_1 \subset C_2$, then

$$\pi(C_1, x) \leq \pi(C_2, x) \Leftrightarrow \mu(C_1) \leq \mu(C_2) \qquad (19)$$

Thus for suitable subsets of $\{1,2,3,\ldots,n\}$, a binary relation with respect to set inclusion exists. One needs to use caution in that if $\emptyset \neq C_1 \cap C_2^c$ and $\emptyset \neq C_2 \cap C_1^c$, no conclusion can be made on the comparison of $\pi(C_1, x)$ and $\pi(C_2, x)$. One can obtain certain quantitative comparisons because the following set inclusion holds: $C_1 \cap C_2 \subset C_1$, $C_1 \cap C_2 \subset C_2$, and $C_i \subset C_1 \cup C_2$ for $i = 1, 2$.

From the other end, given $C \subset \{1,2,3,\ldots,n\}$. the WR and SR conditions also impose a poset structure "$\lhd$" on allocations in X as follows. If $x_i \leq y_i, \forall i \in C$, then

$$x \lhd y \text{ in that } \pi(C, x) \leq \pi(C, y) \qquad (20)$$

Without additional restrictions on (n, π, X), one cannot compare $\pi(C, x)$ to $\pi(C, y)$ if $x_i = y_i, \forall i \in C$. But these above stated properties indeed make $\pi(C, \_)$ a "semi-poset"

$$\text{if } x \lhd y \implies \pi(C, x) \leq \pi(C, y) \qquad (21)$$

In a certain sense, (n, π, X) displays properties of a fibre bundle. That is, (n, π, X) can be considered as a map on the cross product $2^I \times X$ that fibers over the subsets of $\{1,2,3,\ldots,n\}$. Given a subset $C \subset 2^I$, the fibering respects set inclusions. Conversely, when restricted to a particular subset, the fibre splitting respects the poset structure stated above. Given these observations, the following question can be asked: is there a general form of representation theorem for pillage games? In other words, can there exist a real-valued function $f: 2^I \times X \to \mathbb{R}$ and a semi-measure μ such that given $(C, x) \in 2^I \times X$, we have

$$\pi(C, x) = \int_{C,x} f \, d\mu \qquad (22)$$

It is understood that integrability in this case can be made more general than that of Riemann integration or even Lebesgue integrability. We should be advised that if a semi-measure μ exists on $2^I \times X$ such that the power function π can be represented as the integral of this μ-measurable real-valued function f, the integral of f could be unbounded over the product space $2^I \times X$:

$$\left| \int_{2^I \times X} f \, d\mu \right| = \infty \qquad (23)$$

If this representation is tenable even in the greatest generality, it can, for pillage games and for other cooperative games, reduce the study of these games essentially to the properties of the representation function f and the semi-measure μ.

ACKNOWLEDGEMENTS

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