

The Pre-nucleolus for Fuzzy Cooperative Games

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Abstract In this paper invented by D. Schmeidler (1969) for characteristic function games concept of nucleolus has been extended on fuzzy cooperative games. The fuzzy pre-nucleolus defined by a new way. On the set of classical cooperative games proved its coincidence with the already existed one. For a class of fuzzy games the pre-nucleolus exists and unique. The process of finding of pre-nucleolus illustrated on an example of a fuzzy game.

Keywords Fuzzy cooperative games, Fuzzy coalition, Fuzzy pre-nucleolus

1. Introduction

Let $N = \{1, 2, \dots, n\}$ be the set of all players. A *fuzzy coalition* is an n -dimensional vector $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ with $0 \leq \tau_i \leq 1$ for each $i \in N$. A cooperative fuzzy game with the players set N is a pair (T, v) , where $T \subset [0, 1]^n$ is the set of all fuzzy coalitions and v is the characteristic function of that game which maps a real number to each fuzzy coalition.

Cooperative fuzzy games reflect situations in which for players allowed to take part in a coalition with participation levels that may vary from non-cooperation to full cooperation. The obtained reward in this type of games defines depending on the level of cooperation. Participation levels at which players involved in cooperation gets described by fuzzy coalitions.

Aubin (1981) has been explaining use of fuzzy coalitions by following way when he first introduced them in game theory. Every player can choose his level of participation in a coalition instead of whether to participate in it or not. As an example in favor of that approach can be considered case, when individual players reluctant to invest all of the available resources in enterprise where that coalition involved.

For fuzzy cooperative theory extension of classical decision concepts on fuzzy games is an important topic. It is known, that not every concept of classical theory has its natural counterpart for fuzzy games. At the same time some results in classical cooperative games allow to be transformed on fuzzy games with of course significant differences. In this work we aimed to establish an important in classical theory optimality principle, i.e. nucleolus on fuzzy games.

2. Basic Definitions and Results

Together with fuzzy theory of nucleolus we are also going to deal with the classical version of the same concept. For that reason we need to reproduce here some preliminary facts from the classical theory of nucleolus. At the end of this paragraph we will bring the definition of nucleolus for fuzzy cooperative games.

For classical cooperative games D. Schmeidler [1] has defined nucleolus as an imputation what is the best in the sense of a preference relation $<_{\tilde{v}}$. To define the nucleolus we need the following notations. For the game v and the imputation $x \in R^n$, denote $x(S) = \sum_{i \in S} x_i$.

Let $G = \langle N, \tilde{v} \rangle$ is a classical cooperative game and

$$Y(\tilde{v}) = \{x \in R^N / x_i \geq \tilde{v}(\{i\}), \sum_{i=1, n} x_i = \tilde{v}(N)\}$$

is the set of all imputations for the game G .

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1) Results that included in this paper have been part of author's doctoral dissertation written in early 1990's. There is available a copyright certificate from the US Copyright Office.

The difference $e(S, x) = v(S) - x(S)$ is the excess of coalition S regarding to x . Defined that way excess can be interpreted as a measure for complain of coalition S from imputation x . Let consider the vector of excesses $\theta(x, v) = (\theta_1(x, v), \dots, \theta_{2^n}(x, v))$ with components:

$$\theta_m(x, v) = \max_{U \subset 2^N, |U|=m} \min_{S \in U} (v(S) - x(S)).$$

From the definition of $\theta_m(x, v)$ it is clear, that the components of $\theta(x, v)$ ordered decreasingly.

For the game v on the set R^n defined a quasi-order \preceq the following way. Let $x, y \in R^{[N]}$. $x \preceq_{(v)} y$ if $\theta(x, v) \preceq_L \theta(y, v)$, where $\theta(x, v) \preceq_L \theta(y, v)$ is for the lexicographical order. It means, exist a number m such that

$$\theta_k(x, v) = \theta_k(y, v) \text{ for } k = 1, \dots, m-1 \text{ and } \theta_m(x, v) < \theta_m(y, v).$$

Definition 2.1 For $Y \subset R^n$ and characteristic function v , the set $v(Y) \subset R^{[N]}$ is nucleolus for Y if vectors from $v(Y)$ are minimal in the sense of relation $<_v$: i.e.

$$v(Y) = \{x \in Y / x <_v y \text{ for every } y \in Y\}.$$

Theorem (D. Schmeidler, 1969). For every nonempty, convex and compact set the nucleolus exists and consists of only one vector.

Theorem (A. Sobolev, 1976) Let for a game $G = \langle N, \tilde{v} \rangle$ as a set of payoff vectors defined the set of pre-imputaions:

$$X(\tilde{v}) = \{x \in R^N / \sum_{(i=1, n)} x_i = \tilde{v}(N)\}.$$

Then the game G has a nonempty pre-nucleolus

$$v(X) = \{x \in X(\tilde{v}) / x <_{\tilde{v}} y \text{ for every } y \in X\},$$

that contains only one vector.

For outcomes from $X(\tilde{v})$ the condition of individual rationality has been violated. For that reason the set of payoff vectors $X(\tilde{v})$ is not compact and hence, it is different of the set $Y(\tilde{v})$ of imputations. Despite of that the statement about existence and uniqueness for pre-nucleolus continues to remain true.

Fuzzy cooperative games possess infinite number of coalitions. That fact does not allow using the approach based on idea of lexicographic order to extend this concept on fuzzy games. From there arrives a need for a new definition of pre-nucleolus on fuzzy games. To be valid the needed definition should coincide with the existing one for classical games and at the same time to allow extending that concept on fuzzy cooperative games.

Let (T, v) is an arbitrary fuzzy game, where $T \subset [0, 1]^n$ is the set of all fuzzy coalitions and

$$v: T \rightarrow R^1$$

is the characteristic function of that game.

Below we will prove that the newly defined pre-nucleolus coincides with already existing one.

We will consider the set of only collectively rational payoff vectors, i.e. pre-imputations:

$$X(v) = \{x \in R^N / \sum_{i=1, n} x_i = v(1)\}$$

Inductively defined sets X_k, T_k by accepting that

$$X_0 = X, T_0 = \emptyset. \quad (3.1)$$

For $k = 0, 1, \dots, p$ we will define sets X_{k+1} the following way

$$X_{k+1} = \operatorname{argmin}_{x \in X_k} \sup_{\tau \notin T_k} [e(\tau, x) - e_0] / \rho(\tau, T_k) \quad (3.2)$$

and sets T_k for $k = 1, 2, \dots, p$

$$T_k = \{\tau \in T / x\tau = y\tau, \text{ for every } x, y \in X_k\} \quad (3.3)$$

where $e(\tau, x) = v(\tau) - x\tau$, $e_0 = \min_x \max_{\tau} e(\tau, x)$ and $\rho(\tau, T_k)$ is the distance between the point τ and set T_k :

$$\rho(\tau, T_k) = \inf_{\tau' \in T_k} \rho(\tau, \tau').$$

$$\rho(x, y) = \max_i |x_i - y_i|.$$

For sets $\{T_k\}$ true the following: when k increases, T_k does not decrease: $T_{k+1} \supseteq T_k$. If for some k_0 it is turning out that $T_{k_0+1} = T_{k_0}$, then that entails the stabilization of corresponding set X_{k_0} or otherwise, by increasing k , X_k does not decrease any more. The set X_{k_0} obtained that way we will call the prenucleolus for fuzzy game (T, v) .

3. About the Pre-Nucleolus for Classical Cooperative Games

In this paragraph first will be described the new definition of pre-nucleolus for the set of classical cooperative games. For that type of games below has been proved that pre-nucleolus defined both of the ways coincide.

Let the pair $G = \langle N, v \rangle$ means a classical cooperative game, where $N = \{1, 2, \dots, n\}$ is the set of all players and

$$v: 2^N \rightarrow \mathbb{R}^1$$

a characteristic function satisfying to the condition $v(\emptyset) = 0$.

First, let pay attention that in case of classical cooperative games relations (3.1) - (3.3) accept the following view:

$$X_0 = X, T_0 = \emptyset. \quad (3.4)$$

$$X_{k+1} = \operatorname{argmin}_{x \in X_k} \max_{S \in T_k} e(S, x) \quad (3.5)$$

$$T_k = \{S \in 2^N / \sum_{i \in S} x_i = \sum_{i \in S} y_i \text{ for every } x, y \in X_k\} \quad (3.6)$$

Construction of sets X_k, T_k after finite number of steps will be abrupt because finite is the set 2^N . The last set X_k will contain a unique vector, coinciding with the pre-nucleolus in sense of its initial definition. Takes place the following lemma:

Lemma 3.1. Let $x, y \in X, x \neq y$ and

$$T = \{S / x(S) = y(S)\} \neq 2^N,$$

if $\max_{S \in T} e(S, x) < \max_{S \in T} e(S, y)$, then $x <_v y$.

Proof: For a given vector $x \in X$ according to definition we have that

$$\theta_1(x, v) = \max_S (v(S) - x(S)).$$

Let denote by

$$\Sigma_1 = \operatorname{argmax}_{S \in 2^N} e(S, x).$$

Similarly,

$$\theta_2(x, v) = \max_{S \in \Sigma_1} (v(S) - x(S))$$

$$\Sigma_2 = \operatorname{argmax}_{S \in \Sigma_1} e(S, x)$$

Further, for the components of $\Theta(x, v)$, let

$$\theta_k(x, v) = \max_{S \in \Sigma_1 \cup \dots \cup \Sigma_{k-1}} (v(S) - x(S))$$

$$\Sigma_k = \operatorname{argmax}_{S \in \Sigma_1 \cup \dots \cup \Sigma_{k-1}} e(S, x).$$

Let $S' \in \Sigma_1$ be an arbitrary coalition. If $\Sigma_1 \subset T$, then

$$\theta_1(x, v) = v(S') - x(S') = v(S') - y(S') \leq \max_S (v(S) - x(S)) = \theta_1(y, v).$$

If it takes place the strong inequality then the lemma's statement proved.

We will accept now, that for all $S \in \Sigma_1$ takes place only equality. Let also assume, that for some $k < n$ $\Sigma_1 \cup \dots \cup \Sigma_{k-1} \subset T$, $\Sigma_k \not\subset T$ and for every $S_l \in \Sigma_l$ where $l \leq k-1$,

$$\theta_l(x, v) = v(S_l) - x(S_l) = v(S_l) - y(S_l) \leq \theta_l(y, v)$$

Let consider excess

$$\theta_k(x, v) = v(S') - x(S'),$$

where $S' \in \Sigma_k$. If $S' \notin T$, then

$$\theta_k(x, v) = v(S') - x(S') = \max_{S \in T} (v(S) - x(S)) < \max_{S \in T} (v(S) - y(S)) < \max_{S \in \Sigma_1 \cup \dots \cup \Sigma_{k-1}} e(S, y) = \theta_k(y, v).$$

I.e. $x <_v y$.

Let now $\Sigma_k \subset T$ too and $S' \in \Sigma_k$ is some coalition:

$$\theta_k(x, v) = v(S') - x(S') = v(S') - y(S') \leq \max_{S \in \Sigma_1 \cup \dots \cup \Sigma_{k-1}} (v(S) - y(S)) = \theta_k(y, v).$$

Again, if it takes place the strong inequality, then $x <_v y$. If for all k take place only equalities then $x = y$, which contradicts the condition of lemma.

Let numbers $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_p$ are all of different values accepted by the components of vector $\theta(v(X), v)$.

Below we will deal with the sets B_l and Y_l defined following way:

$$B_l = \{S / e(S, v(X)) \geq \alpha_l\}$$

$$Y_l = \{x \in X / e(S, x) = \alpha_m \text{ for } S \in B_m \setminus B_{m-1},$$

if $m \leq l$; and $e(S, x) \leq \alpha_l$ for $S \notin B_l\}$.

Lemma 3.2. If $X_k = Y_l$ then $T_k \supseteq B_l$.

Proof. Let $X_k = Y_l$ and $S \in B_l$ be an arbitrary coalition. It is clear, that for some $m \leq l, S \in B_m \setminus B_{m-1}$. According to definition of Y_l for every $x \in X_k = Y_l, e(S, x) = \alpha_m$. But then $S \in T_k$ which means that $T_k \supseteq B_l$.

Lemma 3.3. For all $k = 1, \dots, q$ exist numbers l_k and sets X_k such that

$$1 = l_1 < l_2 < \dots < l_k, \text{ and } X_k = Y_{l_k} \quad (3.7)$$

Proof. For $k = 1$ the relation (3.7) follows from definitions of sets X_1 and Y_1 . Accept it already has been proved that for some k and l $X_k = Y_l$. According to lemma 2 supposed to take place inclusion $T_k \supset B_l$. If $T_k \neq 2^N$, because $B_p = 2^N$, then exists $l_{k+1} > l_k$ such that $B_{l_k} \subset T_k$ and $B_{l_{k+1}} \not\subset T_k$. Necessary to prove that it will entail the coincidence of sets X_{k+1} and $Y_{l_{k+1}}$: $X_{k+1} = Y_{l_{k+1}}$. Let now for some $l' > l_k$ $T_k \supset B_{l'}$ and $T_k \not\supset B_{l'+1}$. Then will exist set S_0 such that $S_0 \in B_{l'+1}$ and $S_0 \notin T_k$. Subsequently, $S_0 \notin B_{l'}$. Which means, that $S_0 \in B_{l'+1} \setminus B_{l'}$. Because $v(X) \in Y_{l'} = X_k$, so according to lemma 3.1,

$$\min_{x \in X_k} \max_{S \notin T_k} e(S, x) \geq \max_{S \notin T_k} e(S, v(X)) \geq e(S_0, v(X)).$$

By the other side, as far as $S_0 \in B_{l'+1} \setminus B_{l'}$ then

$$e(S_0, v(X)) = \alpha_{l'+1}.$$

From there it follows that

$$\min_{x \in X_k} \max_{S \notin T_k} e(S, x) \geq \alpha_{l'+1}.$$

Further,

$$\max_{S \notin T_k} e(S, v(X)) \leq \max_{S \notin B_{l'}} e(S, v(X)) \leq \alpha_{l'+1},$$

because $T_k \supset B_{l'}$ and $\alpha_{l'+1}$ is the first value of $e(S, v(X))$, which is smaller than $\alpha_{l'}$. As a result,

$$\min_{x \in X_k} \max_{S \notin T_k} e(S, x) = \alpha_{l'+1}.$$

Or otherwise,

$$X_{k+1} = \{x \in X_k / \max_{S \notin T_k} e(S, x) = \alpha_{l'+1}\}.$$

Which means, that

$$X_{k+1} \supset Y_{l'+1}.$$

Let now prove the opposite inclusion, i.e. if $x \in X_{k+1}$, then $x \in Y_{l'+1}$ too. As far as $T_k \supset B_{l'}$ and $v(X) \in X_k$, so for every $S \in B_m \setminus B_{m-1}$, where $m \leq l'$

$$e(S, x) = e(S, v(X)) = \alpha_m.$$

Besides that, for every $S \notin B_{l'}$, $e(S, x) \leq \alpha_{l'+1}$. It is remaining to proof that for arbitrary $S \in B_{l'+1} \setminus B_{l'}$

$$e(S, x) = \alpha_{l'+1}.$$

Accept that for some coalition $S_0 \in B_{l'+1} \setminus B_{l'}$ $e(S_0, x) < \alpha_{l'+1}$. Let consider the vector

$$z = \varepsilon x + (1 - \varepsilon) v(X).$$

For coalitions $S \in B_{l'}$ $e(S, z) = e(S, v(X))$, for $S \in B_{l'+1}$, $e(S, z) \leq e(S, v(X))$, and $S_0 \in B_{l'+1}$ $e(S_0, z) < e(S_0, v(X))$.

From there it follows that for a number $\varepsilon > 0$ small enough, and every $S_1 \in B_{l'+1}$, $S_2 \notin B_{l'+1}$, $e(S_1, z) > e(S_2, z)$.

Which means that the constructed above vector z is more preferable than $v(X)$: $z \prec_v v(X)$. That contradicts to the fact that $v(X)$ is pre-nucleolus for the game $\langle N, v \rangle$. Hence, for every $S \in B_{l'+1} \setminus B_{l'}$ supposed to hold true the equality $e(S, x) = \alpha_{l'+1}$. But then the received equality would mean, that $x \in Y_{l'+1}$, and $Y_{l'+1} \supset X_{k+1}$. The last inclusion concludes the proof of our lemma.

Theorem. 3.1 There is a number q such that $X_q = v(X)$.

Proof. Because $v(X) \in Y_l$ for every l , so according to lemma 3.3, we will have, that also $v(X) \in X_k$. If X_k contains more than one point then it is obvious that $T_k \neq 2^N$. Then based on lemma 3.3 and lemma 3.2, $B_l \neq 2^N$ and according to lemma 1 we will be able to construct the next set X_{k+1} . The constructed that way last set X_{k+1} will consist of only the nucleolus $v(X)$.

4. Fuzzy Games with Finite Sets of Coalitions

4.1. Let (T', v) is a fuzzy cooperative game, where $T' \subset T$ is some finite set of fuzzy coalitions. Below we will prove that in presence of some conditions this type of games possess a unique prenucleolus.

Lemma 4.1. Let X is a convex polytope and χ is the solution for the next linear programming problem:

$$\min_{x \in X} \varepsilon \quad \text{where } x\tau^j + \varepsilon c_j + e_0 \geq a_j \text{ for every } \tau^j \in T'$$

Then exists a vector $\tau^{j_0} \in T'$ such that for every $x, y \in \chi$.

Proof. Let for every $\tau^j \in T'$ exists a vector $x^j \in \chi$ such that

$$e_0 + x^j \tau^j + \varepsilon_0 c_j > a_j,$$

where ε_0 is the solution of mentioned above linear programming problem. Consider now the vector

$$\bar{x} = \frac{1}{m} \sum_{j=1..m} x^j.$$

Because of convexity of the set χ , $\bar{x} \in \chi$ and

$$e_0 + \bar{x} \tau^j + \varepsilon_0 c_j > a_j,$$

for every τ^j , which contradicts to the condition that (ε_0, χ) is the optimal solution for our minimization problem. So, exists a vector $\tau^{j_0} \in T'$ such that for every $x \in \chi$

$$e_0 + \bar{x} \tau^{j_0} + \varepsilon_0 c_{j_0} = a_{j_0}.$$

From there the assertion of lemma 3.4 follows.

Theorem 4.1 Let (T', v) is a fuzzy game, where T' is a finite set of fuzzy coalitions that also contains coalitions $\tau^i = (0, \dots, \tau_i = 1, \dots, 0)$ for arbitrary $i \in N$. Then the game (T', v) possesses a unique pre-nucleolus.

Proof. We need to prove that after finite number of steps the process of construction of sets X_k, T_k will be abrupt and the last set X_k will consists of a unique point.

The set X_1 is solution for the following minimization problem:

$$\min_{x \in X} \varepsilon \quad \text{where } v(\tau^j) - x\tau^j \leq \varepsilon \text{ for every } \tau^j \in T' \quad (3.8)$$

In problem (3.8) the number ε bounded below. Really, if $\tau^i = (0, \dots, \tau_i = 1, \dots, 0)$, then

$$\varepsilon \geq v(\tau^i) - x_i \text{ for } i \in N.$$

Summing all these inequalities by $i \in N$ we will obtain that

$$n\varepsilon \geq \sum_{i \in N} (v(\tau^i) - x_i).$$

from where

$$\varepsilon \geq \frac{1}{n} (\sum_{i \in N} v(\tau_i) - v(1)),$$

what has been required to prove.

When ε accepts its minimal value we obtain the solution of our problem:

$$X_1 = \{x \in X / v(\tau^i) - x\tau^i \leq \varepsilon, \text{ for all } \tau^i \in T'\}$$

The corresponding set T_1 is:

$$T_1 = \{\tau \in T' / x_1 \tau = x_2 \tau \text{ for every } x, y \in X_1\}$$

Further we need to find the

$$\operatorname{argmin}_{x \in X_1} \max_{\tau \notin T_1} [(v(\tau) - x\tau - e_0) / \rho(\tau, T_1)].$$

That is the same as solving the following minimization problem:

$$\min_{x \in X_1} \varepsilon \quad \text{where } v(\tau) - x\tau - e_0 \leq \varepsilon \rho(\tau, T_1) \text{ for } \tau \notin T_1$$

The solution X_2 for this problem is a convex politope and the set T_2 strictly contains the set T_1 : $T_2 \supset T_1$. The same will take place on the following steps too. As far as the set T' is finite, so construction of sets X_k, T_k will be abrupt after finite number of steps.

Let now $T_p = T'$. It is remaining to prove that $|X_p| = 1$. If $x, y \in X_p$ then from $T_p = T'$ will follow that $\tau^l x = \tau^l y$ for arbitrary $\tau^l \in T'$, from where $x = y$. That concludes the proof of our theorem.

4.2. Fuzzy Games with Piece-Wise Affine Characteristic Functions

Below proved a theorem about existence and uniqueness of pre-nucleolus for fuzzy games with piece-wise affine characteristic functions.

Theorem 4.2. Let (T, v) is a fuzzy cooperative game with piece-wise affine characteristic function v . That means, exists a collection of simplexes $\{\Sigma^j\}$ what covers $T: T = \cup \Sigma^j$, $\Sigma^k \cap \Sigma^l = \emptyset$ if $k \neq l$, and for $\tau \in \Sigma^j$, $v(\tau) = u_j(\tau) - \alpha_j$, where $u_j(\tau)$ is a linear function and $\alpha_j \geq 0$. Then the game (T, v) has a pre-nucleolus that consists of a unique point.

Proof. According to definitions of sets X_k , T_k

$$X_m = \operatorname{argmin}_{x \in X_{m-1}} \sup_{\tau \notin T_{m-1}} [(e(\tau, x) - e_0) / \rho(\tau, T_{m-1})]$$

$$T_m = \{\tau \in T \mid x\tau = y\tau, \text{ for every } x, y \in X_m\}$$

The set X_1 is the solution for following minimization problem:

$$\begin{aligned} \min \quad & \varepsilon \\ & v(\tau) - x\tau \leq \varepsilon \\ & x \in X. \end{aligned} \quad (3.9)$$

Let consider the following linear programming problem:

$$\begin{aligned} \min \quad & \varepsilon \\ & v(\tau^j) - x\tau^j \leq \varepsilon \\ & x \in X, \end{aligned} \quad (3.9')$$

where the $\{\tau^j\}$ is the set of all peaks of simplexes $\{\Sigma_k\}$. Accept that the pair (ε'_0, X') is the solution for that problem, where X' is a convex politope. It is clear that $e_0 \geq \varepsilon'_0$.

We will prove that ε'_0 also is solution for the problem (3.9). For that reason we will need to show that the inequality $v(\tau) - x\tau \leq \varepsilon'_0$ holds true for all $\tau \in T$, when $x \in X$.

Let $\tau \in T$ is an arbitrary coalition and Σ^k is a simplex with peaks $\tau^{j_1}, \tau^{j_2} \dots \tau^{j_n}$, which contains τ . Then

$$\tau = \sum_{k=1, n} \lambda_k \tau^{j_k}, \text{ where } \lambda_k \geq 0 \text{ and } \sum_{k=1, n} \lambda_k = 1.$$

Because $v(\tau)$ is an affine function on Σ_k so we will have that

$$v(\tau) - x\tau = \sum_{k=1, n} \lambda_k v(\tau^{j_k}) - x \sum_{k=1, n} \lambda_k \tau^{j_k} = \sum_{k=1, n} \lambda_k (v(\tau^{j_k}) - x\tau^{j_k}) \leq \varepsilon'_0.$$

From there, ε'_0 really is a solution for the problem (3.9). So, we will have that X_1 is the following set:

$$X_1 = \operatorname{argmin}_{x \in X} \sup_{(\tau \in T)} \{v(\tau) - x\tau\}.$$

According to definition of sets T_m , for a fixed $\tau \in T_m$ and every $x \in X_m$, $x\tau = c_\tau$, what means that the product $x\tau$ is constant for every $x \in X_m$. Let now $\tau^1, \tau^2 \in T_m$ are such coalitions that $x\tau^1 = c_1$, and $x\tau^2 = c_2$. From there it will follow that if for some numbers λ_1 and λ_2

$$\lambda_1 \tau^1 + \lambda_2 \tau^2 \in T,$$

then for every $x \in X_m$

$$x(\lambda_1 \tau^1 + \lambda_2 \tau^2) = c_1 \lambda_1 + c_2 \lambda_2.$$

The latter one means that set T_m is the intersection of the set of all coalitions T with some hyperplane and subsequently is a convex set, because of convexity of T .

Next we will rewrite the definition of X_m in a different form:

$$X_m = \operatorname{argmax}_{x \in X_{m-1}} \inf_{\tau \notin T_{m-1}} [(e_0 - e(\tau, x)) / \rho(\tau, T_{m-1})]$$

The set X_m defined that way is solution for the following maximization problem:

$$\begin{aligned} \max \quad & \varepsilon \\ & e_0 - v(\tau) + x\tau \geq \varepsilon \rho(\tau, T_{m-1}) \text{ for every } \tau \notin T_{m-1} \\ & x \in X_{m-1} \end{aligned} \quad (3.10)$$

As it was in the beginning of the proof besides this problem also let consider the corresponding linear programming problem for peaks of simplexes $\{\Sigma^k\}$ that does not belong to T_{m-1} :

$$\begin{aligned} \max \quad & \varepsilon \\ & e_0 - v(\tau^j) + x\tau^j \geq \varepsilon \rho(\tau^j, T_{m-1}) \text{ for every } \tau^j \in \{\Sigma^k\} \setminus T_{m-1} \\ & x \in X_{m-1} \end{aligned} \quad (3.10')$$

The problem (3.10') has a solution because it is a linear programming problem and X_{m-1} is a convex polytope. Let

denote that solution by (ε', X'_m) and prove that inequalities (3.10') remain true for all $x \in X'_m$ and $\tau^j \in \{\sum^k\} \setminus T_{m-1}$. For $\tau^j \in \{\sum^k\} \setminus T_{m-1}$ the inequality (3.10') follows from definition of sets X_m . For $\tau^j \in \{\sum^k\} \cap T_{m-1}$ (3.10') is true because for that kind of τ^j the right side of (5') is equal to 0 and the left side is not negative as far as $X_m \subset X_1$.

Let now $x \in X'$ and $\tau \notin T_{m-1}$. Accept that \sum^k is a simplex for what $\tau \in \sum^k$ and $\tau^{j_1}, \dots, \tau^{j_n}$ are peaks for that simplex. According to the Karatheodory's theorem:

$$\begin{aligned} e_0 - v(\tau) + x\tau &= e_0 - \sum_{k=1,n} \lambda_k v(\tau^{j_k}) + x \sum_{k=1,n} \lambda_k \tau^{j_k} = \sum_{k=1,n} \lambda_k (e_0 - v(\tau^{j_k}) + x\tau^{j_k}) \geq \\ &\geq \varepsilon' (\sum_{k=1,n} \lambda_k \rho(\tau^{j_k}, T_{m-1})) \geq \varepsilon' \rho(\tau, T_{m-1}). \end{aligned}$$

The last inequality in the chain above takes place because of convexity of metric $\rho(x, T)$ by the variable x . As a result, has been proved that the solution (ε', X'_m) for the problem (3.10') also is solution for (3.10). From there according to lemma 4.1 exists j_0 such that $\tau^{j_0} \notin T_{m-1}$ and for arbitrary $x \in X'_m$ takes place equality in (3.10'). Then because $T_m \supset T_{m-1}$ and $\tau^{j_0} \notin T_{m-1}$ so $\tau^{j_0} \in T_m$. As a result to that the dimension of T_m will increase by at least one. From there because as its proved above the sets T_m are convex, so after finite number of steps T_m will coincide with T and the corresponding set X_m will contain only one point.

4.3. An Example for Calculation of Pre-Nucleolus

The paragraph below devoted to finding of the pre-nucleolus for fuzzy game from a parameterized class. Let considered a game $G = \langle [0, 1]^2, v \rangle$ with the following characteristic function $v(\tau)$:

$$v(\tau) = \min \left\{ \frac{\tau_1}{\tau_0}, \frac{\tau_2}{\tau_0}, \frac{1-\tau_1}{1-\tau_0}, \frac{1-\tau_2}{1-\tau_0} \right\}, \text{ for } \tau = (\tau_1, \tau_2) \in [0, 1]^2 \text{ and } \tau_0 < \frac{1}{2}.$$

It is clear that for this game $v(1) = 1$ and $X = \{x \in R^2 / x_1 = -x_2\}$.

Solving of the problem will start from dividing the square $T = [0, 1]^2$ on eight triangle subsets and figuring out values of $v(\tau)$ on each one of them. Let denote these subsets by Ω_i ($i = 1 \dots 8$) and start to describe them.

$$(1) \quad \Omega_1 = \{ \tau \in T / \tau_1 \leq \tau_2 \leq \tau_0 \} \text{ for } \tau \in \Omega_1, v(\tau) = \frac{\tau_1}{\tau_0}.$$

$$(2) \quad \Omega_2 = \{ \tau \in T / \tau_2 \leq \tau_1 \leq \tau_0 \} \text{ for } \tau \in \Omega_2, v(\tau) = \frac{\tau_2}{\tau_0}.$$

$$(3) \quad \Omega_3 = \{ \tau \in T / \tau_1 + \tau_2 \leq 2\tau_0, \tau_2 \geq \tau_0 \}$$

Based on inequalities that define Ω_3 it is obtained that $\tau_2 - \tau_0 \leq \tau_1 - \tau_0$ and $\tau_1 \leq \tau_0 \leq \tau_2$ from there $v(\tau) = \frac{\tau_1}{\tau_0}$.

$$(4) \quad \Omega_4 = \{ \tau \in T / \tau_1 + \tau_2 \leq 2\tau_0, \tau_1 \geq \tau_0 \}$$

For $\tau \in \Omega_4$ are true the following inequalities:

$$\tau_2 - \tau_0 \leq \tau_0 - \tau_1, \tau_2 \leq \tau_0 \leq \tau_1. \text{ From what it follows that } v(\tau) = \frac{\tau_2}{\tau_0}.$$

$$(5) \quad \Omega_5 = \{ \tau \in T / \tau_1 + \tau_2 \geq 2\tau_0, \tau_1(1 - \tau_0) \geq (1 - \tau_1)\tau_0 \}$$

The definition of Ω_5 implies that for $\tau \in \Omega_5$ hold true the inequalities:

$$\tau_2 - \tau_0 \geq \tau_0 - \tau_1, \tau_1 \leq \tau_0 \leq \tau_2; \frac{\tau_1}{\tau_0} \leq \frac{1-\tau_2}{1-\tau_0}; \frac{\tau_1}{\tau_0} \leq 1 \leq \frac{\tau_2}{\tau_0}$$

So, for $\tau \in \Omega_5$: $v(\tau) = \frac{\tau_1}{\tau_0}$.

$$(6) \quad \Omega_6 = \{ \tau \in T / \tau_1 + \tau_2 \geq 2\tau_0, \tau_2(1 - \tau_0) \geq (1 - \tau_2)\tau_0 \}$$

Analogically to $\tau \in \Omega_5$ in this case too

$$\tau_2 - \tau_0 \geq \tau_0 - \tau_1, \tau_2 \leq \tau_0 \leq \tau_1$$

Further, because

$$\frac{\tau_2}{\tau_0} \leq \frac{1-\tau_1}{1-\tau_0}; \frac{\tau_2}{\tau_0} \leq 1 \leq \frac{\tau_1}{\tau_0}, \text{ so } v(\tau) = \frac{\tau_2}{\tau_0}.$$

$$(7) \quad \Omega_7 = \{ \tau \in T / \tau_1 \leq \tau_2, \tau_1(1 - \tau_0) \geq (1 - \tau_2)\tau_0 \}$$

From inequalities that define Ω_7 follows that for $\tau \in \Omega_7$

$$\frac{1-\tau_1}{1-\tau_0} \leq \frac{\tau_1}{\tau_0} \leq \frac{\tau_2}{\tau_0}; \frac{1-\tau_2}{1-\tau_0} \leq \frac{1-\tau_1}{1-\tau_0}. \text{ i. e. } v(\tau) = \frac{1-\tau_2}{1-\tau_0}.$$

$$(8) \quad \Omega_8 = \{ \tau \in T / \tau_2 \leq \tau_1, \tau_2(1 - \tau_0) \geq (1 - \tau_1)\tau_0 \}$$

On Ω_8 hold true the following inequalities:

$$\frac{1-\tau_1}{1-\tau_0} \leq \frac{\tau_2}{\tau_0} \leq \frac{\tau_1}{\tau_0}; \frac{1-\tau_2}{1-\tau_0} \geq \frac{1-\tau_1}{1-\tau_0}. \text{ So, } v(\tau) = \frac{1-\tau_1}{1-\tau_0}.$$

To find sets X_1 and T_1 enough to calculate the following magnitude:

$$\min_x \max_\tau e(\tau, x), \text{ where} \\ e(\tau, x) = v(\tau) + x(\tau_2 - \tau_1).$$

Let now to calculate the magnitude of $\max_\tau e(\tau, x)$ by the scheme below:

$$\max_\tau e(\tau, x) = \max_{(1 \leq i \leq 8)} e(\tau, x).$$

Further by turn will be figured out magnitudes of the following inner maximums:

$$\begin{aligned} \max_{\Omega_1} e(\tau, x) &= \max_{\Omega_1} \left\{ \frac{\tau_1}{\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} x\tau_0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \\ \max_{\Omega_2} e(\tau, x) &= \max_{\Omega_2} \left\{ \frac{\tau_2}{\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} -x\tau_0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \\ \max_{\Omega_3} e(\tau, x) &= \max_{\Omega_3} \left\{ \frac{\tau_1}{\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} x & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \\ \max_{\Omega_4} e(\tau, x) &= \max_{\Omega_4} \left\{ \frac{\tau_2}{\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \\ \max_{\Omega_5} e(\tau, x) &= \max_{\Omega_1} \left\{ \frac{\tau_1}{\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} x & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \\ \max_{\Omega_6} e(\tau, x) &= \max_{\Omega_2} \left\{ \frac{\tau_2}{\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \\ \max_{\Omega_7} e(\tau, x) &= \max_{\Omega_7} \left\{ \frac{1-\tau_2}{1-\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} x & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases} \\ \max_{\Omega_8} e(\tau, x) &= \max_{\Omega_8} \left\{ \frac{1-\tau_1}{1-\tau_0} + x(\tau_2 - \tau_1) \right\} = \begin{cases} 1 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \end{aligned}$$

Now, when the values for $\max_\tau e(\tau, x)$ by subsets Ω_i , already have been found can be calculated value for the preliminary expression:

$$\begin{aligned} \min_x \max_\tau e(\tau, x) &= \min \{ \min_{x \geq 0} \max_\tau e(\tau, x), \min_{x \leq 0} \max_\tau e(\tau, x) \} = \\ &= \min \{ \min_{x \geq 0} \max (x\tau_0, 1, x), \min_{x \leq 0} \max (1, -x\tau_0, -x) \} = \min_x \max \{x, 1\} = 1. \end{aligned}$$

Further, because for x with $|x| > 1$ $\min_{|x| > 1} \max \{ |x|, 1 \} > 1$, so from there it is clear that

$$\operatorname{argmin}_x \max_\tau e(\tau, x) = [1, -1].$$

That value together with definition of the set T_1 gives that $T_1 = \{\tau^0\}$. Let denote

$$e_0 = \min_x \max_\tau e(\tau, x) = 1.$$

To find sets X_2, T_2 should be calculated the magnitude of $\max_{\tau \neq \tau_0} F(\tau, x)$, where

$$F(\tau, x) = \frac{v(\tau) - x(\tau_2 - \tau_1) - e_0}{\max \{ |\tau_1 - \tau_0|, |\tau_2 - \tau_0| \}}.$$

The magnitude of $\max_{(\tau \neq \tau_0)} F(\tau, x)$ also will be calculated by subsets Ω_i , the same way as it has been done with the $\max_\tau e(\tau, x)$.

$$\begin{aligned} \max_{\Omega_1 \setminus \{\tau^0\}} F(\tau, x) &= \max_{(\Omega_1 \setminus \{\tau^0\})} \frac{\frac{\tau_1}{\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_0 - \tau_1} = \max_{(\Omega_1 \setminus \{\tau^0\})} \left\{ -\frac{1}{\tau_0} + \frac{x(\tau_2 - \tau_1)}{\tau_1 - \tau_0} \right\} = \\ &= \begin{cases} -\frac{1}{\tau_0} & \text{if } x \geq 0 \\ -\frac{1}{\tau_0} - x & \text{if } x < 0 \end{cases}. \end{aligned}$$

$$\begin{aligned} \max_{(\Omega_2 \setminus \{\tau^0\})} F(\tau, x) &= \max_{(\Omega_2 \setminus \{\tau^0\})} \frac{\frac{\tau_2}{\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_0 - \tau_2} = \max_{(\Omega_2 \setminus \{\tau^0\})} \left\{ -\frac{1}{\tau_0} + \frac{x(\tau_2 - \tau_1)}{\tau_2 - \tau_0} \right\} = \\ &= \begin{cases} -\frac{1}{\tau_0} - x & \text{if } x < 0 \\ -\frac{1}{\tau_0} + x & \text{if } x \geq 0 \end{cases}. \end{aligned}$$

$$\max_{(\Omega_3 \setminus \{\tau^0\})} F(\tau, x) = \max_{(\Omega_3 \setminus \{\tau^0\})} \frac{\frac{\tau_1}{\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_0 - \tau_1} = \max_{(\Omega_3 \setminus \{\tau^0\})} \left\{ -\frac{1}{\tau_0} + \frac{x(\tau_2 - \tau_1)}{\tau_0 - \tau_1} \right\} = \begin{cases} -\frac{1}{\tau_0}, & \text{if } x \geq 0 \\ -\frac{1}{\tau_0} - x, & \text{if } x < 0 \end{cases}$$

$$\begin{aligned}
\max_{(\Omega_4 \setminus \{\tau^0\})} F(\tau, x) &= \max_{(\Omega_4 \setminus \{\tau^0\})} \frac{\frac{\tau_2}{\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_0 - \tau_2} = \max_{(\Omega_4 \setminus \{\tau^0\})} \left\{ -\frac{1}{\tau_0} + \frac{x(\tau_2 - \tau_1)}{\tau_2 - \tau_0} \right\} = \\
&= \begin{cases} -\frac{1}{\tau_0} + 2x & \text{if } x \geq 0 \\ -\frac{1}{\tau_0} + x & \text{if } x < 0 \end{cases} . \\
\max_{(\Omega_5 \setminus \{\tau^0\})} F(\tau, x) &= \max_{(\Omega_5 \setminus \{\tau^0\})} \frac{\frac{\tau_1}{\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_2 - \tau_0} = \max_{(\Omega_5 \setminus \{\tau^0\})} \left\{ -\frac{1}{\tau_0} + \frac{(-\frac{1}{\tau_0} - x)(\tau_2 - \tau_1)}{\tau_2 - \tau_0} \right\} = \\
&= \begin{cases} -\frac{x+1}{1-\tau_0} & \text{if } -\frac{1}{\tau_0} - x < 0 \\ \text{impossible, if } -\frac{1}{\tau_0} - x \geq 0 \end{cases} . \\
\max_{(\Omega_6 \setminus \{\tau^0\})} F(\tau, x) &= \max_{(\Omega_6 \setminus \{\tau^0\})} \frac{\frac{\tau_2}{\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_1 - \tau_0} = \max_{(\Omega_6 \setminus \{\tau^0\})} \left\{ -\frac{1}{\tau_0} + \left(\frac{1}{\tau_0} - x \right) \frac{(\tau_2 - \tau_1)}{\tau_1 - \tau_0} \right\} = \\
&= \begin{cases} \frac{x-1}{1-\tau_0} & \text{if } \frac{1}{\tau_0} - x > 0 \\ \text{impossible} & \text{if } \frac{1}{\tau_0} - x \leq 0 \end{cases} . \\
\max_{(\Omega_7 \setminus \{\tau^0\})} F(\tau, x) &= \max_{(\Omega_7 \setminus \{\tau^0\})} \frac{\frac{1-\tau_2}{1-\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_2 - \tau_0} = \max_{(\Omega_7 \setminus \{\tau^0\})} \left\{ \frac{1}{\tau_0 - 1} - x \frac{(\tau_2 - \tau_1)}{\tau_1 - \tau_0} \right\} = \\
&= \begin{cases} \frac{x-1}{1-\tau_0} & \text{if } x \geq 0 \\ -\frac{1}{1-\tau_0} & \text{if } \frac{1}{\tau_0} - x < 0 \end{cases} . \\
\max_{(\Omega_8 \setminus \{\tau^0\})} F(\tau, x) &= \max_{(\Omega_8 \setminus \{\tau^0\})} \frac{\frac{1-\tau_1}{1-\tau_0} - x(\tau_2 - \tau_1) - 1}{\tau_1 - \tau_0} = \max_{(\Omega_8 \setminus \{\tau^0\})} \left\{ \frac{1}{\tau_0 - 1} - x \frac{(\tau_2 - \tau_1)}{\tau_1 - \tau_0} \right\} = \\
&= \begin{cases} \frac{x-1}{1-\tau_0} & \text{if } x \geq 0 \\ \frac{1}{1-\tau_0} & \text{if } x < 0 \end{cases} .
\end{aligned}$$

At the end it is remaining to calculate one more magnitude, which will give us the set X_2 :

$$\begin{aligned}
\min_{(x \in [-1, 1])} \max_{\tau \neq \tau_0} F(\tau, x) &= \min \{ \min_{x \in [0, 1]} \max_{\tau \neq \tau_0} F(\tau, x), \min_{x \in [-1, 0]} \max_{\tau \neq \tau_0} F(\tau, x) \} = \\
&= \min \{ \min_{x \in [0, 1]} \max \left[-\frac{1}{\tau_0}, x - \frac{1}{\tau_0}, -x - \frac{1}{\tau_0}, 2x - \frac{1}{\tau_0}, -\frac{x+1}{1-\tau_0}, \frac{x-1}{1-\tau_0} \right], \min_{x \in [-1, 0]} \max \left[-\frac{1}{\tau_0}, -\frac{1}{\tau_0} - x, -\frac{1}{\tau_0} - 2x, -\frac{1}{\tau_0} + x, \right. \\
&\quad \left. \frac{x+1}{1-\tau_0}, \frac{x-1}{1-\tau_0}, -\frac{1}{1-\tau_0} \right] \} = \min \{ \min_{x \in [0, 1]} \max \left[-\frac{1}{\tau_0} + 2x, \frac{x-1}{1-\tau_0} \right], \min_{x \in [-1, 0]} \max \left[-\frac{1}{\tau_0} - 2x, -\frac{x+1}{1-\tau_0} \right] \} = \\
&= \max \left\{ -\frac{1}{\tau_0}, -\frac{1}{1-\tau_0} \right\} = -\frac{1}{1-\tau_0}.
\end{aligned}$$

So, as a result it obtains that

$$X_2 = \operatorname{argmin}_{x \in [-1, 1]} \max_{\tau \neq \tau_0} F(\tau, x) = 0.$$

From there for the pre-nucleolus $v(X)$ of initial game it follows that

$$v(X) = (0, 0).$$

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