

# Bi-Cooperative Network Games: A Solution Concept

Surajit Borkotokey, Loyimee Gogoi\*

Department of Mathematics, Dibrugarh University, Dibrugarh, India

**Abstract** We introduce the notion of a Bi-cooperative network game as a graph restricted Bi-cooperative game where interactions among players with bipolar motives are considered only through some exogenously given networks. Once such a network forms, the challenge rests on obtaining a suitable allocation of the payoff accrued by its members. In classical Network games the Position value is an important link based allocation rule. We extend this idea to define the Position value for the class of Bi-cooperative network games. It is then characterized using two axioms: efficiency (EFF) and balanced link contribution (BLC).

**Keywords** Bi-cooperative game, Bi-cooperative network game, Position value, LG value

## 1. Introduction

In this paper we introduce the notion of a Bi-cooperative network game under a general setup that extends the earlier works of Gogoi et al. [3]. Such games engulf features of both Networks and Bi-cooperative games where the players support or oppose an issue through their links (we call them positive and negative links respectively). We also define the Position value as a link based allocation rule for this class. A characterization of the Position value is done with the axioms of efficiency (EFF) and balanced link contribution (BLC) in the line of Slikker [15].

A network describes the interactions among players through their links. Network games due to Jackson and Wolinsky [7] are network restricted Cooperative games. These games have their roots in graph restricted Cooperative games proposed by Myerson [11] where a Cooperative game is defined over only those coalitions where players are connected by some exogenously given network. Graph restricted games do not account for the network structure. In Network games [7], however the value function is defined over the sub-networks of a given network and thus the structure of the network is important. A Bi-cooperative game due to Bilbao [1] can be thought of as an extension of classical Cooperative game where players can support an issue, oppose it or remain indifferent. It assumes that the value accrued by a group of players (coalition) depends on the action of the remaining players with possibly few absentees. Thus the player set is divided into a partition of three groups: the positive contributors who support an issue, the negative contributors who oppose the issue and the absentees who remain indifferent to that issue. The

characteristic function representing the game assigns a value to the positive contributors that depend on its opponents i.e., the negative contributors. Combining these two structures (networks and bi-cooperative) in [3], a Bi-cooperative network game is defined on a restricted domain where each group of positive and negative contributors is supposed to form components in the given network. These components resemble with the bi-coalitions of a Bi-cooperative game and therefore the model in [3] is seen to mimic the graph restricted games due to Myerson [11] under a bi-cooperative setup. However based on the different network structures owing to physical conditions, in the following we identify three types of situations that engulf the features of both Bi-cooperative and Network games. It can be easily seen that in [3] the situations pertaining to type (b) are only considered.

- (a) Players are positive or negative or indifferent as in ordinary Bi-cooperative games and consequently all the links of a positive player are termed as positive while the links of a negative player are negative and finally those of the absentees are considered indifferent. It is likely to have some links that connect a positive contributor to a negative contributor through which information may pass from one to the other.
- (b) Players and links both are either positive or negative so that we get a positive component or a negative component in the network.
- (c) Players may have links to the supporting network as well as to the opposing network simultaneously.

Note that an essential assumption in the study of Bi-cooperative games is that a player can never be both a positive and negative contributor simultaneously, however in a Bi-cooperative network game that models situations of type (c) can have players who contribute to the network both positively and negatively through her distinct links. Consequently instead of calling her positive or negative we

\* Corresponding author:

loyimeegogoi@gmail.com (Loyimee Gogoi)

Published online at <http://journal.sapub.org/jgt>

Copyright © 2014 Scientific & Academic Publishing. All Rights Reserved

call her links as positive or negative links depending on their nature. This may be the case when players are connected to more than one networks with multiple identities. Examples can be found in [5, 9, 10].

A solution in this paradigm will be a rational distribution of the worth (may be in terms of money, power and influence etc.) accrued by the given network. Network games under cooperative framework were introduced by Jackson and Wolinsky [7]. Two distinct solution concepts: the player based allocation rule and the link based allocation rule are found in the literature. However, unlike the classical Network game where it is the prerogative of the decision maker to choose between the player based and link based rules, in a Bi-cooperative network game such rules cannot be applied arbitrarily. For example, a link based allocation rule can be applied to all the three types while a player based rule applies only to type (a) and (b) but not type (c). It is worth noting that in [3] the Myerson value as a player based allocation rule is obtained along with some standard characterizations. Therefore in order to address the issues of all the three types mentioned above, in this paper we focus on a link based allocation rule: the Position value. Its counterpart in the network literature under cooperative framework is seen to be one of the most significant link-based allocation rules. Characterization of the Position value for Network games are found in [14-16].

Various solution concepts of Bi-cooperative games have so far been proposed [2, 9]. Labreuche and Grabisch [9] have shown that the Bi-cooperative games proposed by Bilbao [1] inherits many identical characteristics from the multi-choice games defined by Hsiao and Raghavan [5] with three levels of participation. The major difference between the notions of Bi-cooperative and multi-choice games however lies in the fact that Bi-cooperative games are bipolar with the options of positive and negative contributions, while in multi-choice games, cooperations are along the same direction. Thus we find in [9] a new model of a Bi-cooperative game (as opposed to Bilbao et al. [2] that preserves such bipolarity and is not isomorphic to a multi-choice game. Labreuche and Grabisch [9] proposed the LG value as a solution concept that represents the share of the wealth obtained by some players after they decided on their participation to the game. Other values for Bi-cooperative games are also defined in [2] etc., however they mostly represent the net influence of a player in switching from one role to the other and is not specific to a particular coalition.

In our present paper, we assume that the network is fixed and remain there for some time so that the players do not change their positions immediately. Therefore, the value given in [9] being specific to a particular partition of positive, negative and absent contributors seems to be more relevant here and thus we follow their approach.

The rest of the paper is arranged as follows. Section 2 discusses the existing literature on Cooperative games, Network games and Bi-cooperative games in details. In section 3 we introduce the notion of a Bi-cooperative

network game. Section 4 includes the main results of the paper and is followed by the concluding remarks in section 5.

## 2. Model Formulation

In this section, we present the definitions and results necessary for development of our model. To a large extent this section is based on Labreuche and Grabisch [9], Jackson and Wolinsky [7] and Jackson [8].

### 2.1. Bi-cooperative Game

Let  $N = \{1, \dots, n\}$  be a finite set of players. Let  $Q(N) = \{(S, T) \mid S, T \subseteq N, S \cap T = \emptyset\}$  be the set of pairs of disjoint coalitions. We call every member  $(S, T)$  of  $Q(N)$  a bi-coalition. The underlying assumption is that players in  $S$  are positive contributors to the game, players in  $T$  are negative contributors and those in  $N \setminus (S \cup T)$  are absentees.

**Definition 1.** A Bi-cooperative game is a function  $b: Q(N) \rightarrow \mathbb{R}$  with  $b(\emptyset, \emptyset) = 0$ . The real number  $b(S, T)$  represents the worth of the bi-coalition  $(S, T)$  when players in  $S$  support an issue, players in  $T$  oppose it and the remaining players are indifferent.

Let  $G(N)$  be the set of all Bi-cooperative games on  $N$ .

**Definition 2.** A one point solution concept or a value for Bi-cooperative games is a function, which assigns to every Bi-cooperative game an  $n$ -dimensional real vector that represents a payoff distribution over the players.

The LG value proposed by Labreuche and Grabisch [9] for Bi-cooperative games, denoted by  $\Phi^{LG}$  is defined as follows.

For any  $b \in G(N)$ ,  $(S, T) \in Q(N)$  such that for all  $i \in N$

$$\Phi_i^{LG}(b)(S, T) = \sum_{K \subseteq (S \cup T) \setminus i} \frac{k!(s+t-k-1)!}{(s+t)!} \begin{bmatrix} V(K \cup i) \\ -V(K) \end{bmatrix}$$

where for  $K \subseteq S \cup T$ ,  $V(K) := b(S \cap K, T \cap K)$ .

The LG value for a positive (respective negative) contributor depends only on her added-value from being indifferent to become a positive contributor (respectively negative contributor). The information regarding how this player behaves when she switches from positive to negative is not relevant, see [9].

In [9], it is argued that a Bi-cooperative game is isomorphic to the multichoice game [5] with three levels of participation under the order relation  $\sqsubseteq$  in  $Q(N)$  implied by monotonicity viz., for  $(S, T), (S', T') \in Q(N)$ ,  $(S, T) \sqsubseteq (S', T')$  iff  $S \subseteq S'$  and  $T' \subseteq T$ . Therefore alternatively they [8] have adopted the product order viz., for  $(S, T), (S', T') \in Q(N)$ ,  $(S, T) \sqsubseteq (S', T')$  iff  $S \subseteq S'$  and  $T \subseteq T'$ . Under this order  $Q(N)$  is an inf-semilattice with  $(\emptyset, \emptyset)$  as the bottom element and all  $(S, N \setminus S)$ ,  $S \subseteq N$ , the maximal elements. The product order relation distances the LG value from its multi-choice counterparts. Moreover it is specific to a particular bi-coalition. Therefore in this paper we follow the order considered in [9].

**Definition 3.** For  $(S, T) \in Q(N)$ , the superior unanimity games  $\{\bar{u}(S, T) \mid (\emptyset, \emptyset) \neq (S, T) \in Q(N)\}$  in  $G(N)$  are given by,

$$\bar{u}_{(S,T)}(A,B) = \begin{cases} 1, & \text{if } (S,T) \subseteq (A,B) \\ 0, & \text{otherwise} \end{cases}$$

The above collection forms a basis of  $G(N)$  and so every  $b \in G(N)$  can be expressed as a linear combination of the superior unanimity games as follows,

$$b = \sum_{(K,L) \in Q(N)} a_{K,L} \bar{u}(K,L) \quad (7)$$

where  $a_{K,L}$  are the real constants.

Following [13], we can associate to every  $b \in G(N)$  and each  $(S,T) \subset Q(N)$ , a Cooperative game  $u$  defined on  $S \cup T$  such that  $u(M) = b(S \cap M, T \cap M), \forall M \subseteq S \cup T$ . So  $u$  has a corresponding representation in terms of the unanimity Cooperative games  $\{U_M \mid M \subseteq N, M \neq \emptyset\}$  as follows, see [4, 13].

$$u = \sum_{\emptyset \neq M \subseteq S \cup T} a_M U_M$$

where

$$a_M = \sum_{L \subseteq M} (-1)^{m-l} u(L)$$

It follows that

$$\begin{aligned} \sum_{\emptyset \neq S \cup T} a_M &= u(S \cup T) = b(S \cap (S \cup T), T \cap (S \cup T)) \\ &= b(S, T) \end{aligned}$$

Now after some simple computations, we find the expression of the LG value for  $b$  in terms of the Harsanyi's dividends [4] for  $\Phi^{sh}$ , the Shapley value [13] of the associated Cooperative game  $(S \cup T, u)$  as follows,

$$\begin{aligned} \Phi_i^{LG}(b)(S, T) &= \Phi_i^{sh}(S \cup T, u) \\ &= \sum_{M \subseteq (S \cup T): i \in M} \frac{a_M}{|M|} \end{aligned} \quad (9)$$

## 2.2. Network Game

Let  $N = \{1, \dots, n\}$  be a finite set of players. Let  $g^N$  be the set of all subsets of  $N$  of size 2. We call  $g^N$  the complete network with  $n$  nodes. Let  $G = \{g \mid g \subseteq g^N\}$  be the set of all possible networks on  $N$ . By  $l \in g$ , we mean the link  $l$  is in the network  $g$ . For  $g \in G$ , we denote by  $l(g)$  the total number of links in  $g$ . Let  $L_i(g)$  be the set of links that player  $i$  is involved in, so that  $L_i(g) = \{ij \in g\}$ . We denote by  $l_i(g)$  the number of links in player  $i$ 's link set. It follows that  $l(g) = \frac{1}{2} \sum l_i(g)$ . Let  $N(g)$  be the set of all players in  $g$ , i.e.,  $N(g) = \{i \in N \mid L_i(g) \neq \emptyset\}$ . For any  $g_1, g_2 \in G$ , denote by  $g_1 + g_2$  the network obtained through adding networks  $g_1$  and  $g_2$  and by  $g_1 / g_2$  the network obtained from  $g_1$  by subtracting its sub-network  $g_2$ . A value function is a function  $v: G \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ , where  $\emptyset$  represents the empty network i.e., network without links. The set of all possible value functions is denoted by  $V$ . The value function specifies the total worth that is generated by a given network structure. It may involve both costs and benefits whenever this information is available.

**Definition 4.** A Network game is a pair  $(N, v)$ , consisting of a set of players  $N$  and a value function  $v$ . If  $N$  is fixed and no confusion arises about this, we denote the Network game by only  $v$ .

**Definition 5.** An allocation rule is a function  $Y: G \times V \rightarrow \mathbb{R}^n$  such that  $Y_i(g, v)$  represents the payoff to player  $i$  with respect to  $v$  and  $g$  and  $\sum_i Y_i(g, v) = v(g)$

**Definition 6.** The unanimity value function  $u_g$  is defined by

$$u_g(g') = \begin{cases} 1, & \text{if } g \subseteq g' \\ 0, & \text{otherwise} \end{cases}$$

The unanimity value functions form a basis for  $V$ . Thus every  $v \in V$  can be written as a unique linear combination of unanimity value functions  $u_g$  i.e.,  $v = \sum_{g \subseteq g^N} a_g u_g$  where  $a_g \in \mathbb{R}$  are called unanimity coefficients of  $v$ .

**Definition 7.** Let  $v$  be a value function with unanimity coefficients  $(a_{g'})_{g' \subseteq g}$  and network  $g$  be given. Then the Position value  $Y_i^{NPV}(g, v)$  is defined by (see [15]).

$$Y_i^{NPV}(g, v) = \sum_{g' \subseteq g} \frac{a_{g'} l_i(g')}{2l(g')} \quad \forall i \in N \quad (10)$$

Let  $v|_g$  denote the associated Cooperative game with respect to the Network game  $v$  considering the links in  $g^N$  as players. It follows that for every link  $l$  of  $g$ , if  $\Phi^{sh}$  denotes the Shapley value [13] of the restriction  $v|_g$  to the subsets of  $g$  then we have,

$$\Phi_l^{sh}(g, v|_g) = \sum_{g' \subseteq g: l \in g'} \frac{a_{g'}}{l(g')} \quad (11)$$

Combining (10) and (11), we obtain

$$Y_i^{NPV}(g, v) = \sum_{l \in L_i(g)} \frac{1}{2} \Phi_l^{sh}(g, v|_g), \forall i \in N \quad (12)$$

## 3. Bi-Cooperative Network Game

In this section we introduce the notion of a Bi-cooperative network game. Let  $g_1, g_2 \in G$  such that  $g_1 \cap g_2 = \emptyset$ . The pair  $(g_1, g_2)$  is called a bi-network. We assume that players in  $g_1$  provide positive contributions and those in  $g_2$  provide negative contributions. The rest are absentees. Let  $Q(g^N) = \{(g_1, g_2) \mid g_1, g_2 \in G \text{ with } g_1 \cap g_2 = \emptyset\}$  be the set of all bi-networks. For  $g_1, g_2 \in G$ , we denote by  $l(g_1, g_2)$  the total number of links in  $g_1$  and  $g_2$ . Let  $L_i(g_1, g_2)$  be the set of links that player  $i$  is involved in  $g_1$  and  $g_2$  i.e.,  $L_i(g_1, g_2) = L_i(g_1) \cup L_i(g_2)$ . We denote by  $N(g_1, g_2)$  the set of all player in  $g_1$  and  $g_2$  i.e.,  $N(g_1, g_2) = N(g_1) \cup N(g_2)$  and  $l_i(g_1, g_2)$  is the number of links of player  $i$  in  $g_1$  and  $g_2$ . A value function is a function  $b: Q(g^N) \rightarrow \mathbb{R}$ , with  $b(\emptyset, \emptyset) = 0$ . Thus a value function assigns every member  $(g_1, g_2)$  of  $Q(g^N)$  a real number its worth for which the nodes in  $g_1$  contribute positively to the network, nodes in  $g_2$  contribute negatively and the other remain indifferent. Note that here a node can have both positive and negative links simultaneously.

**Definition 8.** A Bi-cooperative network game is a pair  $(N, b)$ , of a set  $N$  of players and a value function  $b$  defined

on  $Q(g^N)$ . When there is no ambiguity on the player set  $N$ , we simply denote it by  $b$ . The set of all Bi-cooperative network games is denoted by  $\mathcal{BG}$ .

**Definition 9.** An allocation rule for  $\mathcal{BG}$  is a function  $Y : Q(g^N) \times \mathcal{BG} \rightarrow \mathbb{R}^n$  such that

$$\sum_i Y_i((g_1, g_2), b) = b(g_1, g_2), \forall (g_1, g_2) \in Q(g^N), b \in \mathcal{BG}$$

Where  $Y_i((g_1, g_2), b)$  represents the payoff to the  $i$ -th player with respect to  $b$  and the bi-network  $(g_1, g_2) \in Q(g^N)$ .

The following remark is important.

**Remark 1.** An allocation rule of a Bi-cooperative network game assigns payoff to each of the players that is dependent on the value added to the game due to switching her roles from being indifferent to a positive contribution (or alternatively from negative contribution to being indifferent).

## 4. Main Results

Prior to the main result of the paper, we introduce the associated link game of a Bi-cooperative network game considering the set of links as a player set. Let  $[g^N]$  denote the set of hypothetical players representing the links in  $g^N$ . Given  $(g_1, g_2) \in Q(g^N)$  denote by  $[g_1]$  (similarly  $[g_2]$ ) the set of all hypothetical players representing the links in  $g_1$  (similarly  $g_2$ ). Given a Bi-cooperative network game  $(N, b) \in \mathcal{BG}$ , define the associated link game  $([g^N], b^*)$ ,  $(b^* : Q([g^N]) \rightarrow \mathbb{R})$  of  $(N, b)$  as follows. Given  $(S, T) \in Q([g^N])$  there is a  $(g_1, g_2) \in Q(g^N)$  such that  $S = [g_1]$  and  $T = [g_2]$  and  $b^*(S, T) = b(g_1, g_2)$ . Given  $(g_1, g_2) \in Q(g^N)$ , set  $Q([g_1], [g_2]) = \{(S, T) \mid S \subseteq [g_1], T \subseteq [g_2] : S \cap T = \emptyset\}$ . It follows that for  $(S, T) \in Q([g_1], [g_2])$ , there is a  $(g_1', g_2') \in Q(g^N)$  with  $g_1' \subseteq g_1$  and  $g_2' \subseteq g_2$  such that  $S = [g_1']$ ,  $T = [g_2']$ .

**Definition 10.** An allocation rule  $Y$  is called a link-based allocation rule on  $\mathcal{BG}$  if there is some  $\Psi : Q(g^N) \times \mathcal{BG} \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$  such that for all  $(g_1, g_2) \in Q(g^N)$ ,  $b \in \mathcal{BG}$ ,  $i \in N$

$$\sum_{l \in ([g_1] \cup [g_2])} \Psi_l((g_1, g_2), b) = b(g_1, g_2)$$

and

$$Y_i((g_1, g_2), b) = \sum_{l \in L_i(g_1, g_2)} \frac{\Psi_l((g_1, g_2), b)}{2}$$

**Definition 11.** The Position value  $Y^{BNPV}$  is the allocation rule according to which each player  $i \in N$  in a Bi-cooperative network game  $(N, b)$  receives half of the LG value from each of her links considered as hypothetical players in the associated link game  $([g^N], b^*)$  i.e.,

$$Y_i((g_1, g_2), b) = \sum_{l \in L_i(g_1, g_2)} \frac{1}{2} \Phi_l^{LG}(b^*)([g_1], [g_2])$$

### 4.1. The Characterization of the Position Value

**Definition 12.** An allocation rule  $Y$  is efficient (EFF) if for any  $b \in \mathcal{BG}$  and  $(g_1, g_2) \in Q(g^N)$ ,

$$\sum_{i \in N(g_1, g_2)} Y_i((g_1, g_2), b) = b(g_1, g_2)$$

**Lemma 1.** The Position value satisfies EFF.

**Proof.** The proof follows immediately from the denition of the Position value.

For  $(g_1, g_2) \in Q(g^N)$  and  $l$  a link in  $g_1$  or  $g_2$  denote by  $(g_1, g_2) \setminus l$  the bi-network  $(g_1 \setminus l, g_2)$  whenever  $l \in g_1$  and the bi-network  $(g_1, g_2 \setminus l)$  whenever  $l \in g_2$ .

**Definition 13.** An allocation rule  $Y$  satisfies balanced link contribution (BLC) if for any  $b \in \mathcal{BG}$  and  $(g_1, g_2) \in Q(g^N)$  and  $i, j \in N$ .

$$\begin{aligned} & \sum_{l \in L_j(g_1, g_2)} (Y_i((g_1, g_2), b) - Y_i((g_1, g_2) \setminus l, b)) \\ &= \sum_{l \in L_i(g_1, g_2)} (Y_j((g_1, g_2), b) - Y_j((g_1, g_2) \setminus l, b)) \end{aligned}$$

The property of BLC asserts that the net effect of loosing a link that contributes positively (or negatively) is same for its constituent players.<sup>i</sup>

**Lemma 2.** The Position value satisfies BLC.

**Proof.** Let  $Y_i^{BNPV}, Y_j^{BNPV}$  be the  $i^{\text{th}}$  and  $j^{\text{th}}$  components of the Position value for a given  $b \in \mathcal{BG}$ ,  $(g_1, g_2) \in Q(G)$  and  $i, j \in N$ . It follows from (9) that,

$$\begin{aligned} & \sum_{l \in L_j(g_1, g_2)} \{Y_i^{BNPV}((g_1, g_2), b) - Y_i^{BNPV}((g_1, g_2) \setminus l, b)\} \\ &= \sum_{l \in L_j(g_1, g_2)} \left( \sum_{l_1 \in L_i(g_1, g_2)} \frac{1}{2} \Phi_{l_1}^{LG}(b^*)([g_1], [g_2]) - \sum_{l'_1 \in L_i((g_1, g_2) \setminus l)} \frac{1}{2} \Phi_{l'_1}^{LG}(b^*)([g_1 \setminus \{l\}], [g_2 \setminus \{l\}]) \right) \end{aligned}$$

$$\text{Let } [g_1] = S, [g_2] = T, [g'_1] = [g_1 \setminus \{l\}] = S',$$

$$[g'_2] = [g_2 \setminus \{l\}] = T'$$

$$\begin{aligned}
 &= \sum_{l \in L_j(g_1, g_2)} \left( \sum_{l_1 \in L_i(g_1, g_2)} \frac{1}{2} \Phi_{l_1}^{sh}(S \cup T, u) - \sum_{l'_1 \in L_i(g'_1, g'_2)} \frac{1}{2} \Phi_{l'_1}^{sh}(S' \cup T', u) \right) \\
 &= \frac{1}{2} \sum_{l \in L_j(g_1, g_2)} \left( \sum_{M \subseteq S \cup T} \frac{a_M}{|M|} |M_i| - \sum_{M' \subseteq S' \cup T'} \frac{a_{M'}}{|M'|} |M'_i| \right) \\
 &= \sum_{l \in L_i(g_1, g_2)} \{Y_j^{BNPV}((g_1, g_2), b) - Y_j^{BNPV}((g_1, g_2) \setminus l, b)\}
 \end{aligned}$$

Where for each  $K \subseteq N$ ,  $|K_i|$  denotes the number of hypothetical players in  $K$  representing the links that involve player  $i \in N$ .

We have shown that the Position value satisfies two properties viz., EFF and BLC. Now we will establish that there is only one allocation rule satisfying these two properties.

**Theorem 2.** *The Position value  $Y^{BNPV}$  is uniquely determined by the axioms of EFF and BLC.*

**Proof.** The proof is by induction on  $l(g_1, g_2)$  for  $(g_1, g_2) \in Q(g^N)$ . Let  $N = \{i, j\}$  such that  $l(g_1, g_2) = 1$ . Then either of the following holds.

Case (a) :  $g_1 = \{ij\}$ ,  $g_2 = \emptyset$  and

Case (b) :  $g_1 = \emptyset$ ,  $g_2 = \{ij\}$ .

Case (a) : If possible let,  $Y_1, Y_2 : Q(g^N) \times \mathcal{BG} \rightarrow \mathbb{R}^n$  be two different allocation rules satisfying EFF and BLC. It follows trivially from BLC that,  $Y_1^1((g_1, g_2), b) = Y_1^1((g_1, g_2))$ , and  $Y_2^2((g_1, g_2), b) = Y_2^2((g_1, g_2))$  so that  $Y_1 = Y_2$ . Case (b) follows immediately from Case (a) and so the proof is omitted. Thus the result holds for a single link of positive or negative contributors.

Let  $l(g_1, g_2) > 1$ . For  $l(g_1, g_2) \leq k - 1$ , suppose that the allocation rule which satisfies EFF and BLC is unique for  $b$ . BLC, we obtaining the following.

$$\begin{aligned}
 \sum_{l \in L_2(g_1, g_2)} (Y_1((g_1, g_2), b) - Y_1((g_1, g_2) \setminus l, b)) &= \sum_{l \in L_1(g_1, g_2)} (Y_2((g_1, g_2), b) - Y_2((g_1, g_2) \setminus l, b)) \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 \sum_{l \in L_n(g_1, g_2)} (Y_1((g_1, g_2), b) - Y_1((g_1, g_2) \setminus l, b)) &= \sum_{l \in L_1(g_1, g_2)} (Y_n((g_1, g_2), b) - Y_n((g_1, g_2) \setminus l, b))
 \end{aligned}$$

This would further imply that

$$\begin{aligned}
 l(L_2(g))Y_1 - l(L_1(g))Y_2 &= \sum_{l \in L_2(g)} (Y_1((g_1, g_2), b) - Y_2((g_1, g_2) \setminus l, b)) \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 l(L_n(g))Y_1 - l(L_1(g))Y_n &= \sum_{l \in L_n(g)} (Y_1((g_1, g_2), b) - Y_n((g_1, g_2) \setminus l, b))
 \end{aligned}$$

And by EFF, we have,

$$\sum_{i \in N(g_1, g_2)} (Y_i((g_1, g_2), b)) = b(g_1, g_2).$$

The above system of  $n$  equations has  $n$  independent variables  $Y_1, Y_2, \dots, Y_n$ . It is customary to show that the system has a unique solution. Therefore, for  $l(g_1, g_2) = k$ , the allocation rule which satisfies EFF and BLC is the unique allocation rule.

## 5. Conclusions

In this paper, we have introduced the notion of a Bi-cooperative game. The Position value as a link based solution concept is proposed. It is then characterized by the

axioms of EFF and BLC. We have kept the nomenclature from their cooperative counterparts; however the concept and the formulation differ due to the presence of bi-polarity among the players within a network. In a future work we propose to study alternative characterizations in the line of [16].

## ACKNOWLEDGEMENTS

The work done in this paper is under the UGC Major Research Project UGC-India 42-26/2013(SR).

## REFERENCES

- 
- [1] J. M., Bilbao, Cooperative Games on Combinatorial Structures, Kluwer Academic Publishers, Boston, 2000.
  - [2] Bilbao, J. M., Fernandez, J.R., Jimenez, N. and Lopez, J. J., 2008a, Biprobabilistic values for bi-cooperative games, *Disc. App. Math.* 156, 2698-2711.
  - [3] Gogoi, L., Borkotokey, S., and Kumar, S., 2014, Bi-cooperative network games: A note, *Journal of Assam Academy of Mathematics*, 6, 17-31.
  - [4] Harsanyi, J. C., 1963, A simplified bargaining model for the n-person cooperative game, *International Economic Review*, 4, 194-220.
  - [5] Hsiao, C. R., and Raghavan, T.E.S., 1993, Shapley value for multichoice cooperative games, *Games and Economic Behaviour*, 5, 240-256.
  - [6] C. Isby, David, World War II: Double Agent's D-Day Victory, World War II, June 2004; accessed at <http://www.historynet.com/world-war-ii-double-agents-d-day-victory.htm>, 2004.
  - [7] Jackson, M. O., and Wolinsky, A., 1996, A Strategic Model of Social and Economic Networks, *Journal of Economic Theory*, 71(4) 44-74.
  - [8] Jackson, M. O., 2005, Allocation rules for network games, *Games and Economic Behavior*, 51 (1), 128-154.
  - [9] Labreuche, C., and Grabisch, M., 2008, A Value for bi-cooperative games, *International Journal of Game Theory*, 37 (3), 409-438.
  - [10] Manea, M., 2011, Bargaining in Stationary Networks. *American Economic Review*. 101 (5), 2042-80.
  - [11] Myerson, R., 1977, Graphs and Cooperation in Games, *Mathematics of Operations Research*, 2(3), 225-229.
  - [12] Park, J-H., 2000, International trade agreements between countries of asymmetric size. *Journal of International Economics*. 50, 473-495.
  - [13] L. S., Shapley, A value for n-person games, in Kuhn, H., Tucker, A.W., *Contribution to the Theory of games II*, Princeton, New Jersey: Princeton University Press, 307-317, 1953.
  - [14] Slikker, M., 2005<sup>a</sup>, Link monotonic allocation schemes, *International Game Theory Review*, 7(4), 419-429.
  - [15] Slikker, M., 2005<sup>b</sup>, A Characterization of the Position value, *International Journal of Game Theory*, 33, 505-514.
  - [16] van den Nouweland, A., and Slikker, M., 2012, An axiomatic characterization of the position value for network situations, *Mathematical Social Sciences*, 64, 266-271.

---

<sup>i</sup> Note that the BLC in classical Network games is different from that of a Bi-cooperative network game in the sense that the net effects of the links are considered over both positive and negative contributions of the game simultaneously on the basis of their presence in  $g_1$  or  $g_2$ .