

Does Linearity of the Lorentz Transformation Follow from the two Postulates of Special Relativity

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Abstract One would be hard pressed to find a derivation of the Lorentz transformation (LT) that uses *only* the two postulates of special relativity, and that is *entirely* mathematically rigorous. Most, if not all, derivations presume that, in addition to the two postulates, the LT has a particular form such as linearity. We will give several examples of such derivations from several esteemed textbooks. We will also discuss how Einstein derived the LT in his revolutionary 1905 paper, “On the Electrodynamics of Moving Bodies.” Then, we will provide a rigorous derivation of the LT using *only* the two postulates.

Keywords Lorentz Transform, Mathematically Rigorous Derivation, Special Relativity, Postulates of Relativity

1. Introduction

It is well known that two inertial frames in special relativity are related via the LT. What appears to be unknown is how to derive the LT without making any additional assumptions about the form of the transformation. In most cases, authors presume that the transformation is linear, i.e., that it has the form

$$\begin{aligned}x' &= \alpha x + \beta ct \\ c t' &= \lambda x + \kappa ct\end{aligned}\quad (1)$$

where $\alpha, \beta, \lambda, \kappa$ are constants to be determined. As will be elaborated upon in the following section, some authors will, at best, give a “hand-waving” argument as to why the transformation is linear. Failure to provide a rigorous proof of why the two postulates imply linearity leaves open the possibility that there may be a nonlinear transformation that is consistent with the two postulates. Our ultimate goal is to eliminate nonlinearity as a possibility.

In the following section we will discuss several example derivations of the LT that appear in the literature. Subsequently, we will provide a rigorous proof that the LT is a consequence of the two postulates without making any additional assumptions.

2. Example Derivations

Let us first describe the familiar setting that is common to all of the examples that we intend to discuss. Consider two

inertial frames, S and S' . We'll assume that the S' frame is moving at a speed v in the direction of increasing values of x . Moreover, our derivation will be based on the conventional that $x = 0 = x'$ whenever $t = 0 = t'$.

Since there is no motion in either the y or z direction it is taken that $y' = y$ and $z' = z$. So all that remains to determine the interrelatedness between t', t, x' , and x .

2.1. Spacetime Physics

Arguably, Spacetime Physics by Taylor and Wheeler, [1], is among one of the most popular texts on special relativity. On page 100, it is asserted that the LT has the form (1).

Let us now discuss the argument given in [1] of why the two reference frames are related via a linear transformation. An observer at rest in frame S observes three events that occur in the same spatial location at three equally spaced times, say $t=1$, $t=2$, and $t=3$. The time dilation formula, $t'=\gamma t$, was previously established with rigor. Hence, an observer at rest in S' will also see these same three events occur at equally spaced times, $t' = \gamma$, $t' = 2\gamma$, and $t' = 3\gamma$.

Subsequently, the transformation given by $t' = Kt^2$, where K is a constant, is considered. Under this particular transformation, the authors demonstrate that an observer in S' does not see the three events occurring at three equally spaced times.

The fact that equally spaced times in one frame yields equally spaced times in the other may indeed only be possible if the frames are related via a linear transformation. But to provide a rigorous proof, the task of ruling out an *arbitrary* nonlinear transformation must be completed.

2.2. Hyperbolic Trigonometric Functions

The invariance of the speed of light leads to the equation

$$c^2 t^2 - x^2 = 0 \quad (2)$$

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where (ct, x, y, z) are the coordinates of an arbitrary inertial reference frame. This motivates the likes of [2] and [3] to consider a transformation of the form

$$\begin{aligned} x' &= x \cosh(\Psi) + ct \sinh(\Psi) \\ ct' &= x \sinh(\Psi) + ct \cosh(\Psi) \end{aligned} \quad (3)$$

What motivates (3) is the fact that $c^2(t')^2 - (x')^2 = 0$ follows from both (2) and (3). However, once again, we have a situation whereby Ψ is tacitly assumed to be independent of both x and t . So not only is (3) a linear transformation, it's a linear transformation of a particular form. It's worth re-emphasizing that we are not arguing that the (3) is incorrect. We will ultimately *prove* that it is correct. But if Ψ is simply assumed to be independent of x and t , then a transformation of the form (3) in which Ψ depends on x and/or t remains possible.

It is worth pursuing the argument presented in [2] a bit further. In letting $x = 0$, it follows from (3) that

$$x'/ct' = \tanh(\Psi).$$

But $x'/t' = v$ is the velocity of S' relative to the S . Thus,

$$\tanh(\Psi) = v/c \quad (4)$$

From this it is easily shown that

$$\sinh(\Psi) = \gamma v/c \text{ \& } \cosh(\Psi) = \gamma \quad (5)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is known as the Lorentz factor.

Substituting (5) into (3) yields the familiar LT. The drawback of this approach is that there's still no guarantee that Ψ is independent of x and t . For example, in choosing $\Psi = xt$, the above argument can be implemented to draw the same conclusion.

2.3. Special Relativity & Classical Field Theory

Next, we discuss [4] which is one of four books of Susskind and Friedman's "The Theoretical Minimum" series. They begin by pointing out that $x' = 0$ if and only if $x = vt$. This is certainly true in the way we described the frames S and S' . Then they claim that

$$x' = (x - vt)f(v) \quad (6)$$

where f is a function to be determined. Clearly, (6) certainly implies that $x' = 0$ if and only if $x = vt$. The converse, however, is not true. As a random example,

$$x' = \sin(x - vt)f(v)$$

also ensures that $x' = 0$ if and only if $x = vt$. Yet again we have a situation in which prior knowledge of the LT is used to derive the LT.

2.4. On the Electrodynamics of Moving Bodies

On page 6 of [5], it is written "In the first place it is clear that the equations must be linear on account of the properties of homogeneity which we attribute to space and time." There are two points to be made about this assertion. The first is that Einstein is now going beyond his two postulates to support his claim.

Secondly, it isn't immediately clear how homogeneity guarantees that two inertial reference frames are related linearly. A common definition of the term *homogeneous* is that there is no special place in the universe. More precisely, at any given point in time, the universe is everywhere the same assuming that we consider sufficiently large spatial regions. In my best effort to put this in mathematical terms, let $\phi(x)$ denote a physical quantity of some kind at a spatial location x . Homogeneity requires the function ϕ to be invariant under spatial translations, $\phi(x + a) = \phi(x)$. How this implies that two inertial reference frames are related by a linear transformation is unclear.

3. A Rigorous Derivation of the Lorentz Transformation

In this section we will derive the Lorentz Transformation using only the two postulates of special relativity. We will do so without making the presupposition that the transformation has any particular form.

For easy reference, let's state the two postulates.

1. **The Principle of Relativity:** The laws of physics are the same in all inertial reference frames.
2. **Invariance of the Speed of Light:** The speed of light in a vacuum is constant and independent of the motion of the light source or the observer.

The length contraction and time dilation formulas are easily and *rigorously* derived in essentially any source on the topic. Thus, we will simply state these two formulas.

The time between two distinct events that occur at the same spatial location in the S' frame is related to the time in the S frame by the equation

$$t = \gamma t'. \quad (7)$$

Furthermore, suppose that a rod that is at rest in the S' frame, and is of length l' as measured by an observer who is at rest in that frame. An observer in the S frame will determine the length of the rod to be

$$l = l'/\gamma. \quad (8)$$

We will now proceed to determine how x' can be written in terms of x and t . As asserted in Section 2, S' travels in the direction of increasing values of x at a speed v . That is, given an arbitrary constant K , the plane $x' = K$ in the S' frame travels in the direction of increasing values of x at speed v . So a stationary observer in S will describe the motion of such a traveling plane via the kinematic equation $x = vt + x_0$, where $|x_0|$ is the distance between the traveling plane and the plane $x = 0$ at $t = 0$ according to the observer in S .

Let us consider a specific example in which $x' = 17$.

Figure 1 depicts a snapshot of the plane $x' = 17$ at $t = 0 = t'$, the moment in which the planes $x = 0$ and $x' = 0$ align. While a stationary observer in S' at $x' = 0$ will see the plane $x' = 17$ as stationary and 17 units away, a stationary observer in S at $x = 0$ will see the plane at a

contracted length of $17/\gamma$ units away at that moment. Thus, we substitute $17/\gamma$ for x_0 in the kinematic equation. Doing so yields

$$x = vt + 17/\gamma,$$

which can be rewritten as

$$x = vt + 17/\gamma,$$

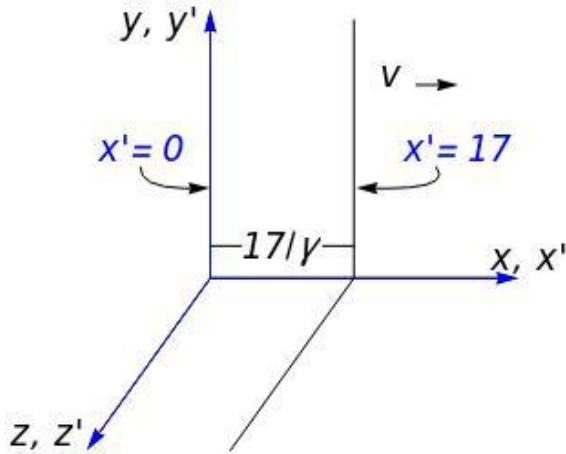


Figure 1. The plane $x' = 17$ at time $t = 0 = t'$

Of course there is nothing special about the number 17. An arbitrary but fixed value of x' refers to a stationary plane in S' and a plane traveling at speed v in S . Thus, the very same argument can be applied to x' that was applied to 17 to obtain

$$x' = \gamma(x - vt). \quad (9)$$

Our next task will be to obtain ct' in terms of x and t . In proving (9) we viewed planes of the form $x' = K$ as traveling in the direction of increasing values of x at speed v . It is equally legitimate to consider S as the moving frame and S' as the stationary frame. That is, we can repeat the same method that we used to obtain (9) to deduce

$$x = \gamma(x' + vt'). \quad (10)$$

It follows from (10) that $vt' = -x' + x/\gamma$, or equivalently

$$ct' = -cx'/v + cx/(\gamma v).$$

By (9), we can substitute $\gamma(x - vt)$ for x' in the above equation. Doing this and a little algebra yields

$$ct' = \gamma(ct - vx/c) \quad (11)$$

as desired.

Proving (9) and (11) was our primary objective for this section. It's worth noting that (9) and (11) can be combined to conclude that

$$(ct')^2 - (x')^2 = (ct)^2 - x^2$$

which holds whenever $(ct, x) = (ct', x')$.

4. Conclusions

We have presented several textbook examples of how the LT is derived. In every case, the author took advantage of their prior knowledge regarding the LT's form. This raises the question of whether the LT *must* be of the stated form. In particular, on the topic of whether the transformation must be linear, a hand-waving argument is presented at best. We also discussed the shortcomings of Einstein's reasoning of why the LT is linear as presented in [5].

Next, we provided a rigorous derivation of the Lorentz transform without presupposing that it has any particular form from the start.

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