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The Transformation of Spacetime Coördinates between Inertial Frames

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Abstract We make only the following two requirements: (1) inertial invariance and (2) that the product of two boosts in a given direction yields a boost in the same direction. It is shown that there are three (consistent) possibilities: (a) a Galilean transform, (b) a Lorentz transform, or (c) a rotation in Euclidean spacetime. For the case of the Lorentz transform, the relativistic rule for composition of velocities is obtained, with the velocity of light arising as a constant of integration.

Keywords Lorentz Transformation, Gallilei Transformation, Special Relativity

1. Introduction

Physics in the twentieth century may be characterized by an understanding of the rôle played by the symmetries of physical system. Although the subject became prominent in the 1920s through the work of Hermann Weyl in the subject of quantum mechanics, an important precursor of group theory in physics was proposed in 1911, in the study of the group properties of space-time transformations by Phillip Frank and Hermann Rothe.[1] They showed the unique rôle of the Galilei and Lorentz transformations in this topic, inasmuch as only these transformations in two dimensional space-time (x,t) were consistent with the group transformation property. Unfortunately, although that approach could have led to highlighting the rôle of symmetry groups in all physical systems, it did not happen that way.

We have reconsidered the matter studied by Frank and Rothe in this work, requiring that the composition of two boosts in space-time be a boost of similar character. We find three possibilities, corresponding to a Galilei transformation, a Lorentz transformation, and a rotation in a Euclidean space-time. In addition we find in Section II that a space-time dilation may be present in each of these three cases, but a physical reason for eliminating the dilatations is described. The issues are discussed in Section III. The Euclidean rotation occurs as mathematical possibility, but it can be rejected upon physical grounds, as we discuss. While the symmetries imposed by group compositions provide an important limitation upon physical systems, they must be supplemented by physical requirements, in this and many other cases.

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2. Analysis

The most general linear transformation of spacetime coördinates (x,t) into (x',t') that transforms the origin $(0,0)$ into itself is

$$x' = a\,x + b\,t$$
$$t' = c\,x + d\,t$$

The parameters a,b,c,d depend only on the velocity v of the two frames, which we introduce by requiring that the points $x'=0$ lie along the line $x = vt$, so that

$$\begin{aligned} x' &= a(v)\,[x - v\,t] \\ t' &= c(v)\,x + d(v)t \end{aligned} \tag{1}$$

Correspondingly, we require that the curve $x=0$ correspond to the line $x' = -vt'$, so that

$$-v\,a(v) = -v\,d(v)$$

and $a(v) = d(v)$. Let us write these equations in matrix form:

$$\begin{vmatrix} x' \\ t' \end{vmatrix} = M(v) \begin{vmatrix} x \\ t \end{vmatrix} \tag{2}$$

where

[1] Ü ber die Transformation der Raumzeitkoordinationen von ruhenden auf bewegte Systeme, Annalen der Physics, 3, 325-355, 1911. Our English translation of this article appears in this journal, 11, 141-152 (2021).

$$M(v) = \begin{vmatrix} a(v) & -va(v) \\ c(v) & a(v) \end{vmatrix} \tag{3}$$

The composition of two such transformations v and v' is given by

$$\begin{vmatrix} x'' \\ t'' \end{vmatrix} = M(v'') \begin{vmatrix} x \\ t \end{vmatrix}$$

where

$$M(v'') = M(v')M(v) = \begin{vmatrix} a(v') & -v'a(v') \\ c(v') & a(v') \end{vmatrix} \cdot \begin{vmatrix} a(v) & -va(v) \\ c(v) & a(v) \end{vmatrix}$$
$$= \begin{vmatrix} a(v')a(v) - v'a(v')c(v) & -(v+v')a(v')a(v) \\ c(v')a(v) + a(v')c(v) & -vc(v')a(v) + a(v')a(v) \end{vmatrix}$$
$$\equiv \begin{vmatrix} a(v'') & -v''a(v'') \\ c(v'') & a(v'') \end{vmatrix}$$

From the requirement that the diagonal elements of $M(v'')$ are equal we obtain $-v'a(v')c(v) = -vc(v')a(v)$, so that

$$\frac{c(v')}{v'a(v')} = \frac{c(v)}{va(v)} = -\kappa \tag{4}$$

where κ is a constant independent of v or v'. Thus $c(v) = -\kappa va(v)$. Furthermore, from the requirement $M_{12}(v'') = -v''M_{11}(v'')$ we obtain

$$-(v'+v)a(v')a(v) = -v''[a(v')a(v) + \kappa v'va(v')a(v)] \tag{5}$$

or

$$v'' = \frac{v'+v}{1+\kappa v'v} \tag{6}$$

The coefficient $M_{11}(v'')$ leads to the nonlinear functional equation

$$a(v'') = a(v')a(v)\,[1+\kappa v'v] \tag{7}$$

In particular, when $v'=0$, then $v''=v$ and $a(v) = a(0)a(v)$, or $a(0)=1$. Let us differentiate Eq. (7) with respect to v' and then set $v'=0$ to obtain

$$\dot{a}(v)[1-\kappa v^2] = a(v)[\dot{a}(0) + \kappa v] \tag{8}$$

This is a linear first order differential equation for $a(v)$, with the parameter $\dot{a}(0) = \alpha$ present. We may write it as

$$\frac{\dot{a}(v)}{a(v)} = \frac{\alpha + \kappa v}{1-\kappa v^2} \tag{9}$$

The parameter κ may be positive, zero, or negative.

For the simplest case $\kappa = 0$ we obtain the solution $a(v) = \exp[\alpha v]$, or

$$\begin{aligned} x' &= e^{\alpha v}\,[x - vt] \\ t' &= e^{\alpha v}\,t \end{aligned} \qquad v'' = v + v' \tag{10}$$

For $\alpha = 0$ we obtain the Galilei transformation, whereas the parameter α also introduces a space-time dilation or contraction.

For positive κ we write $\kappa = 1/c^2$. where c is the characteristic velocity. The solution to Eq. (9) is

$$a(v) = \left[\frac{c+v}{c-v}\right]^{c\alpha/2} \frac{1}{\sqrt{1-v^2/c^2}}$$

We may express these relations more elegantly in terms of the 'rapidity' parameter r, with $v/c = \tanh r$:

$$a(r) = e^{\alpha cr} \cosh r \qquad r'' = r + r'$$

Equivalently

$$\begin{aligned} x' &= e^{\alpha cr}\,[x\cosh r - ct\sinh r] \\ ct' &= e^{\alpha cr}\,[ct\cosh r - x\sinh r] \\ v'' &= \frac{v+v'}{1+vv'/c^2} \end{aligned} \tag{11}$$

Evidently, the parameter α produces a scale transformation, as before. With $\alpha = 0$ we obtain the Lorentz boost with velocity v.

Finally, for negative κ we write $\kappa = -1/c^2$. where c is the velocity scale. The solution is

$$a(v) = e^{\alpha c \tan^{-1}(v/c)} \frac{1}{\sqrt{1+v^2/c^2}}$$

The solution to Eq. (9) is conveniently expressed in terms of the angle θ, with $v/c = \tan\theta$:

$$a(v) = e^{\alpha c\theta} \cos\theta \qquad \theta'' = \theta + \theta'$$

Equivalently

$$\begin{aligned} x' &= e^{\alpha c\theta}\,[x\cos\theta - ct\sin\theta] \\ ct' &= e^{\alpha c\theta}\,[ct\cos\theta + x\sin\theta] \\ v'' &= \frac{v+v'}{1-vv'/c^2} \end{aligned} \tag{12}$$

For $\alpha = 0$ this corresponds to a rotation by angle θ in the Euclidean (x, ct) plane. Again, the parameter α induces a scale transformation.

3. Discussion

First we consider the Galilei transformation, Eq. (9), that was obtained in the previous section. For the three dimensional case this can be generalized to obtain

$$\begin{aligned} \vec{r}\,' &= \exp[\vec{a}\cdot\vec{v}\,]\,[\vec{r} - \vec{v}\,t] \\ t' &= \exp[\vec{a}\cdot\vec{v}\,]\,t \end{aligned} \tag{13}$$

This particular transformation is not spatially isotropic, because it picks out a special direction identified by the vector $\vec{a}$. Upon this basis, we must make the replacement $\vec{a} = \vec{0}$ to recover the pure Galilei transform. These transforms yield an Abelian group.

For the case of the Euclidean rotation in space-time, we should similarly eliminate the scale factor $\exp[\alpha c\theta]$ by setting $\alpha = 0$, to obtain the pure space-time rotation

$$\begin{aligned} x' &= x\cos\theta - ct\sin\theta \\ ct' &= ct\cos\theta + x\sin\theta \end{aligned} \tag{14}$$

This equation clearly satisfies the group composition property, and it can readily be generalized to rotations in four-dimensional space-time, leading to the group $O(4)$.

Although for this case a characteristic velocity c arises, there is no limit upon the velocity v. In fact, the rule for composition of velocities, Eq. (12), permits arbitrarily large velocities. Furthermore, the composition of two positive velocities v_1 and v_2 is negative, provided that $v_1 v_2 > c^2$. This bizarre circumstance requires rejection of this transformation as a physical possibility. In the Euclidean transformation, the characteristic velocity would not be inertially invariant, in contradiction to the inference of the Michelson-Morley experiment. In fact, it would yield a length dilation and time contraction, in contradiction with experiment.

By similar reasoning, one must set $\alpha = 0$, or the scale factor $\exp[\alpha cr] = 1$ in Eq.(11) to obtain the Lorentz transformation. As promised, the inertial invariance of the characteritic velocity arises as a consequence of the group composition property for this case. For consistency with Maxwell's equations, that velocity must be interpreted as the velocity of light.

Note that the Lorentz transformations in three spatial dimensions do not form a group, since the product of two transformations with respective velocities $\vec{v}_1$ and $\vec{v}_2$ with $\vec{v}_1 \times \vec{v}_2 \neq \vec{0}$ does not yield a Lorentz transformation, but in addition a spatial rotation in the plane perpendicular to $\vec{v}_1$ and $\vec{v}_2$ is needed. The full Lorentz group coonsists of spatial rotations and Lorentz boosts, the rotations forming a subgroup.