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About the Existence of Black Holes

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Abstract The existence of black holes is based on singularity in the Schwarzschild metric at the Schwarzschild radius. Another singularity is at r = 0. These singularities of a spherically symmetric non-rotating, uncharged mass of radius R, are considered here. Considering Newton's shell theorem, the gravitational potential falls off linearly with r for $r < R$. The point r = 0 is an infinitesimally small location in space. No mass can be indefinitely condensed to this point. Thus, when investigating the concept of a mass, one has to consider its finite radius. By including the shell theorem for $r < R$, the singularity at $r \to 0$ is removed. It is also shown, that a situation where $r_s < R$ but $r_{ph} > R$ is possible. Therefore, the existence of a photon sphere light ring does not necessarily indicate a black hole. It is shown, that the condition for a gravitational collapse is $R < \frac{3}{2} r_s$, and not $R < r_s$. Further in this work, the question of maximal density is considered and compared to the quantum limit of mass density put by Planck's units as dictated from dimensional analysis. There are two main claims here: 1. Black holes (if exist) will result only if the Schwarzschild radius is larger than 2/3R. 2. General relativity leads to quantization of gravity.

Keywords Newton's Shell theorem, Schwarzschild singularities, Photon sphere, Planck's units, Quantization of Gravity

1. Introduction

The Schwarzschild solution appears to have singularities at r = 0 and $r = r_s$, with $r_s = \frac{2GM}{c^2}$ which cause the metric to diverge at these radii. There is no problem as long as $R > r_s$.

The singularity at $r = r_s$ divides the Schwarzschild coordinates in two disconnected regions. The exterior Schwarzschild solution with $r > r_s$ is the one that is related to the gravitational fields of stars and planets. The interior Schwarzschild solution with $0 \leq r < r_s$, which contains the singularity at r = 0, is completely separated from the outer region by the so-called singularity at $r = r_s$. The Schwarzschild coordinates therefore give no physical connection between these two regions.

The singularity at $r = r_s$ is an illusion however; it is an instance of what is called a coordinate singularity. As the name implies, the singularity arises from a bad choice of coordinates or coordinate conditions. When changing to a different coordinate system, the metric becomes regular at $r = r_s$ and can extend the external region to values of r smaller than r_s. Using a different coordinate transformation one can then relate the extended external region to the inner region.

As will be shown later on, this so-called singularity is completely removed even for the Schwarzschild coordinates.

The case r = 0 is different, however. If one asks that the solution be valid for all r, one runs into a true physical singularity, or gravitational singularity, at the origin. To see whether this is a true singularity one must look at quantities that are independent of the choice of coordinates. One such important quantity is the Kretschmann invariant [19], which is given by $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$. For a Schwarzschild black hole of mass M and radius R, the Kretschmann invariant is $K_r = \frac{48G^2M^2}{c^4R^6} \approx \frac{842G^2}{\rho^2c^4}$. Obviously it is independent of the radius R, and is always greater than 0 (provided we accept the physical requirement that the density ρ is never infinite).

At r = 0 the curvature can become infinite, if and only if, it represents a point particle of zero mass. One cannot compress a finite mass into an infinitesimal point (of radius R = 0). Any finite mass should have a finite radius R > 0. Therefore, when looking into the point r = 0, one needs to consider the effects of the outer shells of matter which surround the r = 0 point.

General relativity predicts that any object collapsing beyond a certain point (for stars this is the Schwarzschild radius) would form a black hole, inside which a singularity (covered by an event horizon) would be formed.

As will be shown in the following, no such singularities exist.

2. Spherically Symmetric, Non-rotating, Uncharged Spherical Body

Consider a spherically symmetric, non-rotating, uncharged mass (a stellar object or an elementary particle).

Under spherical symmetry, and at a remote distance in empty space outside the object, the Schwarzschild metric [1,2,3] is given by:

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$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{2GM}{rc^2}\right) & & & \\ & \frac{-1}{\left(1-\frac{2GM}{rc^2}\right)} & & \\ & & -r^2 & \\ & & & -r^2 sin^2\theta \end{pmatrix} \quad (1)$$

This metric describes very well the gravitational effects of that mass on gravitational field curvature, and dynamics of moving particles, in the empty regions outside the object.

However, it suffers of two problems:

1. *At the Schwarzschild radius $r_s = \frac{2GM}{c^2}$, the metric has a singularity.*
2. *At r → 0, the metric diverges.*

The first singularity is not a real one, as it can be removed by appropriate coordinate transformations [5-13].

The second singularity cannot be removed by coordinate transformation. It is an inherent singularity at r → 0 in any inverse square law central force problem of gravitational field). One so far have not been able to relate to the divergency problem of force which is inversely proportional to the square of the distance r from center of the field source.

To solve this problem, we must investigate the behavior of the field inside the mass.

If any mass source is finite in size, the solution to the potential as a function of distance must be modified, so that it includes the region where r < R, the radius of the mass.

In addition, the Einstein field equation should be related to in non-empty space. But to a very good approximation, the Einstein equation in the inside of even the heaviest stellar objects can be considered same as empty space.

This assumption will only be valid for a perfect fluid solution, where of the $T_{\mu\nu}$ tensor, only energy density is considered, while momentum density, shear stress, momentum flux and pressure are either null or irrelevant.

3. Shell Theorem

Let R be the radius of an uncharged non-rotating spherically symmetric mass, of given density $\rho(r)$, with r being the radial distance from center (at r = 0).

Assuming this object has a volume $v = \frac{4\pi r^3}{3}$ and radius R, its average density will be $\langle\rho\rangle = \frac{3M}{4\pi R^3}$ and since $M = \int_0^R 4\pi r^2 \rho(r)\, dr$ it is straightforward to see that

$$M = \frac{4\pi}{3} \lim_{R\to 0} R^3\, \rho(0)$$

This means that unless the density at the center $\rho(0)$ is infinite, the mass must vanish (M = 0).

The only case of M > 0 is possible if and only if $\rho(0)$ is infinite. And if we assume physical continuity, then we may write

$$\lim_{R\to 0} R^3\, \rho(R) = \frac{3M}{4\pi}$$

In other words

$$\rho(R) = \frac{3M}{4\pi R^3}$$

And for any finite mass M

$$\lim_{R\to 0} \rho(R) \longrightarrow \infty$$

However, as will be shown later, the density of any object must have an upper limit and it cannot become infinitely large. To resolve this contradiction, one must accept, that no mass can be condensed to a singular point r = 0. A mass will always have some finite density and a finite radius (volume).

Therefore, when considering gravitation at r = 0, one must consider the effects of surrounding mass. Condensed as it might be, but still of finite density and surrounding r = 0.

In classical mechanics, the shell theorem, proven already by Isaac Newton [14] states that:

A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.

If the body is a spherically symmetric shell (i.e., a hollow ball), no net gravitational force is exerted by the shell on any object inside, regardless of the object's location within the shell.

A corollary is that inside a solid sphere of constant density, the gravitational *force* within the object varies linearly with distance from the center, becoming zero by symmetry at the center of mass.

Due to the shell theorem, the gravitational potential of a spherically symmetric object of mass M and radius R, as a function of distance r, from the object's center is given by:

$$\Phi(r) = -\frac{GM}{R^3} \begin{cases} \frac{1}{2}(3R^2 - r^2)\ for\ r \le R \\ \frac{R^3}{r} \qquad for\ r > R \end{cases} \quad (2)$$

The result for r < R is obtained by summation of the potential at some point p inside the mass from its spherical shell between R and r, and the remaining mass inside sphere of radius r.

The potential at point p due to the shell is given by

$$\Phi_{shell}(p) = -2\pi\varrho G(R^2 - r^2) \quad (3)$$

The potential at same point p due to the remaining inner sphere is given by

$$\Phi_{inner}(p) = -4\pi\varrho G r^2 \quad (4)$$

The total potential at point p inside the sphere is a superposition of both $\Phi_{shell}(p)$, and $\Phi_{inner}(p)$. Thus, (for r < R):

$$\Phi_p(r) = \Phi_{shell}(r) + \Phi_{inner}(r) = -4\pi\rho G\left(\frac{R^2}{2} - \frac{r^2}{6}\right) \quad (5)$$

$$\Phi_p(r) = -\frac{3GM}{2R^3}\left(R^2 - \frac{1}{2}r^2\right) \quad (6)$$

4. The Identification of $\Phi(r)$ as the Gravitational Potential

In the Schwarzschild metric, the g_{00} term is a function of the distance r and one may write:

$$g_{00}(r) = \left(1 + \frac{2\Phi(r)}{c^2}\right) \quad (7)$$

First argument – dimensions

$\Phi(r)$ is a function of distance r, and must have the dimensions of c^2. It must obey the constraint of $\lim_{r\to\infty}\Phi(r) \longrightarrow 0$ as should be the case at infinite distance in empty flat space.

Its units should be [G][Kg]/[m] in order to make $\frac{\Phi(r)}{c^2}$ dimensionless.

We therefore have a good reason to assume that $\Phi(r)$ is the gravitational potential due to a spherically symmetric object, independent of the size and of the density of that object. Exactly as described by eq. 2.

Second argument- Einstein equation under spherical symmetry

For a curved spacetime, based on the Riemann tensor, it can be shown that $\Phi(r)$ must have the form of a gravitational potential (see Appendix 1).

Following these arguments, it is justified to identify $\Phi(r)$, as the gravitational potential of the field created by a spherically symmetric non-rotating uncharged mass.

5. Inside the Sphere (r < R)

In the case where r < R (inside the mass), $\mathbb{K} \neq 0$ where $\mathbb{K} = \frac{16\pi G}{c^2}$

G=$6.6743 \times x10^{-11}$ and so $\mathbb{K} \approx 37\ x10^{-27}$, so unless the internal density ρ is of the order of 10^{27} Kg/m^3, the equation can be solved with the assumption of $\mathbb{K}\rho = 0$.

Even the heaviest neutron stars have overall densities of the order of $6\ x10^{17}$ Kg/m^3. For elementary particles, the neutron for instance, has density of approximately $3\ x10^{17}$ Kg/m^3. So, for all practical calculations one may assume $\mathbb{K} = 0$ inside any mass (that is, for r < R).

For the internal pressure p, as long as the internal pressure is much less than $10^{24}\ Pa$, $\mathbb{K}p = 0$. For the Sun for instance, the internal core pressure is $p10^{11}\ Pa$.

Finding the solution of the homogeneous differential equation with $\mathbb{K}\rho = 0, \mathbb{K}p = 0$, will lead to the non-homogeneous solution with $\mathbb{K} = \text{constant}$. But we will concentrate on the homogeneous solution, since $\mathbb{K} \approx 0$.

In case of a constant $\mathbb{K}$, the non-homogeneous solution can be found.

This is the case for a perfect fluid.

Under the assumption of $\mathbb{K} \approx 0$, for the interior of the sphere, the time separation inside the sphere is given by $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$:

$$d\tau^2 = \left(1 + \frac{2\Phi(r)}{c^2}\right)dt^2 - \frac{1}{c^2}(h(r)r^2dr - r^2d\theta^2 - r^2sin^2\theta d\varphi^2) \quad (8)$$

6. Modified Schwarzschild Metric

Inserting the expression for $\Phi(r)$:

$\Phi(r) = -4\pi\rho G\left(\frac{R^2}{2} - \frac{r^2}{6}\right)$ (for (r $\leq$ R)

$\Phi(r) = -4\pi\rho G\left(\frac{R^3}{3r}\right)$ (for (r $>$ R)

The Schwarzschild metric inside the sphere (r $\leq$ R) becomes

$$d\tau^2 = \left(1 - \frac{3}{2}\frac{r_s}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right)dt^2 - \frac{1}{c^2}(h(r)r^2dr - r^2d\theta^2 - r^2sin^2\theta d\varphi^2) \quad (9)$$

while outside the sphere (r > R) it becomes

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \frac{1}{c^2}(h(r)r^2dr - r^2d\theta^2 - r^2sin^2\theta d\varphi^2) \quad (10)$$

There are now three different possibilities, depicted in the following:

A a situation where both r_s and r_{ph} are inside the objects radius R. $r_s < r_{ph} <$ R. This is a standard object.

B - a situation where r_s is inside the object's radius R (a standard object) while r_{ph} is outside the object. $r_s <$ R $< r_{ph}$.

C - a situation of a black hole, where both r_s and r_{ph} are outside R. $R < r_s < r_{ph}$. Therefore, B represents a possible situation where a photon sphere light ring exists around a standard (not a black hole) body.

For said cases, the photon sphere radius $r_{ph} = \frac{3}{2}r_s$ may fall inside or outside R.

A	Standard object	r_s <R	r_{ph} <R
B	Standard object surrounded by white ring	r_s <R	R < r_{ph}
C	Black hole	r_s >R	r_{ph} >R

Case A is a standard object, with both r_s, and r_{ph} falling inside R.

Case B is a white object, with r_s falling inside R but r_{ph} falling outside R. Light is trapped in the photon sphere, causing the object to have a white glowing ring around it. There is no gravitational collapse.

Case C is a true black body, with both r_s and r_{ph} falling outside R.

Case A, both r_s and r_{ph} are inside R (holds for most stellar objects – but not for black holes). One obtains:

$$d\tau^2 = \left(1 - \frac{3}{2}\frac{r_s}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right)dt^2 - \frac{1}{c^2}(h(r)r^2dr - r^2d\theta^2 - r^2sin^2\theta d\varphi^2) \quad (11)$$

Picking the coordinate system so that the radius r is along the x axis, the θ and φ terms are zero

In other words, for r > R

$$d\tau^2 = dt^2\left(1 - \frac{r_s}{r}\right) \quad (12)$$

while for r ≤ R:

$$d\tau^2 = dt^2\left(1 - \frac{3}{2}\frac{r_s}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right) \quad (13)$$

Obviously, for r = R, $d\tau^2 = \left(1 - \frac{r_s}{R}\right)$, for both.

Investigating the condition

$$1 - \frac{r_{ph}}{R}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right) > 0$$

we see that in Case B, for all $r \in [0, R]$ and as long as $r_s \leq \frac{2}{3}R$ there is no singularity in the Schwarzschild metric.

At $r_s = \frac{2}{3}R$ (and $r_{ph} = R$). $d\tau^2$ becomes zero, and the object evaporates into photons ($d\tau = 0$). Thus, $r_s > \frac{2}{3}R$ is the condition for a gravitational collapse. As long as $r_s < \frac{2}{3}R$ the object is a standard object and does not undergo any gravitational collapse. This, in contrast to the current assertion (for r > R), is based on Eq. 37, which states that $r_s > R$, is the condition for a gravitational collapse.

We see that as long as the photon sphere is inside the object ($r_{ph} \in [0, R]$), the Schwarzschild radius $r_s \in [0, \frac{2}{3}R]$. In other words, as long as the photon sphere does not show, the object is stable and does not suffer gravitational collapse. Once the photon sphere shows up ($r_{ph} > R$), the object undergoes gravitational collapse.

Anywhere in the region where $r_s \in [\frac{2}{3}R, R]$ the object undergoes gravitational collapse, and since $r_{ph} = 3/2\,R$, the photon sphere is outside the object, creating a bright light ring outside R, at r = r_{ph}.

Near the sphere's center, where r → 0 and r < R, one has

$$d\tau^2 = dt^2\left(1 - \frac{r_{ph}}{R}\right) \quad (14)$$

And, as long as, $r_{ph} < R$, $d\tau > 0$. It may be re-written as:

$$d\tau = dt\sqrt{\left(1 - \frac{4\pi\rho G}{c^2}R^2\right)} \quad (15)$$

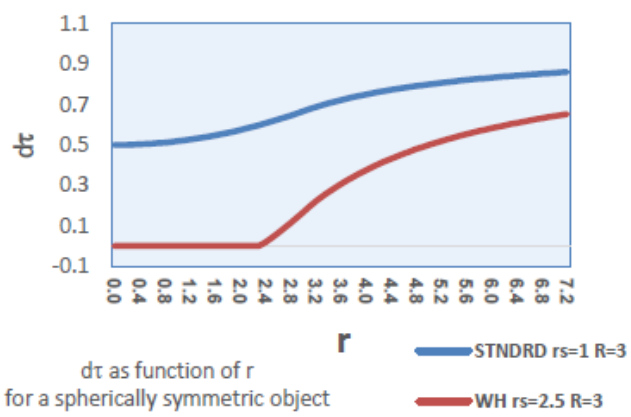

Figure 1. For an object of radius R=3(*). The lines represent the behavior of dτ, for the cases where r_s < R (A-STNDRD) and r_s > 2/3R (B-WH). As can be seen, in case B, the inner part of the object is below the horizon (dτ = 0), but some of the the external part is not. This case allows for the photon sphere to be visible outside the body and yet the object is not a black hole yet. In this figure, dτ < 0 is set to dτ = 0. *(**The dimensions are relative only**)

In the case of infinitesimally small r, we see that the time component becomes independent of the distance r from the origin. In any case, the divergence at r → 0 is removed.

The proper time $d\tau$, must remain well defined. Therefore, an upper limit on the density must exist for a given radius R, otherwise the expression for $d\tau$ becomes undefined.

Notice that this assertion leads to the constraint

$$\rho \leq \frac{c^2}{4\pi G R^2} \quad (16)$$

In other words, there is an upper limit on the density ρ, of any object with radius R. When $R \to 0$, the density may increase indefinitely, but as will be shown next, there is an upper limit on the density.

7. Density must have an Upper Limit

Simple dimensional arguments show that the physical phenomena where quantum gravitational effects become relevant are those characterized by the Planck length $\ell_p = \sqrt{\frac{\hbar G}{c^3}} = 1.616x10^{-35}$ m. Here $\hbar$ is the Planck constant that governs the scale of the quantum effects, G is the Newton constant that governs the strength of the gravitational force, and c is the speed of light, that governs the scale of the relativistic effects. The Planck length is many times smaller than what current technology is capable of observing. physical effects at scales that are so small. Because of this, we have no direct experimental guidance for building a quantum theory of gravity [16].

Suppose there exists a quantum minimum for distance. We call it the Planck length and denote it by ℓ_p.

It is given by

$$\ell_p = \sqrt{\frac{\hbar G}{c}} = 1.616x10^{-35} \text{ m.}$$

Define in addition the Planck mass

Planck's mass $m_p = \sqrt{\frac{\hbar c}{G}} = 2.176x10^{-8}$ Kg.

If this assumption is true, then the minimal spherical volume possible is

$$V = \frac{4\pi {\ell_p}^3}{3}$$

Let m' denote the mass of this volume, so its density will be given by

$$\rho_P = \frac{3m'}{4\pi {\ell_p}^3}$$

Since by assumption ℓ_p is the minimal length possible in nature, then for any mass m' the density ρ_P is the maximum possible.

It can be showed, that for a classical spherically symmetric object of mass m' and radius R, the general relativistic limit gives

$$d\tau = dt\sqrt{\left(1 - \frac{4\pi\rho G}{c^2}R^2\right)} \quad (17)$$

Since the expression in brackets must be real, we arrive at

the restriction:

$$1 - \frac{4\pi\rho G}{c^2} R^2 \geq 0 \tag{18}$$

And therefore

$$\rho \leq \frac{c^2}{4\pi G R^2} \tag{19}$$

For an object of any given mass m' and radius R we have

$$\rho = \frac{m'}{V} = \frac{3m'}{4\pi R^3} \tag{20}$$

Since for any mass m' of radius R one must have, by Eq. 16 above,

$$\rho = \frac{3m'}{4\pi R^3} \leq \frac{c^2}{4\pi G R^2} \tag{21}$$

The result is that for any mass m'

$$m'(R) \leq \frac{Rc^2}{G} \tag{22}$$

Obviously, the smaller the radius R, the smaller the allowed mass m'.

For the minimal possible length (according to Planck) $R = \ell_p$ one obtains:

$$m' \leq \frac{\ell_p c^2}{G} \tag{23}$$

Recall now, by Planck's dimensionality analysis) that $\ell_p = \sqrt{\frac{\hbar G}{c}}$ and $m_p = \sqrt{\frac{\hbar c}{G}}$ so

$$\frac{m_p}{\ell_p} = \frac{c^2}{G} \tag{24}$$

Therefore

$$m' \leq \frac{\ell_p c^2}{G} = m_p \tag{25}$$

Therefore, for any mass m' (with radius ℓ_p)

$$m' \leq m_p \tag{26}$$

And since

$$\rho = \frac{m'}{V} \leq \frac{m_p}{V} = \rho_P \tag{27}$$

We have the result that ρ_P is the maximal possible density, namely Planck density.

In other words, for any given radius R, the mass m'(R) becomes smaller and smaller with R, but when one reaches the smallest possible radius ℓ_p, the mass must be smaller than the Planck mass m_p, and so, the density will always be smaller than the Planck density ρ_P.

One can reduce the radius R, but the density will never exceed the Planck density.

$$lim_{R \to \ell_p} \rho(R) = \left(\frac{\ell_p}{R}\right)^3 \rho_P \tag{28}$$

Assume next, that the sphere has density $\rho(r)$, which varies with distance r from its center.

Assume the sphere is of minimal possible radius ℓ_p.

We need to calculate the radius R of a quantized particle by its average normalized density so that we obtain a reduced average radius.

By comparing the integrated variable density over the Planck radius, to a volume, with constant density ρ_0 one obtains:

$$4\pi \int_0^{\ell_p} r^2 \rho(r) dr = \frac{4\pi \rho_0 {\ell_p}^3}{3} \tag{29}$$

By definition, the average classical distance $\langle r^2 \rangle$ is given by the integral over the normalized density:

$$\langle r^2 \rangle \stackrel{\text{def}}{=} \frac{1}{\ell_p} \int_0^{\ell_p} r^2 \rho(r)/\rho_0 \, dr \tag{30}$$

Thus

$$4\pi \ell_p \rho_0 \langle r^2 \rangle = \frac{4\pi \rho_0 {\ell_p}^3}{3} \tag{31}$$

and so

$$\langle r \rangle = \sqrt{\langle r^2 \rangle} = \frac{1}{\sqrt{3}} \ell_p \tag{32}$$

Hence, the actual measured classical radius R is given by

$$R = \langle r \rangle = \frac{1}{\sqrt{3}} \ell_p \tag{33}$$

The above result shows how the lower limit of the classical gravitation theory by Einstein, is related to the Planck length, which is a quantum phenomenon posed by dimensional analysis of the universe constants.

Therefore, classical relativity and the relationship between the universal constants leads to quantization of space.

8. Quantization by Planck's Units and Maximal Density Limit

In the limit where R $\to 0$, one needs to consider quantum limits and the uncertainty principle.

Planck's mass $m_p = \sqrt{\frac{\hbar c}{G}} = 2.176x10^{-8}$ Kg.

Planck's length $\ell_p = \sqrt{\frac{\hbar G}{c}} = 1.616x10^{-35}$ m.

However, dimensional analysis can only determine the Planck's units up to a dimensionless multiplicative factor. For instance, use h instead of $\hbar$.

Based on Planck's units one obtains Planck's maximal density ρ_P as Planck's mass/Planck's volume (assuming ℓ_p is the lowest possible physical distance, leads to the maximal possible physical density assumption).

$$\rho_P = \frac{m_p}{4\pi {\ell_p}^3/3} = 1.23074x10^{+96} \tag{34}$$

(If one uses the Planck constant instead of the reduced Planck's constant, the Planck density will be modified by a factor of 2π).

Since one must assume that Planck's density is the maximum possible density allowed, we now ask, what should the minimum classical radius R be, in order for $\rho \leq \rho_P$. (any classical density cannot exceed the Planck density).

Therefore:

$$\rho = \frac{c^2}{4\pi G R^2} \leq \rho_P = \frac{3m_p}{4\pi {\ell_p}^3} \tag{35}$$

Doing the calculation, we find that one must have R $\geq \frac{1}{\sqrt{3}} \ell_p = 0.93325x10^{-35}$ m.

Compared to Planck's length $\ell_p = 1.61625x10^{-35}$ m, the classically derived radius R is smaller than the allowed Planck's length ℓ_p, by an unexplained factor of 0.5774.

These calculations were based on the assumptions of negligible $\mathbb{K} = \frac{16\pi G}{c^2}\rho$, and R $\rightarrow 0$ (r < R).

It seems now, that the classical solution to the metric of a spherically symmetric object, puts a lower limit on the radius R of an object. It looks like there is a sort of quantization limit to small scale objects in spacetime.

Instead of having $R \geq \ell_p$, we find $R \geq 0.5774\,\ell_p$.

The reason for this 0.5774 factor is due to the fact that we have assumed a constant density inside the sphere. If instead we assume variation in density with radius (highest density at r $\rightarrow 0$and then it falls off to zero at ℓ_p). We need to calculate the radius R of a quantized particle by its average density so that we obtain a reduced average radius.

By comparing the integrated variable density over the Planck radius, to the same volume, but with a constant average density ρ_0 one obtains (see paragraph VI on density upper limit)):

$$R = \langle r \rangle = \frac{1}{\sqrt{3}}\ell_p \tag{36}$$

Since 0.5774 = $\frac{1}{\sqrt{3}}$, it is now clear where the reduction factor came from.

The above result shows how the lower limit of the classical gravitation theory by Einstein, (Schwarzschild radius), and the dimensional constraints between the constants of the universe, result in a quantized material space.

9. A Tunneled Sphere "Thought Experiment"

Consider a sphere with a thin penetrating tunnel through center of sphere between opposite surfaces.

The sphere of radius R and of mass M will exert a gravitational attractive force on a remote particle. The force will be directed towards the center of the sphere, as if all of its mass is concentrated at its center (where r = 0).

The attracted particle will be accelerated until it reaches the surface, but then will continue its path through the hole. It will continue with no acceleration inside the shell and then come out through the opposite hole.

On its journey, the particle will pass twice the Schwarzschild radius $r_s = \frac{GM}{c^2}$ on its way towards the center and on its way out.

The metric of this shell will be similar to that of a spherical mass. Only that this time, even though r_s is inside the body, one can reach the Schwarzschild radius independently of whether it is a black hole or a standard mass.

One way of doing the experiment would be to send a photon through the hole in the mass and compare its time of arrival with a simultaneously emitted photon traveling outside the mass. One would then be able to compare differences in the time of arrival for the two photons.

This gedanken experiment can reveal whether the approach to r_s will last forever in the eyes of the external observer. If it does, then the particle will never be able to come out through the opposite hole.

10. Dynamics Near Center

It is interesting to see what are the equations of motion near center (where r<< R).

Assuming spherical symmetry [21,22], it is allowed to investigate it along the radial distance r, and assume it is the x axis.

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{37}$$

where

$$\Gamma^\lambda_{\gamma\mu} = \frac{1}{2}g^{\lambda r}\left(\partial_\mu g_{r\gamma} + \partial_\gamma g_{r\mu} - \partial_r g_{\gamma\mu}\right) \tag{38}$$

The total time of flight in the classical case will be given by (see Appendix 2):

$$2T = \frac{2}{\sqrt{Q_0}}cos^{-1}\left[2\left(1 - \frac{Q_0R^3}{GM}\right)\right]$$

Whereas in the general relativistic case it will be:

$$2T = \frac{2}{\sqrt{Q}}cos^{-1}\left[2\left(1 - \frac{QR^3}{GM}\right)\right]$$

Comparing the two results, $T_{Gen.Rel.} > T_{class}$.

As an example, for the Sun, with R = $6.963x10^8$m and Sun density $\rho_\odot = 150\,x10^3$ Kg/m^3 the predicted general relativistic cross time will be 6.6 mSec longer than the classical calculated cross time. This result is based on average Q over r from 0 to R.

For an average Neutron star of radius R=11,000 m and density $\rho(r) = 4x10^{17}$ Kg/m^3, the passage time under general relativity prediction, will last 0.13 mSec longer than under classical calculation.

11. Conclusions

Using Newton's classical shell theorem, the Schwarzschild metric was modified. This removed the singularity at r = 0 for a standard object (not a black hole).

For all practical matters, r < R can be treated as an empty space even for the densest known stellar objects (neutron stars) and also for elementary particles (neutrons for instance).

It was demonstrated how general relativity evidently leads to quantization of space-time. Both classical and quantum mechanical limits on density give the same result.

Appendix 1

For a curved spacetime we define the Riemann tensor:

$$R^\mu_{\alpha\beta\gamma} = \partial_\beta\Gamma^\mu_{\alpha\gamma} - \partial_\gamma\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta} \quad [1]$$

Where $\Gamma^{\mu}_{\alpha\beta} = g^{\mu\sigma}\Gamma_{\sigma\alpha\beta}$ are the Christoffel symbols:

$$\Gamma_{ikl} = \frac{1}{2}\left(\partial_l g_{ik} + \partial_k g_{il} - \partial_i g_{kl}\right) \quad (39)$$

The Ricci tensor $R_{\alpha\beta}$ is defined by:

$$R_{\alpha\beta} = g^{\sigma\varrho}\, R_{\alpha\varrho\sigma\beta} \quad (40)$$

And the curvature scalar R is defined by:

$$\boldsymbol{\mathcal{R}} = g^{\alpha\beta}\, R_{\alpha\beta} = g^{\alpha\beta} g^{\sigma\varrho}\, R_{\alpha\beta\sigma\varrho} \quad (41)$$

For the Einstein equation one has:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, \boldsymbol{\mathcal{R}} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (42)$$

In empty space $T_{\mu\nu} = 0$ everywhere and therefore $R_{\mu\nu} = 0$ everywhere.

Due to spherical symmetry, the metric must be of the form

$$g_{\mu\nu} = \begin{pmatrix} g(r) & & & \\ & h(r) & & \\ & & r^2 & \\ & & & r^2 sin^2\theta \end{pmatrix} \quad (43)$$

Where g(r) and h(r) are functions of the distance r from the coordinate center (located at the center of the mass.

So, the infinitesimal proper time interval $d\tau$ between two events along a time-like path is given by

$$d\tau^2 = g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

$$= g(r)dt - \frac{1}{c^2}\left(h(r)r^2 dr - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2\right) \quad (44)$$

With the flat space metric $g_{\mu\nu} = (+ - - -)$.

In order to solve Einstein's equation everywhere, one can no longer assume that the curvature $\boldsymbol{\mathcal{R}}$ is null anywhere. This is true only outside the mass and so $\boldsymbol{\mathcal{R}} = 0$ only for r > R.

Assuming the mass density of the object being homogeneous and at total freeze of motion, $T_{\mu\nu} = 0$, except for $T_{00} = \rho c^2$, with ρ being the mass density.

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, \boldsymbol{\mathcal{R}}\right) = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} \quad (45)$$

Under perfect fluid conditions, $T_{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$,

where p is the pressure.

Define $\mathbb{K} = \frac{16\pi G}{c^2}$, and so,

$$R_{00} = \frac{16\pi G}{c^2}\rho = \mathbb{K}\rho \quad (46)$$

For all other terms, $R_{11} = R_{22} = R_{33}$:

$$R_{ii} = \frac{16\pi G}{c^2} p = \mathbb{K} p \quad (47)$$

From the definitions of the Riemann tensor and the Ricci tensor we obtain

$$R_{\mu\gamma} = \partial_\delta \Gamma^{\delta}_{\mu\gamma} - \partial_\gamma \Gamma^{\delta}_{\mu\delta} + \Gamma^{\delta}_{\delta\lambda}\Gamma^{\lambda}_{\gamma\mu} - \Gamma^{\delta}_{\gamma\lambda}\Gamma^{\lambda}_{\delta\mu} \quad (48)$$

$$R_{00} = \partial_\delta \Gamma^{\delta}_{00} - \partial_0 \Gamma^{\delta}_{0\delta} + \Gamma^{\delta}_{\delta\lambda}\Gamma^{\lambda}_{00} - \Gamma^{\delta}_{0\lambda}\Gamma^{\lambda}_{\delta 0} \quad (49)$$

Following some tedious mathematical work, one arrives at [4]:

$$\partial_r ln\big(h(r)g(r)\big) = -\frac{16\pi G}{c^2}\rho\frac{rh(r)}{g(r)} \quad (50)$$

And so

$$R_{22} = \frac{1}{h(r)} - 1 - \frac{r}{2h(r)}\left(\frac{h'(r)}{h(r)} - \frac{g'(r)}{g(r)}\right)$$

$$= \frac{1}{h(r)} - 1 - \frac{r}{2h(r)}\,\partial_r ln\left(\frac{h(r)}{g(r)}\right) \quad (51)$$

We have then two equations:

$$\partial_r ln\big(h(r)g(r)\big) = -\mathbb{K}\frac{rh(r)}{g(r)} \quad (52)$$

$$R_{22} = \frac{1}{h(r)} - 1 - \frac{r}{2h(r)}\,\partial_r ln\left(\frac{h(r)}{g(r)}\right) \quad (53)$$

Outside the mass (r>R)

$\mathbb{K} = 0$ and so:

$h(r)g(r) = \alpha$ (some constant).

Substitute in eq. 2 to obtain:

$$R_{22} = \frac{g(r)}{\alpha} - 1 - \frac{rg(r)}{2\alpha}\,\partial_r ln\left(\frac{\alpha}{g^2(r)}\right) = 0 \quad (54)$$

And after few steps:

$$\alpha = \partial_r(rg(r)) \quad (55)$$

So finally:

$$g(r) = \alpha r + \beta \quad (56)$$

$$h(r) = \frac{\alpha}{\alpha r + \beta} \quad (57)$$

The metric becomes:

$$g_{\mu\nu} = \begin{pmatrix} \alpha r + \beta & & & \\ & \frac{\alpha}{\alpha r + \beta} & & \\ & & r^2 & \\ & & & r^2 sin^2\theta \end{pmatrix} \quad (58)$$

Comparing this metric with the asymptotic *weak-field* metric ($r \gg R$):

$$g_{\mu\nu} = \begin{pmatrix} \left(1 - \frac{2GM}{rc^2}\right) & & & \\ & \frac{-1}{\left(1 - \frac{2GM}{rc^2}\right)} & & \\ & & -r^2 & \\ & & & -r^2 sin^2\theta \end{pmatrix} \quad (59)$$

It is suggested that

$$\alpha r + \beta = \left(1 + \frac{2\Phi(r)}{c^2}\right) \quad (60)$$

Therefore,

$$\beta = 1 \quad (61)$$

$$\alpha = \frac{2\Phi(r)}{rc^2} \quad (62)$$

This shows us that the structure of the Schwarzschild metric in Eq. 7, is not only a result of dimensional considerations discussed above, but also a result of spherical symmetry considerations of the Einstein Equation, Eq. 17.

This result was obtained based on symmetry arguments only, and it is independent of the weak-field approximation.

Inside the mass (r < R)

$\mathbb{K} > 0$:

The only approximation made was the assumption on empty space solution. For non-empty space, it will be valid

for a perfect fluid of zero pressure.

However, one may use a model where the pressure p is proportional o the density [18], and so $p = \omega\rho$ where ω is a proportionality factor.

Appendix 2

Since by coordinates choice and symmetry, only Γ^{λ}_{00} and Γ^{λ}_{11} are non-zero (assuming constant gravitational field $\partial_t g_{r\mu} = 0$).

$$\frac{d^2x}{d\tau^2} = -\Gamma^1_{00}\left(\frac{dt}{d\tau}\right)^2 - \Gamma^1_{11}\left(\frac{dx}{d\tau}\right)^2 \quad (63)$$

$$\Gamma^1_{00} = \frac{1}{2}\, g^{1r}(\partial_0 g_{r0} + \partial_0 g_{r0} - \partial_r g_{00})$$

$$= \frac{1}{2}\, g^{11}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}) = -\frac{1}{2}\, g^{11}\, \partial_1 g_{00} \quad (64)$$

and likewise

$$\Gamma^1_{11} = \frac{1}{2}\, g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = \frac{1}{2}\, g^{11}\, \partial_1 g_{11} \quad (65)$$

The equation of motion in the x direction becomes

$$\frac{d^2x}{d\tau^2} = \left(\frac{1}{2}\, g^{11}\, \partial_1 g_{00}\right)\left(\frac{dt}{d\tau}\right)^2 c^2 - \left(\frac{1}{2}\, g^{11}\, \partial_1 g_{11}\right)\left(\frac{dx}{d\tau}\right)^2 \quad (66)$$

Since the choice of coordinates is such that x is represented by r, one can write

$$\frac{d^2r}{d\tau^2} = \frac{1}{2}\, g^{11}\, \partial_r g_{00}\left(\frac{dt}{d\tau}\right)^2 c^2 - \frac{1}{2}\, g^{11}\, \partial_r g_{11}\left(\frac{dr}{d\tau}\right)^2 \quad (67)$$

From the proper-time result (which holds inside the sphere, where r < R)

$$d\tau^2 = dt^2\left(1 - \frac{4\pi G\rho R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right) = -g_{00}dt^2 \quad (68)$$

Define for convenience

$$g_{00} = A^2(r) = 1 - \frac{4\pi G\rho R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right) \quad (69)$$

And so

$$\frac{d\tau}{dt} = A \quad (70)$$

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \frac{1}{A}\frac{dr}{dt} \quad (71)$$

And

$$\frac{d^2r}{d\tau^2} = \frac{1}{A^2}\frac{d^2r}{dt^2} \quad (72)$$

From the metric definition:

$$d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dx^0dx^0 + g_{11}dx^1dx^1 \quad (73)$$

and since

$$g_{00} = A^2 \quad (74)$$

$$g_{11} = -\frac{1}{g_{00}} = -\frac{1}{A^2} \quad (75)$$

One obtains:

$$\partial_r g_{11} = \frac{1}{{g_{00}}^2}\partial_r g_{00} \quad (76)$$

Inserting into Eq. [49] one obtains:

$$\frac{1}{A^2}\frac{d^2r}{dt^2} = \frac{1}{2}\, g^{11}\, \partial_r g_{00}\left[\frac{1}{A^2}c^2 - \frac{1}{{g_{00}}^2}\,\frac{1}{A^2}\left(\frac{dr}{dt}\right)^2\right] \quad (77)$$

And so

$$\frac{d^2r}{dt^2} = -\frac{1}{2}\,\frac{1}{g_{00}}\,\partial_r g_{00}\left[c^2 - \frac{1}{{g_{00}}^2}\left(\frac{dr}{dt}\right)^2\right] \quad (78)$$

For $\frac{r}{R} \ll 1$

$$g_{00} = \frac{4\pi G\rho R^2}{c^2} - 1 \quad (79)$$

And so $\partial_r g_{00} = 0$, which means that $\frac{d^2r}{dt^2} = 0$.

The acceleration near the center is zero.

However, if one moves away from the center:

$$g_{00} = \frac{4\pi G\rho R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right) - 1 \quad (80)$$

And so

$$\partial_r g_{00} = \frac{4\pi G\rho R^2}{c^2}\left(-\frac{2r}{3R^2}\right) = -\frac{8\pi G\rho}{3c^2}\, r \quad (81)$$

$$\frac{d^2r}{dt^2} = -\frac{1}{2}\,\frac{1}{g_{00}}\,\partial_r g_{00}\left[c^2 - \frac{1}{{g_{00}}^2}\left(\frac{dr}{dt}\right)^2\right] \quad (82)$$

$$\frac{d^2r}{dt^2} = \frac{1}{2}\,\frac{1}{A^2}\left(-\frac{8\pi G\rho}{3c^2}\, r\right)\left[c^2 + \frac{1}{A^4}\left(\frac{dr}{dt}\right)^2\right] \quad (83)$$

$$\frac{d^2r}{dt^2} = -\frac{4\pi G\rho}{3c^2A^2}\, r\left[c^2 + \frac{1}{A^4}\left(\frac{dr}{dt}\right)^2\right] \quad (84)$$

$$\frac{d^2r}{dt^2} = -\frac{4\pi G\rho}{3A^2}\, r\left[1 + \frac{1}{c^2A^4}\left(\frac{dr}{dt}\right)^2\right] \quad (85)$$

Near center (where r → 0), $A^2 = 1 - \frac{4\pi G\rho R^2}{c^2}$ depends on r only through the density $\rho(r)$.

We may write

$$\frac{d^2r}{dt^2} = Q(r)\, r\left(1 + W(r)\left(\frac{v(r)}{c}\right)^2\right) \quad (86)$$

Here, $v(r)$ is the approach velocity. Q and W depend on r and are defined by:

$$Q(r) \stackrel{\text{def}}{=} \frac{4\pi G\rho\,(r)}{3A^2(r)} \quad (87)$$

$$W(r) \stackrel{\text{def}}{=} \frac{1}{A^4(r)} \quad (88)$$

The result shows how on approach to center (r → 0), where $\rho(r)$ is assumed constant, the acceleration decreases in proportion with r on one hand, but depends also on the square of the approach velocity.

For small approach-to-center velocity ($v/c \ll 1$)

$$\frac{d^2r}{dt^2} = -Q\, r \quad (89)$$

One may approximate though for a slow approach velocity ($v/c \ll 1$):

$$\frac{d^2r}{dt^2} = -Q(r)\, r \quad (90)$$

where

$$Q(r) = \frac{4\pi G\rho\,(r)}{3\left[1 - \frac{4\pi G\rho\,(r)R^2}{c^2}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right]} \quad (91)$$

(Recall that by the assumption on the maximal density (Eq. 38), $\frac{4\pi G\rho(r)R^2}{c^2} < 1$, and so, $Q(r) \geq 0$. Also, note that

$[Q] = 1/sec^2]$).

One may solve this under certain approximation:

$\rho(r) = \rho_0$ constant density inside r < R.

In this case Q(r) is nearly a constant independent of r, and the equation becomes

$$\frac{d^2r}{dt^2} = -Q\, r \tag{92}$$

Which general solution is

$$r(t) = B + Ccos(\sqrt{Q}\, t) + Dsin(\sqrt{Q}\, t) \tag{93}$$

Even though $Q(r)$ is not constant, the term $\frac{4\pi G\rho(r)R^2}{c^2}$ in the denominator of $Q(r)$ is of the order of, or less than, $9.2x10^{-9}$. Q(r) may be approximated as:

$$Q(r) \cong \frac{4\pi G\rho\,(r)}{3\left[1-9.2x10^{-9}\left(1-\frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right]} \cong \frac{4\pi G\rho\,(r)}{3} \tag{94}$$

$$\dot{r}(t) = -C\sqrt{Q}\, sin(\sqrt{Q}\, t) + D\sqrt{Q}\, cos(\sqrt{Q}\, t) \tag{95}$$

Assuming zero start velocity, $\dot{r}(0) = -\sqrt{Q}$ D one must have D = 0.

Hence

$$r(t) = B - Ccos(\sqrt{Q}\, t) \tag{96}$$

Also, $r(0) = R$.

Therefore

$$R = B - C \tag{97}$$

$$r(t) = B - (B - R)cos(\sqrt{Q}\, t) \tag{98}$$

Assume the acceleration $\ddot{r}(0) = -\frac{GM}{R^2}$,

$$\ddot{r}(t) = B - Ccos(\sqrt{Q}\, t) \tag{99}$$

Using these assumptions, one obtains:

$$r(t) = R + \frac{GM}{QR^2}\left(cos(\sqrt{Q}\, t) - 1\right) \tag{100}$$

Furthermore, if the object arrives at the center (r = 0) at some later time T, one has:

$$r(T) = 0 = R + \frac{GM}{QR^2}\left(cos(\sqrt{Q}\, T) - 1\right) \tag{101}$$

The time to cross the diameter 2R (from surface to surface through the center) is thus

$$2T = \frac{2}{\sqrt{Q}} cos^{-1}\left[1 - \frac{QR^3}{GM}\right] \tag{102}$$

Where

$$Q(r) = \frac{4\pi G\rho(r)}{3\left[1-\frac{4\pi G\rho(r)R^2}{c^2}\left(1-\frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right]}$$

In this case

$$\frac{d^2r}{dt^2} = -\frac{4\pi G}{3}\, r\, \rho(r) \tag{103}$$

Classically, a falling object of mass m, starting at r < R, will be accelerated towards center according to the potential:

$$\Phi(r) = -\frac{GM}{2R^3}(3R^2 - r^2) \tag{104}$$

And the acceleration will be

$$F = m\frac{d^2r}{dt^2} = -m\frac{\partial\Phi(r)}{\partial r} = -m\frac{GM}{2R^3}2r \tag{105}$$

$$\frac{d^2r}{dt^2} = -\frac{GM}{R^3}r = -\frac{G4\pi\rho_0R^3}{3R^3}\, r = -\frac{4\pi G\rho_0}{3}\, r \tag{106}$$

Classically, the acceleration will be

$$\frac{d^2r}{dt^2} = -\frac{4\pi G}{3}\rho_0\, r = -Q_0 r \tag{107}$$

with $Q_0 = \frac{4\pi G}{3}\rho_0$

This, in comparison to the general relativistic solution:

$$\frac{d^2r}{dt^2} = -Q(r)\, r \tag{108}$$

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