

Buy-Ballot Estimates for Mixed Model in Descriptive Time Series

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Abstract This paper examines the Buy-Ballot estimation procedure for time series decomposition when the trending curve is linear and admits mixed model. The main objective is to obtain the row, column and the overall means of Buy-Ballot table for the mixed model. The method adopted is Buy-Ballot procedure developed for choice model, estimation of trend parameter and seasonal indices by [1]. Results of the computed values of the estimated trend parameters and seasonal indices are given in table 3.

Keywords Descriptive time series, Mixed model, Trend Parameter, Seasonal Indices, Buy-Ballot Estimate

1. Introduction

[2] proposed two methods of estimating the parameter of a linear trend-cycle component from the periodic means of the Buy-Ballot table (table 1). This method was initially developed for short period of time in which the trend-cycle component (M_t) is jointly combined and can be represented by linear equation

$M_t = a + b_t$, $t = 1, 2, \dots, n$ where a is the intercept, b is the slope and t is the time point.

The models most commonly used for time series decomposition are the

Additive Model:

$$X_t = T_t + S_t + C_t + e_t \quad (1)$$

Multiplicative Model:

$$X_t = T_t \times S_t \times C_t \times e_t \quad (2)$$

and Mixed Model

$$X_t = T_t \times S_t \times C_t + e_t \quad (3)$$

If short period of time are involved, the cyclical component is superimposed into the trend [3] and the observed time series ($X_t, t = 1, 2, \dots, n$) can be decomposed into the trend-cycle component (M_t), seasonal component (S_t) and the irregular/residual

component (e_t). Therefore, the decomposition models are

Additive Model:

$$X_t = M_t + S_t + e_t \quad (4)$$

Multiplicative Model:

$$X_t = M_t \times S_t \times e_t \quad (5)$$

and Mixed Model

$$X_t = M_t \times S_t + e_t \quad (6)$$

It is always assumed that the seasonal effect, when it exists, has period s , that is, it repeats after s time periods.

$$S_{t+s} = S_t, \text{ for all } t \quad (7)$$

For Equation (4), it is convenient to make the further assumption that the sum of the seasonal components over a complete period is zero, ie,

$$\sum_{j=1}^s S_{t+j} = 0. \quad (8)$$

Similarly, for Equations (5) and (6), the convenient variant assumption is that the sum of the seasonal components over a complete period is s .

$$\sum_{j=1}^s S_{t+j} = s. \quad (9)$$

1.1. Buy-Ballot Procedure for Time Series Decomposition

According to [3] the successive periodic mean (\bar{X}_i) provide a simple description of the underlying trend from the methods of monthly or quarterly means. It was shown that the estimates of the seasonal indices can be obtained from the

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column means $\bar{X}_{.j}$. Therefore, while the periodic means give estimate of the trend, the column means gives estimates of seasonal indices.

It has been observed that a time series theoretically contains four components, However, if short period time are involved, the trend components is estimated into the cyclical and we obtain a trend cycle components denoted by m_t . Under these conditions, it can be stated that estimates of trend-cycle and seasonal components can be obtained from the row and column means, respectively, of the Buys-Ballot table. These estimates have been designated “Buys-Ballot” estimates in this study the details of the procedure for estimation of the trend-cycle component (m_t) are presented in section 2 for the mixed model.

The method has developed, we have restricted our discussions to the case, where the trend-cycle component is a straight line. That is

$$m_t = a + b_t$$

We also assumed that the length of periodic interval s .

The Buys - Ballot table helps in the assessment of the trend – cycle and seasonal indices of time series data. The row means ($\bar{X}_{i.}$) estimate trend, and the differences ($\bar{X}_{.j} - \bar{X}_{..}$) or the ratio ($\frac{\bar{X}_{.j}}{\bar{X}_{..}}$) between the column means

($\bar{X}_{.j}$) and the overall mean ($\bar{X}_{..}$) estimate the seasonal indices. [4], showed seasonal variation (Table 1) of the Buys - Ballot table. According to them, each row is one period (usually a year) and each column is a season of the period/year (4 quarter, 12 months etc). A cell, (i, j) of the table contains the mean value for all observations made during the period i at the season j . To analyse the data, it is important to include the period and seasonal totals ($T_{i.}$) and ($T_{.j}$), period and seasonal averages ($\bar{X}_{i.}$) and ($\bar{X}_{.j}$), period and seasonal standard deviations ($\hat{\sigma}_{i.}$) and ($\hat{\sigma}_{.j}$) as part of the Buys – Ballot table. Also included for purposes of analysis are the grand total ($T_{..}$), grand mean ($\bar{X}_{..}$) and pooled standard deviation ($\hat{\sigma}_{..}$).

2. Buys-Ballot Estimates for Mixed Model

This section considers the Buys-Ballot estimation procedure for trend and seasonal indices for mixed model in descriptive time series analysis when trend-cycle component is linear. Section 2.2 treats Buys-Ballot estimation procedure for trend, Section 2.3 discuss the Buys-Ballot estimation procedure for seasonal indices.

Table 1. Buys - Ballot Table for Seasonal time series

Rows/ Period (i)	Columns (season) j								
	1	2	...	j	...	s	$T_{i.}$	$\bar{X}_{i.}$	$\hat{\sigma}_{i.}$
1	X_1	X_2	...	X_j	...	X_s	$T_{1.}$	$\bar{X}_{1.}$	$\hat{\sigma}_{1.}$
2	X_{s+1}	X_{s+2}	...	X_{s+j}	...	X_{2s}	$T_{2.}$	$\bar{X}_{2.}$	$\hat{\sigma}_{2.}$
3	X_{2s+1}	X_{2s+2}	...	X_{2s+j}	...	X_{3s}	$T_{3.}$	$\bar{X}_{3.}$	$\hat{\sigma}_{3.}$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
i	$X_{(i-1)s+1}$	$X_{(i-1)s+2}$...	$X_{(i-1)s+j}$...	X_{is}	$T_{i.}$	$\bar{X}_{i.}$	$\hat{\sigma}_{i.}$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
m	$X_{(m-1)s+1}$	$X_{(m-1)s+2}$...	$X_{(m-1)s+j}$...	X_{ms}	$T_{m.}$	$\bar{X}_{m.}$	$\hat{\sigma}_{m.}$
$T_{.j}$	$T_{.1}$	$T_{.2}$...	$T_{.j}$...	$T_{.s}$	$T_{..}$		
$\bar{X}_{.j}$	$\bar{X}_{.1}$	$\bar{X}_{.2}$...	$\bar{X}_{.j}$...	$\bar{X}_{.s}$		$\bar{X}_{..}$	
$\hat{\sigma}_{.j}$	$\hat{\sigma}_{.1}$	$\hat{\sigma}_{.2}$...	$\hat{\sigma}_{.j}$...	$\hat{\sigma}_{.s}$			$\hat{\sigma}_{..}$

In this arrangement each time period t is represented in terms of the period i (e.g. year) and season j (e.g. month of the year), as $t = (i-1)s + j$. Thus, the period (row), season (column) and overall totals, means and variances are defined as

$$T_{i.} = \sum_{j=1}^s X_{(i-1)s+j}, \quad \bar{X}_{i.} = \frac{T_{i.}}{s}, \quad \hat{\sigma}_{i.}^2 = \frac{1}{s-1} \sum_{j=1}^s (X_{ij} - \bar{X}_{i.})^2$$

$$T_{.j} = \sum_{i=1}^m X_{(i-1)s+j}, \quad \bar{X}_{.j} = \frac{T_{.j}}{m}, \quad \hat{\sigma}_{.j}^2 = \frac{1}{m-1} \sum_{i=1}^m (X_{ij} - \bar{X}_{.j})^2$$

$$T_{..} = \sum_{i=1}^m \sum_{j=1}^s X_{(i-1)s+j},$$

2.1. Buys-Ballot Procedure for Estimation of Trend

In developing this procedure we have restricted our discussion to the case where the trending curve is linear. We also assume that the length of periodic interval is s . Based on these, we obtain the following row and overall averages for the mixed model. (6). Recall, for the mixed model,

$\sum_{j=1}^s S_{t+j} = s$, $S_{t+j} = S_j$ and $e_t \sim N(0,1)$. When arranged in a Buys-Ballot table, with m -rows and s -columns

$t = (i-1)s + j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, s$. Therefore X_t in (6) becomes

$$X_{(i-1)s+j} = M_{(i-1)s+j} \times S_{(i-1)s+j} + e_{(i-1)s+j} \quad (10)$$

For convenience, let $X_{ij} = X_{(i-1)s+j}$, $M_{ij} = M_{(i-1)s+j}$ and $e_{ij} = e_{(i-1)s+j}$. Hence,

$$M_{ij} = a + b[(i-1)s + j] \quad (11)$$

and

$$X_{ij} = M_{ij}S_j + e_{ij} = \{a + b[(i-1)s + j]\}S_j + e_{ij} \quad (12)$$

The row total is

$$\begin{aligned} T_i &= \sum_{j=1}^s X_{ij} \\ &= \sum_{j=1}^s \{a + [b(i-1)s + j]\}S_j + \sum_{j=1}^s e_{ij} \\ &= \sum_{j=1}^s [a + bs(i-1)]S_j + b \sum_{j=1}^s jS_j + \sum_{j=1}^s e_{ij} \\ &= s[a + bs(i-1)] + b \sum_{j=1}^s jS_j + \sum_{j=1}^s e_{ij} \end{aligned} \quad (13)$$

Hence, the row mean is

$$\begin{aligned} \bar{X}_i &= \frac{1}{s} \sum_{j=1}^s X_{ij} \\ &= \frac{1}{s} \left\{ s[a + bs(i-1)] + b \sum_{j=1}^s jS_j + \sum_{j=1}^s e_{ij} \right\} \\ &= [a + bs(i-1)] + \frac{b}{s} \sum_{j=1}^s jS_j + \bar{e}_i. \end{aligned} \quad (14)$$

where $\bar{e}_i = \frac{1}{s} \sum_{j=1}^s e_{ij}$.

2.2. Buys-Ballot Procedure for Estimation of Seasonal Component

We restrict our discussion of the Buys- Ballot procedure for estimation of seasonal component to the case where trend-cycle component of time series linear. The length of periodic interval is also taken to be s . Based on these, we obtain the following column and overall totals and averages for the mixed model.

The column total is

$$T_{.j} = \sum_{i=1}^m X_{ij}$$

$$\begin{aligned} &= \sum_{i=1}^m (M_{ij}S_j + e_{ij}) \\ &= \left\{ ma + b \left[s \sum_{i=1}^m (i-1) + \sum_{i=1}^m j \right] \right\} S_j + \sum_{i=1}^m e_{ij} \\ &= \left\{ ma + b \left[\frac{sm(m-1)}{2} + mj \right] \right\} S_j + \sum_{i=1}^m e_{ij} \\ &= m \left\{ a + b \left[\frac{sm-s}{2} + j \right] \right\} S_j + \sum_{i=1}^m e_{ij} \\ &= m \left\{ a + b \left(\frac{n-s}{2} \right) + bj \right\} S_j + \sum_{i=1}^m e_{ij} \\ &= m \left\{ \left[a + b \left(\frac{n-s}{2} \right) + bj \right] S_j \right\} + \bar{e}_{.j} \end{aligned} \quad (15)$$

Hence, the column mean is

$$\begin{aligned} \bar{X}_{.j} &= \frac{1}{m} \sum_{i=1}^m X_{ij} \\ \bar{X}_{.j} &= \left[a + b \left(\frac{n-s}{2} \right) + bj \right] S_j + \bar{e}_{.j} \end{aligned} \quad (16)$$

where $\bar{e}_{.j} = \frac{1}{m} \sum_{i=1}^m e_{ij}$

overall mean

$$\begin{aligned} \bar{X}_{..} &= \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^s X_{(i-1)s+j} \\ &= \frac{1}{s} \sum_{j=1}^s \bar{X}_{.j} = \frac{1}{m} \sum_{i=1}^m \bar{X}_i \\ &= \frac{1}{m} \sum_{i=1}^m \left\{ \left[a + b(i-1)s + \frac{b}{s} \sum_{j=1}^s jS_j \right] + \bar{e}_i \right\} \\ &= \frac{1}{m} \left[am + bs \sum_{i=1}^m (i-1) + \frac{bm}{s} \sum_{j=1}^s jS_j \right] + \bar{e}_{..} \end{aligned}$$

where $\bar{e}_{..} = \frac{1}{m} \sum_{i=1}^m \bar{e}_i$.

$$\begin{aligned} \bar{X}_{..} &= a + bs \frac{m(m-1)}{2} + \frac{b}{s} \sum_{j=1}^s jS_j + \bar{e}_{..} \\ &= a + \frac{b}{2}(n-s) + \frac{b}{s} \sum_{j=1}^s jS_j + \bar{e}_{..} \end{aligned} \quad (17)$$

3. Estimation of Trend Parameters

We use the row means and overall means to estimate the parameters of the trend line. For mixed model

$$a - b(s - c_1) + (bs)_i \quad (18)$$

$$\equiv \alpha + \beta_i$$

where $\alpha = a - b(s - c_1)$, $\beta = bs$

$$\therefore \hat{a} = \alpha + \hat{b}(s - c_1) \quad (19)$$

$$\hat{b} = \frac{\beta}{s} \quad (20)$$

Again it reduces to

$$\hat{a} = X_{..} \quad (21)$$

when there is no trend. That is, when $b = 0$, a is estimated using the grand mean $\bar{X}_{..}$

3.1. Estimates of S_j , $j = 1, 2, \dots, s$

Here, we use column averages and overall averages to estimate the seasonal indices.

For mixed model, using (16)

$$\bar{X}_{.j} = \left[a + b \left(\frac{n-s}{2} \right) + b_j \right] S_j$$

$$\equiv [\alpha + \beta_j] + S_j \quad (22)$$

where $\alpha = a + b \left(\frac{n-s}{2} \right)$, $\beta = b$ (23)

$$\beta = b \quad (24)$$

$$\therefore \hat{S}_j = \frac{\bar{X}_{.j}}{a + b \left(\frac{n-s}{2} \right) + b_j} \quad (25)$$

when there is no trend $b = 0$, we obtain from (16) that

$$\hat{S}_j = \frac{\bar{X}_{.j}}{\bar{X}_{..}} \quad (26)$$

4. Real Life Example

The real life example is based on monthly data on number of road traffic accident collected from Federal Road Safety Commission (FRSC) Owerri, Imo State, Nigeria for the period 2009 to 2018 shown in Appendix A. The series of 120 observations has been arranged in a Buys-Ballot table as monthly data ($s = 12$) and for 10 years ($m = 10$). The time plots for the original and transformed series are given in figures 4.1 and 4.2 respectively. The expression of linear trend and seasonal indices for the mixed model is given as:

Hence, $\bar{X}_{.j} = 3.214 - 0.0052j$ (27)

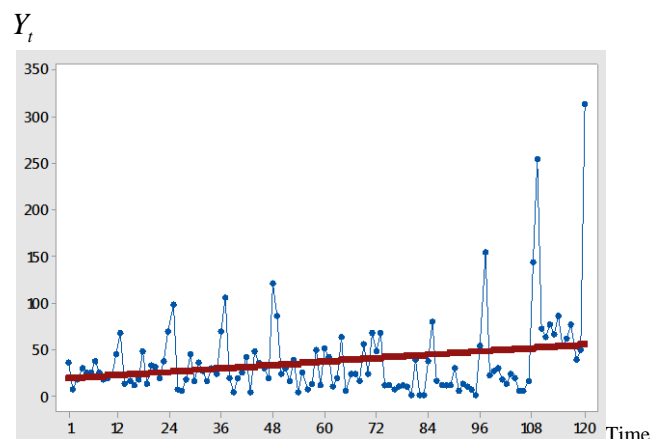


Figure 4.1. Time plot of the actual series on the number of road accidents between (2009-2018)

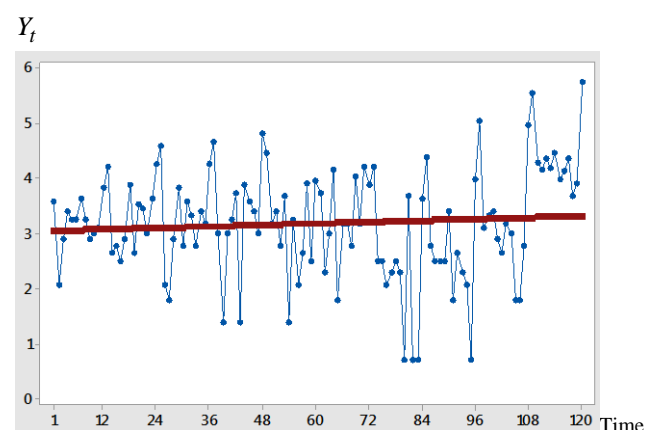


Figure 4.2. Time plot of the transformed series on the number of road accidents between (2009-2018)

Table 2. Estimates of Seasonal indices

j	$\bar{X}_{.j}$	\hat{S}_j
1	4.442	1.3843
2	2.790	0.8709
3	2.770	0.8661
4	3.1030	0.9718
5	3.0580	0.9592
6	3.1180	0.9796
7	2.8940	0.9108
8	2.9290	0.9233
9	3.1520	0.9968
10	2.8140	0.8899
11	2.7680	0.8768
12	4.330	1.3705
$\sum_{j=1}^s \hat{S}_j$		12.0000

Note: mixed model satisfies $\sum_{j=1}^s S_j = s$ as in (9)

Table 3. Estimates of Parameters of Trend and Seasonal Indices for Mixed model

Parameter	Values
\hat{a}	3.4948
\hat{b}	-0.0052
\hat{S}_1	1.3823
\hat{S}_2	0.8683
\hat{S}_3	0.8622
\hat{S}_4	0.9659
\hat{S}_5	0.9521
\hat{S}_6	0.9709
\hat{S}_7	0.9012
\hat{S}_8	0.9122
\hat{S}_9	0.9818
\hat{S}_{10}	0.8766
\hat{S}_{11}	0.8624
\hat{S}_{12}	1.3492

Using (23), (24) and (25)

$$\hat{b} = -0.0052$$

$$\hat{a} = 3.214 - (-0.0052) \left(\frac{120-12}{2} \right)$$

$$\hat{a} = 3.4948$$

$$\hat{S}_j = \frac{\bar{X}_{.j}}{3.214 - 0.0052j}$$

5. Summary, Conclusions and Recommendations

We have outlined the Buys-Ballot Procedure for the estimation of trend-cycle and seasonal indices for time series decomposition. This procedure is computationally simple when compared with other descriptive methods. The values of the estimated trend parameters and seasonal indices are easily computed and the results given in Table 3. No attempt has been made to discuss this method when the trend-cycle component is not linear or when the trend is separated from the cyclical component. Therefore, further research in these directions are recommended.

Appendix A

Buys-Ballot table for the data on the number of road accidents (2009-2018)

Months/ Year	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC	$\bar{X}_{I.}$	σ_i^2
2009	36	8	18	30	26	26	38	26	18	20	22	46	26.2	105.8
2010	68	14	16	12	18	48	14	34	32	20	38	70	32.0	425.5
2011	98	8	6	18	46	16	36	28	16	30	24	70	33.0	731.3
2012	106	20	4	20	26	42	4	48	36	30	20	122	39.8	1386.5
2013	86	24	30	16	40	4	26	8	14	50	12	52	30.2	555.2
2014	42	10	20	64	6	24	24	16	56	24	68	48	33.5	452.5
2015	68	12	12	8	10	12	10	2	40	2	2	38	18.0	408.0
2016	80	16	12	12	12	30	6	14	10	8	2	54	21.3	531.2
2017	154	22	28	30	18	14	24	20	6	6	16	144	40.2	2644.0
2018	254	72	64	78	66	86	54	62	78	40	50	314	101.5	7595.0
$\bar{X}_{.j}$	99.2	20.6	21.0	28.8	26.8	30.2	23.6	25.8	30.6	23.0	25.4	95.8		
$\sigma_{.j}^2$	4085.5	358.3	298.9	557.5	352.2	567.5	251.4	336.4	520.9	232.2	442.7	7080.4		

Source: Federal Road Safety Corps Owerri, Imo State Nigeria (2009 – 2018)

Appendix B

Buy-Ballot table for the transformed data on the number of road accidents (2009-2018)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	\bar{X}_I	σ_i^2
2009	3.5835	2.0794	2.8904	3.4002	3.2581	3.2581	3.6376	3.2581	2.8904	2.9956	3.0910	3.8286	3.181	0.207
2010	4.2195	2.6391	2.7726	2.4849	2.8904	3.8712	2.6391	3.5264	3.4657	2.9957	3.6376	4.2485	3.283	0.394
2011	4.5850	2.0794	1.7918	2.8904	3.8286	2.7726	3.5835	3.3322	2.7726	3.4012	2.9957	4.2485	3.205	0.665
2012	4.6634	2.9957	1.3863	2.9957	3.2581	3.7377	1.3863	3.8712	3.5835	3.4012	2.9957	4.8040	3.257	1.119
2013	4.4544	3.1781	3.4012	2.7726	3.6889	1.3863	3.2581	2.0794	2.6391	3.9120	2.4849	3.9512	3.101	0.760
2014	3.7377	2.3026	2.9957	4.1589	1.7918	3.1781	3.1781	2.7726	4.0254	3.1781	4.2195	3.8712	3.284	0.574
2015	4.2195	2.4849	2.4849	2.0794	2.3026	2.4849	2.3026	0.6932	3.6889	0.6932	0.6932	3.6376	2.314	1.391
2016	4.3820	2.7726	2.4849	2.4849	2.4849	3.4012	1.7918	2.6391	2.3026	2.0794	0.69312	3.9889	2.625	0.951
2017	5.0370	3.0910	3.3322	3.4012	2.8904	2.6391	3.1781	2.9957	1.7918	1.7918	2.7726	4.9698	3.158	1.014
2018	5.5373	4.2767	4.1588	4.3567	4.1897	4.4544	3.9889	4.12713	4.3567	3.6889	3.9120	5.7494	4.400	0.384
\bar{X}_j	4.442	2.790	2.770	3.103	3.058	3.118	2.894	2.929	3.152	2.814	2.768	4.330		
σ_j^2	0.329	0.429	0.638	0.538	0.546	0.732	0.714	0.976	0.648	0.994	1.465	0.429		

Source: Federal Road Safety Corps Owerri, Imo State (2009-2018)

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