

Deriving the (0+2) Cubic Nonlinear Schrödinger Equation from the Theory of Subsonic Compressible Aerodynamics

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Abstract Three discoveries of profound significance are described using three-dimensional aerodynamic theory of compressible flow in unbounded domains with a fixed-to-body reference frame and cylindrical-polar coordinates. The first discovery demonstrates how special relativity expressions are obtainable from within the formulation of the steady-rotating source problem. The second follows from developing a conservative, induced velocity body force for curved filament vortices when simulating a harmonic-oscillating source that rotates and translates along its centerline. The third demonstrates how the focusing (0+2) cubic nonlinear Schrödinger equation is exactly contained within the associated convected wave equation for a source that is rotating, translating, and oscillating.

Keywords Compressible, Convected wave equation, Differential geometry, Harmonic oscillator, Hasimoto transform, Incompressible, Induced velocity, Localized induction approximation, Lorentz transform, NLS, Riemannian geometry, Special relativity

1. Introduction

Three important discoveries are presented using aerodynamic theory for subsonic conditions in unbounded domains. The first discovery follows from proving that the expressions of special relativity are obtained exactly by solving the nonlinear convected wave equation for a steady, rotating source in compressible flow. The problem is solved with absolute 3D cylindrical-polar and absolute time coordinates in a non-inertial, fixed-to-body reference frame. This has previously [1] only been shown for a steady, translating source with absolute 3D Cartesian and absolute time coordinates in a fixed-to-body reference frame.

The second discovery involves adding a conservative body force, that should never have been ignored in the first place, from the Navier-Stokes equation – the induced velocity contribution. Its formulation is based on the localized induction approximation (LIA) theory for curved vortex filaments. This paper addresses the effect of vortices that remain attached to slender, solid bodies translating and rotating through a compressible fluid. Even though it is not discussed here, an induced velocity body force can also be generated by the presence of curved vortex filaments that terminate and remain attached to surface boundaries (e.g.,

solid-fluid or fluid1-fluid2) or to hydrodynamic surfaces such as Lamb surfaces that form, break free, and move with the fluid flow system [2]. This could be a potential source of seed-points for turbulence.

The third discovery comes about after laying bare what has always been present in plain sight – the embedding of the exact cubic nonlinear Schrödinger (NLS) equation within the convected wave equation for compressible flow. However, it requires one to resist the parsimonious logic of eliminating cross-derivatives from partial differential equations in the pursuit of mathematical beauty and over-emphasis on simplicity. The embedding of the cubic NLS equation means that frictionless solitons and other nonlinear vortex processes are predicted to be present within equations describing simple laminar flow systems. Such complex features can't be simulated or even approximated by summing linear perturbations in either analytical or numerical simulations of incompressible flow equations.

2. Background & Review

2.1. Motivation

It was demonstrated in a 2017 paper [1] that the unsteady, nonlinear convected wave equation exactly represented the disturbance created by a steady translating source using an absolute 3D Cartesian and absolute time coordinate system. The source was assumed to translate at a constant speed along a straight line through an initially motionless, compressible fluid in an unbounded domain. Furthermore,

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it was demonstrated that the classical Lorentz transforms (i.e., special relativity) used for velocity, acceleration, momentum, energy, and mass are mathematical artefacts that arise from ignoring nonlinear cross-derivative terms in the convected wave equation and assuming the fluid was incompressible.

This paper will present two major findings for unbounded, compressible fluids using an absolute 3D cylindrical-polar and absolute time coordinate system. Both findings appear to have never been presented or suggested before:

- 1) To show that the convected wave equation analysis and findings of the 2017 paper [1] can be exactly extended to a source attached to a constant-speed rotating reference frame;
- 2) To derive the (0+2) cubic nonlinear Schrödinger equation for a harmonic oscillating source attached to a constant-speed rotating reference frame that simultaneously is also translating at a constant speed along the frame's centerline axis.

The second finding is developed after including an additional body-force term in the Navier-Stokes equations. This conservative body-force term is the negative gradient of a potential. This term also represents a potential energy source that is proportional to the bending stiffness of an elastic, curved filament vortex subjected to a self-induced velocity using the theory of the localized induction approximation. In aerodynamics, this body force arises from the bounded horseshoe vortices that trail from each wingtip surface. These vortices are the ones that engine exhaust makes visible in high altitude contrails.

2.2. Navier-Stokes Equation

The 3D *Navier-Stokes equation* for an inviscid fluid can be written in the following form:

$$\frac{\partial \mathbf{V}_f}{\partial t} + \mathbf{V}_f \cdot \text{grad} \mathbf{V}_f + \frac{(\text{grad } p_f)}{\rho_f} = \mathbf{F}_{BF} \quad (1)$$

Term \mathbf{F}_{BF} , with units L/T^2 , represents a *body force* per unit mass that acts on the fluid. Replace term $\mathbf{V}_f \cdot \text{grad} \mathbf{V}_f$ with the equivalent vector expression $\frac{1}{2} \text{grad} (\mathbf{V}_f \cdot \mathbf{V}_f) + (\text{curl } \mathbf{V}_f) \times \mathbf{V}_f$. If we restrict ourselves to body forces that are conservative, we can then replace term \mathbf{F}_{BF} with the negative gradient of a potential term Λ_{BF} , with units L^2/T^2 , (e.g., $\Lambda_{BF} = gh$ for gravity). The Navier-Stokes equation (1) reduces as:

$$\begin{aligned} \frac{\partial \mathbf{V}_f}{\partial t} + \frac{1}{2} \text{grad} (\mathbf{V}_f \cdot \mathbf{V}_f) + \frac{(\text{grad } p_f)}{\rho_f} \\ + \text{grad } \Lambda_{BF} = - (\text{curl } \mathbf{V}_f) \times \mathbf{V}_f \end{aligned} \quad (2)$$

The problem is further restricted if the $\text{grad } p_f$ term is replaced with a barotropic relationship when $\rho_f = \rho_f(p_f)$,

such that:

$$\begin{aligned} \frac{\partial \mathbf{V}_f}{\partial t} + \text{grad} \left(\frac{1}{2} \mathbf{V}_f \cdot \mathbf{V}_f + \int_{p'=p_{f0}}^{p_f} \frac{dp'}{\rho_f(p')} + \Lambda_{BF} \right) \\ = - (\text{curl } \mathbf{V}_f) \times \mathbf{V}_f \end{aligned} \quad (3)$$

Under the special condition of irrotational flow, the $\text{curl } \mathbf{V}_f$ term (i.e., vorticity) vanishes. However, vorticity will vanish everywhere except along infinitesimally thin vortex lines. In addition, irrotational flow means the velocity vector \mathbf{V}_f in (3) can be replaced with the gradient of a scalar velocity potential term Φ_f , with units L^2/T , such that:

$$\begin{aligned} \text{grad} \left(\frac{\partial \Phi_f}{\partial t} + \frac{1}{2} \mathbf{V}_f \cdot \mathbf{V}_f + \int_{p'=p_{f0}}^{p_f} \frac{dp'}{\rho_f(p')} + \Lambda_{BF} \right) \\ = 0 \end{aligned} \quad (4)$$

The following is a summary of assumptions for the fluid represented by the Navier-Stokes expression given in (4):

- a) Inviscid (negligible viscosity)
- b) Barotropic (density is a function of pressure)
- c) Subjected to a conservative body force
- d) Unsteady flow
- e) Irrotational (except along thin vortex lines)
- f) Potential based flow
- g) Compressible (density varies in time and space)
- h) Homentropic (uniform & constant entropy)

An isentropic flow means the entropy level of each infinitesimal fluid volume does not change with time but may vary from volume element to volume element. Thus, homentropic flow is isentropic but an isentropic flow is not necessarily homentropic.

2.3. Bernoulli Function

The expression within the brackets shown in (4) is called the Bernoulli function H_{BF} , with units L^2/T^2 , thus:

$$\text{grad } H_{BF} = 0 \quad (5)$$

$$H_{BF}(t, p_f, p_{f0}) =$$

$$\frac{\partial \Phi_f}{\partial t} + \frac{1}{2} \mathbf{V}_f \cdot \mathbf{V}_f + \int_{p'=p_{f0}}^{p_f} \frac{dp'}{\rho_f(p')} + \Lambda_{BF} \quad (6)$$

The $\text{grad } H_{BF} = 0$ condition means the Bernoulli function is independent of location along a streamline.

Compressibility of a fluid can be defined as the relative change in the local fluid density ρ_f , with units M/L^3 , in response to a change in the local fluid pressure p_f , with units $M/L/T^2$. An adiabatic compression means the entropy content of the fluid remains approximately constant

during a compression event. The freestream characteristic speed of the fluid, called the *speed of sound* c_∞ , with units L/T , can be defined [3], such that:

$$c_\infty^2 = \left(\frac{\partial p}{\partial \rho} \right) \bigg|_{\rho=\rho_\infty} \quad (7)$$

The ∞ subscript indicates fluid properties are to be evaluated outside the zone of disturbance.

2.4. Momentum and Continuity Equations

The momentum equation can be written for a fluid subjected to a conserved body force [4] in terms of a material derivative $D(\cdot)/Dt$ as:

$$\rho_f \frac{D\mathbf{V}_f}{Dt} = -\text{grad} (p_f + \rho_f \Lambda_{BF}) \quad (8)$$

$$\frac{D(\phi)}{Dt} = \frac{\partial(\phi)}{\partial t} + \mathbf{V}_f \cdot \text{grad} (\phi) \quad (9)$$

The continuity equation represents the change in the quantity of fluid contained in a differential volume element d^3x in the time interval dt . This change is set equal to the amount of fluid flowing in the volume element minus the amount flowing out of the volume element [3], such that:

$$\frac{D\rho_f}{Dt} = -\rho_f \text{div} (\mathbf{V}_f) \quad (10)$$

The gradient of a velocity potential Φ_f can be expressed in terms of the fluid velocity for irrotational flow, such that:

$$\mathbf{V}_f = \text{grad} \Phi_f \quad (11)$$

2.5. Compressible, Nonlinear, Convected-Wave Equation

The momentum and continuity equations can be combined and flow velocity \mathbf{V}_f replaced with velocity potential Φ_f for irrotational, inviscid, barotropic, isentropic flow [5] [6] subjected to a body force:

$$c_\infty^2 \nabla^2 \Phi_f = \frac{\partial^2 \Phi_f}{\partial t^2} + \frac{\partial (\nabla \Phi_f \cdot \nabla \Phi_f)}{\partial t} + \frac{1}{2} \nabla \Phi_f \cdot \nabla (\nabla \Phi_f \cdot \nabla \Phi_f) + \frac{D\Lambda_{BF}}{Dt} \quad (12)$$

Consider for the moment a fixed-to-body Cartesian coordinate system. The above wave equation (12) is an exact expression for the unsteady, nonlinear flow of a compressible fluid. It is written in terms of velocity potential Φ_f that is valid for both subsonic and supersonic flow conditions. However, (12) does not hold for transonic velocities since additional terms are needed to account for compression-shock and temperature losses. Replace the velocity potential term Φ_f with a subscript symbol 'c', Φ_c , to indicate compressible conditions. One can then write (12) as follows:

$$\begin{aligned} & \Phi_{c\,xx} \left(1 - \left(\Phi_{c\,x}/c_\infty \right)^2 \right) + \Phi_{c\,yy} \left(1 - \left(\Phi_{c\,y}/c_\infty \right)^2 \right) \\ & + \Phi_{c\,zz} \left(1 - \left(\Phi_{c\,z}/c_\infty \right)^2 \right) = \\ & \frac{1}{c_\infty^2} \Phi_{c\,tt} + 2 \frac{1}{c_\infty^2} \left(\Phi_{c\,x} \Phi_{c\,xt} + \Phi_{c\,y} \Phi_{c\,yt} + \Phi_{c\,z} \Phi_{c\,zt} \right) \\ & + 2 \frac{1}{c_\infty^2} \left(\Phi_{c\,x} \Phi_{c\,y} \Phi_{c\,xy} + \Phi_{c\,x} \Phi_{c\,z} \Phi_{c\,xz} + \Phi_{c\,y} \Phi_{c\,z} \Phi_{c\,yz} \right) \\ & + \frac{1}{c_\infty^2} \frac{D\Lambda_{BF}}{Dt} \end{aligned} \quad (13)$$

In the format given in (13), the reference frame is attached to the leading edge of the vehicle (or any slender, solid object). The positive X-axis is pointed towards the downstream end of the vehicle. Because of the fixed-to-body reference frame, the vehicle is stationary while the ambient fluid moves with freestream speed $|\mathbf{V}_\infty|$, directed parallel to the positive X-axis. The wave equation (13) is classified as a hyperbolic partial differential equation in terms of spatial coordinate X and time coordinate t for subsonic speeds.

2.6. Convected-Wave Equation in Cylindrical-Polar Coordinates

The unsteady convected wave equation (12) can also be written in a cylindrical-polar coordinate system, such that:

$$\begin{aligned} & \frac{\Phi_{c\,\theta\theta}}{\rho_c^2} \left(1 - \left(\frac{\Phi_{c\,\theta}}{\rho_c c_\infty} \right)^2 \right) + \Phi_{c\,\rho\rho} \left(1 - \left(\frac{\Phi_{c\,\rho}}{c_\infty} \right)^2 \right) \\ & + \Phi_{c\,zz} \left(1 - \left(\frac{\Phi_{c\,z}}{c_\infty} \right)^2 \right) = \frac{1}{c_\infty^2} \Phi_{c\,tt} \\ & + \frac{2}{c_\infty^2} \left(\Phi_{c\,\rho} \Phi_{c\,\rho t} + \frac{\Phi_{c\,\theta}}{\rho_c^2} \Phi_{c\,\theta t} + \Phi_{c\,z} \Phi_{c\,zt} \right) \\ & + \frac{2}{c_\infty^2} \left(\frac{\Phi_{c\,\rho}}{\rho_c^2} \Phi_{c\,\theta} \Phi_{c\,\rho\theta} + \Phi_{c\,z} \frac{\Phi_{c\,\theta}}{\rho_c^2} \Phi_{c\,z\theta} + \Phi_{c\,z} \Phi_{c\,\rho} \Phi_{c\,z\rho} \right) \\ & - \frac{\Phi_{c\,\rho}}{\rho_c} \left(1 + \left(\frac{\Phi_{c\,\theta}}{\rho_c c_\infty} \right)^2 \right) + \frac{1}{c_\infty^2} \frac{D\Lambda_{BF}}{Dt} \end{aligned} \quad (14)$$

Term ρ_c is the polar radius; θ_c is the azimuth angle; $x_c = \rho_c \cos \theta_c$ is the Cartesian X-coordinate; and $y_c = \rho_c \sin \theta_c$ is the Cartesian Y-coordinate. The cylindrical-polar velocity components are as follows:

$$\begin{aligned} V_{c\,z} &= \frac{\partial \Phi_c}{\partial z_c} ; & V_{c\,\theta} &= \frac{1}{\rho_c} \frac{\partial \Phi_c}{\partial \theta_c} ; & V_{c\,\rho} &= \frac{\partial \Phi_c}{\partial \rho_c} \\ &= \frac{\partial z_c}{\partial t} ; & &= \rho_c \frac{\partial \theta_c}{\partial t} ; & &= \frac{\partial \rho_c}{\partial t} \end{aligned} \quad (15)$$

Term V_{cz} is the longitudinal velocity component parallel to the longitudinal unit vector \hat{e}_z ; term $V_{c\theta}$ is the circumferential velocity component parallel to the tangential unit vector \hat{e}_θ ; and term $V_{c\rho}$ is the radial velocity component parallel to the polar-radial unit vector \hat{e}_ρ . Hence, one can write the velocity vector for compressible flow conditions V_c as the sum of the three vector components:

$$V_c = V_{c\rho}\hat{e}_\rho + V_{c\theta}\hat{e}_\theta + V_{cz}\hat{e}_z \quad (16)$$

The quotient of a velocity component with that of the freestream speed of sound c_∞ from (7) is defined as the *Mach number* for that velocity component, such that:

$$\frac{V_{c\rho}}{c_\infty} = M_\rho; \quad \frac{V_{c\theta}}{c_\infty} = M_\theta; \quad \frac{V_{cz}}{c_\infty} = M_z \quad (17)$$

The total Mach number M_f for a fluid velocity is defined as the sum of its squared components. It must be less than one for subsonic flow conditions:

$$M_f = \sqrt{M_\rho^2 + M_\theta^2 + M_z^2} \quad (18)$$

$$M_f < 1 \quad (19)$$

Only the cylindrical-polar coordinate version of the wave equation (14) will be discussed in the remainder of the paper.

3. Case 1 - Rotating but Non-Translating Reference frame with a Vanishing Body Force for Subsonic Velocities

The first case will demonstrate that the 3D nonlinear convected wave equation in cylindrical-polar coordinates for compressible flow conditions with a non-inertial, fixed-to-body reference frame in an unbounded domain reduces to a 2D equation in Cartesian coordinates. This occurs when:

- 1) A source is assumed to consist of a small, slender, solid object;
- 2) Source is fixed to a non-inertial reference frame that rotates at a constant, subsonic, angular speed about the centerline axis;
- 3) Fluid in an unbounded domain is initially at rest;
- 4) Fluids are limited to those that are inviscid (i.e., viscosity $\mu_c \equiv 0$), irrotational (i.e., $\text{curl } \mathbf{u} \equiv 0$), barotropic (i.e., $p_c \equiv f(\rho_c)$), isentropic (i.e., constant entropy), and compressible;
- 5) No body force acts on the surrounding fluid.

Using the above assumptions, we will set the following terms for a rotating but non-translating reference frame and a negligible body force:

$$V_{f\theta} \neq 0; V_{f\rho} \equiv 0; V_{fz} \equiv 0; \text{ \& } \Lambda_{BF} \equiv 0 \quad (20)$$

3.1. Linearization of Wave Equation

The fluid density ρ_f , fluid pressure p_f , fluid velocity V_f , and velocity potential Φ_f will be linearized as follows with a perturbation component:

$$\begin{aligned} \rho_f(\mathbf{x}, t) &= \rho_\infty + \hat{\rho}(\mathbf{x}, t) \\ p_f(\mathbf{x}, t) &= p_\infty + \hat{p}(\mathbf{x}, t) \\ V_f(\mathbf{x}, t) &= V_\infty + \hat{V}(\mathbf{x}, t) \\ \Phi_f(\mathbf{x}, t) &= \Phi_\infty + \hat{\phi}(\mathbf{x}, t) \end{aligned} \quad (21)$$

The subscript ∞ is used to indicate terms that are evaluated at freestream or undisturbed conditions.

It will be assumed that the magnitude of the perturbed velocity \hat{V} is much smaller than the magnitude of the freestream velocity V_∞ and that the magnitude of the freestream velocity is much smaller than the characteristic speed of the fluid c_∞ :

$$\left| \frac{\hat{V}}{V_\infty} \right| \ll 1; \quad \left| \frac{V_\infty}{c_\infty} \right| < 1 \quad (22)$$

Substitute the linearized variables from (21) and the Mach number expression from (17) into the general wave equation (14) for compressible flow conditions. The unsteady, convected wave equation for the perturbed velocity potential in 3D cylindrical-polar coordinates will then reduce as follows for Case 1 conditions in a fixed-to-body reference frame:

$$\begin{aligned} \frac{\hat{\phi}_{\theta\theta}}{\rho^2} (1 - M_\theta^2) + \hat{\phi}_{\rho\rho} + \frac{\hat{\phi}_\rho}{\rho} (1 + M_\theta^2) + \hat{\phi}_{zz} \\ = \frac{1}{c_\infty^2} \hat{\phi}_{tt} + \frac{2}{c_\infty} \frac{M_\theta}{\rho} \hat{\phi}_{\theta t} \end{aligned} \quad (23)$$

Term ρ , with units L , is the polar-radius coordinate in the cylindrical-polar coordinate system for compressible flow. All third- and fourth-order perturbation terms have been dropped in (23).

3.2. Change in Variables

Define the dimensionless parameter $\beta_\theta = \sqrt{1 - M_\theta^2}$, spatial variable s , with units L , and temporal variable ε , with units L , for subsonic velocity conditions, where:

$$s = (\theta - \theta_0)\rho \quad (24)$$

$$\varepsilon = (t - t_0) V_{\infty\theta} \quad (25)$$

The s variable represents an estimate of the circumferential arclength traversed when the azimuth angle changes by the angle $\theta - \theta_0$.

Take the following partial derivatives in terms of the s

variable using the chain-rule of differentiation:

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial s} \rho \\ \frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial^2 f}{\partial s^2} \rho^2\end{aligned}\quad (26)$$

Take the following partial derivatives in terms of the ε variable using the chain-rule of differentiation:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial \varepsilon} V_{\infty \theta} \\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial^2 f}{\partial \varepsilon^2} V_{\infty \theta}^2\end{aligned}\quad (27)$$

The wave equation (23) reduces as follows for Case 1 after substitution of the derivatives from (26) and (27):

$$\begin{aligned}\beta_\theta^2 \hat{\phi}_{ss} + \hat{\phi}_{\rho\rho} + \frac{\hat{\phi}_\rho}{\rho} \left(1 + M_\theta^2\right) + \hat{\phi}_{zz} \\ = M_\theta^2 \hat{\phi}_{\varepsilon\varepsilon} + 2M_\theta^2 \hat{\phi}_{s\varepsilon}\end{aligned}\quad (28)$$

Consider a new perturbation velocity potential term $\phi(\mathbf{x}, \varepsilon)$ and its derivatives, such that

$$\begin{aligned}\hat{\phi}(\mathbf{x}, \varepsilon) &= \phi(\mathbf{x}, \varepsilon) \rho^a \\ \hat{\phi}_\rho &= \left(\phi_\rho + a \frac{\phi}{\rho}\right) \rho^a \\ \hat{\phi}_{\rho\rho} &= \left(\phi_{\rho\rho} + 2\frac{a}{\rho} \phi_\rho + a(a-1) \frac{\phi}{\rho^2}\right) \rho^a\end{aligned}\quad (29)$$

Term a in (29) is an arbitrary dimensionless coefficient.

Replace the potential $\hat{\phi}$ and its derivatives in the wave equation (28) with the new potential ϕ and divide out the common term ρ^a when finished, such that:

$$\begin{aligned}\beta_\theta^2 \phi_{ss} + \phi_{\rho\rho} + \frac{\phi_\rho}{\rho} \left(2a+1 + M_\theta^2\right) \\ + \frac{\phi}{\rho^2} \left(a(a-1) + a(1 + M_\theta^2)\right) + \phi_{zz} \\ = M_\theta^2 \phi_{\varepsilon\varepsilon} + 2M_\theta^2 \phi_{s\varepsilon}\end{aligned}\quad (30)$$

The first bracketed term in (30) can be reduced to a standard radial coordinate form by setting the a coefficient equal to $a \equiv -\frac{1}{2}M_\theta^2$. The wave equation for a fixed-to-body reference frame (30) simplifies as follows upon back substitution with the new a coefficient, such that:

$$\begin{aligned}\beta_\theta^2 \frac{\partial^2 \phi_c}{\partial s_c^2} + \frac{\partial^2 \phi_c}{\partial \rho_c^2} + \frac{1}{\rho_c} \frac{\partial \phi_c}{\partial \rho_c} + \frac{\partial^2 \phi_c}{\partial z_c^2} - \frac{M_\theta^4}{4} \frac{\phi_c}{\rho_c^2} \\ = M_\theta^2 \frac{\partial^2 \phi_c}{\partial \varepsilon_c^2} + 2M_\theta^2 \frac{\partial^2 \phi_c}{\partial s_c \partial \varepsilon_c}\end{aligned}\quad (31)$$

The subscript c has been added in (31) to remind us that this equation is based on a compressible fluid assumption.

3.3. Separation-of-Variables

Consider the following separation-of-variables solution $\phi_c = X_c R_c$, where only the X_c component varies with time:

$$\phi_c(s, \rho_c, z_c, \varepsilon_c) \equiv X_c(s, z_c, \varepsilon_c) R_c(\rho_c) \quad (32)$$

Substitute the transform (32) into the compressible wave (31) and divide the resultant expression by the term $X_c R_c$:

$$\begin{aligned}\frac{1}{X_c} \left(\beta_\theta^2 \frac{\partial^2 X_c}{\partial s_c^2} + \frac{\partial^2 X_c}{\partial z_c^2} - M_\theta^2 \frac{\partial^2 X_c}{\partial \varepsilon_c^2} - 2M_\theta^2 \frac{\partial^2 X_c}{\partial s_c \partial \varepsilon_c} \right) \\ = - \frac{1}{R_c} \left(\frac{\partial^2 R_c}{\partial \rho_c^2} + \frac{1}{\rho_c} \frac{\partial R_c}{\partial \rho_c} - \frac{M_\theta^4}{4} \frac{R_c}{\rho_c^2} \right) \\ = \lambda_{c0}\end{aligned}\quad (33)$$

Term λ_{c0} is an unknown constant coefficient. The X_c and R_c terms are independently set equal to λ_{c0} , giving the coupled differential equations:

$$\frac{\partial^2 R_c}{\partial \rho_c^2} + \frac{1}{\rho_c} \frac{\partial R_c}{\partial \rho_c} - \frac{M_\theta^4}{4} \frac{R_c}{\rho_c^2} + \lambda_{c0} R_c \equiv 0 \quad (34)$$

$$\begin{aligned}\beta_\theta^2 \frac{\partial^2 X_c}{\partial s_c^2} + \frac{\partial^2 X_c}{\partial z_c^2} - M_\theta^2 \frac{\partial^2 X_c}{\partial \varepsilon_c^2} - 2M_\theta^2 \frac{\partial^2 X_c}{\partial s_c \partial \varepsilon_c} \\ - \lambda_{c0} X_c \equiv 0\end{aligned}\quad (35)$$

3.4. Wave Equation for the Bessel Laplacian

The first equation (34), which is not a function of time, is called the wave equation for the Bessel Laplacian. It is of order $\nu = \frac{1}{2}M_\theta^2$, which has non-integer values. The details of the solution will not be presented here but they will be shown in Case 2 to equal:

$$R_c(\rho_c) = C_{cr} J_\nu(\rho_c \sqrt{\lambda_{c0}}) \quad (36)$$

Term C_{cr} is an unknown, real valued constant. It should be obvious that the parameter λ_{c0} must be positive valued for a physically meaningful solution when R_c is based on the Bessel function of the first kind $J_\nu(\cdot)$.

3.5. 2D Wave Equation and Special Relativity

We will now examine the transient, 2D wave equation given in (35) that includes the term $\lambda_{c0} X_c$. It should be noted that (35) is in the proper form of a 2D Cartesian coordinate system representing a compressible fluid. The coordinate set $\{s_c, z_c, \varepsilon_c\}$ for compressible flow conditions will be transformed to an equivalent coordinate set $\{s_{ic}, z_{ic}, \varepsilon_{ic}\}$ for incompressible flow conditions:

$$X_c(s_c, z_c, \varepsilon_c) = \beta_\theta X_{ic}(s_{ic}, z_{ic}, \varepsilon_{ic}) \quad (37)$$

$$s_{ic} = a_c s_c + b_c \varepsilon_c \quad (38)$$

$$z_{ic} = z_c \quad (39)$$

$$\varepsilon_{ic} = c_c s_c + d_c \varepsilon_c \quad (40)$$

Term β_θ is defined as $\beta_\theta = \sqrt{1 - M_\theta^2}$ and a_c, b_c, c_c, d_c are unknown coefficients of the compressible-to-incompressible coordinate transform.

The 2D wave equation with term $\lambda_{c0} X_{ic}$ can be written in Cartesian coordinates for an incompressible fluid medium with a fixed-to-body reference frame as:

$$\frac{\partial^2 X_{ic}}{\partial s_{ic}^2} + \frac{\partial^2 X_{ic}}{\partial z_{ic}^2} = M_\theta^2 \frac{\partial^2 X_{ic}}{\partial \varepsilon_{ic}^2} + \lambda_{c0} X_{ic} \quad (41)$$

The coordinate transform linking the partial differential equation (35) with (41) is written in matrix form as:

$$\begin{pmatrix} s_{ic} \\ \varepsilon_{ic} \end{pmatrix} = \frac{1}{\beta_\theta} \begin{pmatrix} 1 & 0 \\ M_\theta^2 & \beta_\theta^2 \end{pmatrix} \cdot \begin{pmatrix} s_c \\ \varepsilon_c \end{pmatrix} \quad (42)$$

The matrix in (42) is in fact the *Miles transform* [6]. The classical *Lorentz transform* [7] from Special Relativity is related to (42) by pre-multiplying the Galilean transform matrix with that of the Miles transform matrix:

$$\frac{1}{\beta_\theta} \begin{pmatrix} 1 & 0 \\ M_\theta^2 & \beta_\theta^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{\beta_\theta} \begin{pmatrix} 1 & 1 \\ M_\theta^2 & 1 \end{pmatrix} \quad (43)$$

(Miles) (Galilean) (Lorentz)

Note that the coefficient λ_{c0} played no role in the derivation of the transform matrices shown in (42) and (43).

All details of the derivations for (42) and (43) can be found in the 2017 [1] paper. Furthermore, [1] shows how the resultant PDE can be converted between sixteen fixed-to-body, fixed-in-space, compressible, and incompressible reference frames.

3.6. Separation-of-Variables: Again

The wave equation in (41) can be further solved by using the separation-of-variables method a second time by replacing term X_{ic} with the following triple product:

$$X_{ic}(s_{ic}, z_{ic}, \varepsilon_{ic}) = S(s_{ic})Z(z_{ic})T(\varepsilon_{ic}) \quad (44)$$

Partial differential equations that can be solved using the method of separation-of-variables leads to solutions that are products of exponential functions with either real or imaginary arguments [8].

Substitute (44) into (41) and divide the resultant expression by the triple product $S Z T$:

$$\frac{1}{S} \frac{\partial^2 S}{\partial s_{ic}^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z_{ic}^2} = \frac{M_\theta^2}{T} \frac{\partial^2 T}{\partial \varepsilon_{ic}^2} + \lambda_{c0} \quad (45)$$

Set the right-hand side expression equal to another constant γ_0 since it is independent of the left-hand side expression:

$$\begin{aligned} \frac{1}{S} \frac{\partial^2 S}{\partial s_{ic}^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z_{ic}^2} &= \frac{M_\theta^2}{T} \frac{\partial^2 T}{\partial \varepsilon_{ic}^2} + \lambda_{c0} \\ &= \gamma_0 \end{aligned} \quad (46)$$

Set the right-hand side term from the first line in (46) equal the right-hand side term from the second line in (46) and rearrange terms, such that:

$$\frac{\partial^2 T}{\partial \varepsilon_{ic}^2} + \frac{(\lambda_{c0} - \gamma_0)}{M_\theta^2} T = 0 \quad (47)$$

There are four possible solutions to (47) that depend upon the sign of the terms $\lambda_{c0} - \gamma_0$, $\gamma_0 - \alpha_0$, α_0 , and whether the Mach number M_θ vanishes or not. Three phase angles are defined in terms of the unknown constants:

$$\begin{aligned} \Theta_T &= \frac{\sqrt{|\lambda_{c0} - \gamma_0|}}{M_\theta} \\ \Theta_Z &= \sqrt{|\gamma_0 - \alpha_0|} \\ \Theta_S &= \sqrt{|\alpha_0|} \end{aligned} \quad (48)$$

The four possible analytical solutions for term T are as follows:

i. If $(\lambda_{c0} - \gamma_0) > 0$ and $M_\theta > 0$

$$T(\varepsilon_{ic}) = A_c \cos(\varepsilon_{ic} \Theta_T) + A_s \sin(\varepsilon_{ic} \Theta_T) \quad (49)$$

ii. If $(\lambda_{c0} - \gamma_0) < 0$ and $M_\theta > 0$

$$T(\varepsilon_{ic}) = A_p e^{\varepsilon_{ic} \Theta_T} + A_n e^{-\varepsilon_{ic} \Theta_T} \quad (50)$$

iii. If $(\lambda_{c0} - \gamma_0) = 0$ and $M_\theta > 0$

$$T(\varepsilon_{ic}) = A_{00} + A_{01} \varepsilon_{ic} \quad (51)$$

iv. If $M_\theta = 0$

$$T(\varepsilon_{ic}) = 1 \quad (52)$$

The terms $A_c, A_s, A_p, A_n, A_{00}$, & A_{01} are unknown constants of integration that are solved based on the boundary conditions in an unbounded domain.

Rearrange the left-hand side of equation (46) and solve for the Z component:

$$\begin{aligned} \frac{1}{Z} \frac{\partial^2 Z}{\partial z_{ic}^2} &= - \left(\frac{1}{S} \frac{\partial^2 S}{\partial s_{ic}^2} - \gamma_0 \right) \\ &= \alpha_0 \end{aligned} \quad (53)$$

Set the right-hand side of the first line equal the right-hand side of the second line with unknown coefficient α_0 and

solve for Z :

$$\frac{\partial^2 Z}{\partial z_{ic}^2} - (\gamma_0 - \alpha_0)Z = 0 \quad (54)$$

There are three possible solutions to (54), depending upon the sign of the expression $\gamma_0 - \alpha_0$:

i. If $(\gamma_0 - \alpha_0) > 0$

$$Z(z_{ic}) = B_p e^{z_{ic}\Theta_Z} + B_n e^{-z_{ic}\Theta_Z} \quad (55)$$

ii. If $(\gamma_0 - \alpha_0) < 0$

$$Z(z_{ic}) = B_c \cos(z_{ic}\Theta_Z) + B_s \sin(z_{ic}\Theta_Z) \quad (56)$$

iii. If $(\gamma_0 - \alpha_0) = 0$

$$Z(z_{ic}) = B_{00} + B_{01} z_{ic} \quad (57)$$

The terms B_c , B_s , B_p , B_n , B_{00} , & B_{01} are unknown constants of integration that are solved based on the boundary conditions in an unbounded domain.

Rearrange the left-hand side of equation (53) and solve for the S component:

$$\frac{\partial^2 S}{\partial s_{ic}^2} - \alpha_0 S = 0 \quad (58)$$

There are three possible solutions to (58), depending upon the sign of the coefficient α_0 :

i. If $\alpha_0 > 0$

$$S(s_{ic}) = C_p e^{s_{ic}\Theta_S} + C_n e^{-s_{ic}\Theta_S} \quad (59)$$

ii. If $\alpha_0 < 0$

$$S(s_{ic}) = C_c \cos(s_{ic}\Theta_S) + C_s \sin(s_{ic}\Theta_S) \quad (60)$$

iii. If $\alpha_0 = 0$

$$S(s_{ic}) = C_{00} + C_{01} s_{ic} \quad (61)$$

The coefficients C_c , C_s , C_p , C_n , C_{00} , & C_{01} are unknown constants of integration that are solved based on the boundary conditions in an unbounded domain.

3.7. Final Solution to Case 1

The final solution of Case 1 for the velocity potential Φ_f of compressible flow in a fixed-to-body reference frame can be written in the following form:

$$\Phi_f(\mathbf{x}, t) = \Phi_\infty + \beta_\theta \frac{C_{cr}}{\rho_c^v} J_v(\rho_c \sqrt{\lambda_{c0}}) S(s_{ic}) Z(z_{ic}) T(\varepsilon_{ic}) \quad (62)$$

The conversion of incompressible coordinates s_{ic} , z_{ic} , and ε_{ic} to compressible coordinates s_c , z_c , and ε_c are listed in (39) and (42). Order ν of the Bessel function is

defined as $\nu = \frac{1}{2} M_\theta^2$.

The purpose of presenting Case 1 is to show how the original 3D, nonlinear, convected wave equation (23) in cylindrical-polar coordinates for a compressible fluid in a rotating, but non-translating, fixed-to-body reference frame can be transformed to a system of two partial differential equations. One of the partial differential equation (PDE) expressions is solved as a steady wave equation for the Bessel Laplacian. The other PDE expression is solved as a 2D, transient, wave equation in Cartesian coordinates. This was made possible by converting the azimuth coordinate θ to a circumferential arclength coordinate s . It was pointed out in a 2017 paper [1] that the resultant PDE could be converted between sixteen fixed-to-body, fixed-in-space, compressible, and incompressible reference frames. The transformations are based on the Miles, Galilean, and Lorentz matrices.

4. Case 2 - Rotating and Translating Reference Frame Subjected to a Conservative Body Force for Subsonic Velocities

4.1. Introduction to Case 2

The second case will demonstrate that the 3D nonlinear convected wave equation for compressible flow conditions with a non-inertial, fixed-to-body reference frame in an unbounded domain reduces exactly to the (0+2) focusing cubic NLS equation. This occurs when:

- 1) A source is assumed to consist of a small, slender, solid object;
- 2) Source is a harmonic oscillator of the form $\phi_A(\vec{x}) e^{-i\omega(t-t_0)}$;
- 3) Source is fixed to a non-inertial reference frame that rotates at a constant, subsonic, angular speed about the centerline axis;
- 4) Source simultaneously translates at a constant, subsonic speed in a direction parallel to the centerline axis;
- 5) Fluid in an unbounded domain is initially at rest;
- 6) Fluids are limited to those that are inviscid (i.e., viscosity $\mu_c \equiv 0$), irrotational (i.e., $\text{curl } \mathbf{u} \equiv 0$, except along infinitely thin vortex lines), barotropic (i.e., $p_c \equiv f(\rho_c)$), isentropic (i.e., constant entropy), and compressible;
- 7) One or more curved vortex filaments are instantly generated by a source. The filaments remain attached to the source, but they extend into the downwind direction;
- 8) Each curved vortex filament generates an induced velocity in a direction perpendicular to the centerline. The induced velocity produces a conservative body force that acts on the surrounding fluid.

4.2. Wave Equation to be Solved in Case 2

As previously stated, we shall assume in Case 2 a rotating and translating reference frame subjected to a conservative body force and a harmonic oscillator source:

$$V_f \theta \neq 0; V_f \rho \equiv 0; V_f z \neq 0; \& \Lambda_{BF} \neq 0 \quad (63)$$

The corresponding convected wave equation to be solved is given in (14) and after applying the conditions of (63):

$$\begin{aligned} & \frac{\Phi_c \theta \theta}{\rho_c^2} \left(1 - \frac{V_\theta^2}{c_\infty^2} \right) + \Phi_c \rho \rho + \Phi_c z z \left(1 - \frac{V_z^2}{c_\infty^2} \right) \\ &= \frac{1}{c_\infty^2} \Phi_c t t + \frac{2}{c_\infty} \left(\frac{V_\theta}{c_\infty} \frac{\Phi_c \theta t}{\rho_c} + \frac{V_z}{c_\infty} \Phi_c z t \right) \\ &+ 2 \frac{V_z}{c_\infty} \frac{V_\theta}{c_\infty} \frac{\Phi_c z \theta}{\rho_c} - \frac{\Phi_c \rho}{\rho_c} \left(1 + \frac{V_\theta^2}{c_\infty^2} \right) + \frac{1}{c_\infty^2} \frac{D \Lambda_{BF}}{D t} \end{aligned} \quad (64)$$

4.3. Linearization of Wave Equation

The fluid density ρ_f , fluid pressure p_f , fluid velocity V_f , and velocity potential Φ_f will be linearized with a perturbation component in the same way as shown in (21) and (22) and substituted into (64). Upon dropping all third and fourth-order perturbation terms, the convected wave equation from (64) reduces to the following:

$$\begin{aligned} & \frac{\hat{\phi} \theta \theta}{\rho_c^2} \left(1 - M_\theta^2 \right) + \hat{\phi} \rho \rho + \hat{\phi} z z \left(1 - M_z^2 \right) \\ &= \frac{1}{c_\infty^2} \hat{\phi} t t + \frac{2}{c_\infty} \left(M_\theta \frac{\hat{\phi} \theta t}{\rho_c} + M_z \hat{\phi} z t \right) \\ &+ 2 M_\theta M_z \frac{\hat{\phi} \theta z}{\rho_c} - \frac{\hat{\phi} \rho}{\rho_c} \left(1 + M_\theta^2 \right) \\ &+ \frac{1}{c_\infty^2} \frac{D \Lambda_{BF}}{D t} \end{aligned} \quad (65)$$

The Mach coefficients M_θ and M_z are defined in (17).

Before proceeding with the solution of (65) for the nonlinear Schrödinger equation, it is worthwhile to review the Madelung transform that has been used since 1927.

4.4. Brief History of Madelung Transforms

The Madelung transform was first introduced by Madelung [9]. It defines the complex valued wavefunction Ψ_m in polar form as $\Psi_m = \sqrt{r_m} e^{i \Phi_m}$. The argument of the wavefunction equals term Φ_m and the modulus is term $\sqrt{r_m}$. The fluid density ρ_f is then identified with term $\sqrt{r_m}$ and the fluid velocity \vec{V}_f is identified with the gradient operator as $\vec{V}_f = \text{grad } \Phi_m$. The Madelung transform only satisfies the (1+1) linear Schrödinger equation. It also brings about an unusual term called the

Bohm quantum potential U_m that appears in the resultant momentum equation [10], where $U_m = -\nabla^2 |\Psi_m| / |\Psi_m|$.

A more recent and relevant use of the Madelung transformation is given in [10]. He assumes an ideal gas law and temperature T that is a function of time only. The resultant (1+1) nonlinear Schrödinger equation formulation contains the *Bohm-quantum potential* U_m [11] and the *Bialynicki-Birula logarithmic potential* [12]. Several exact solutions for inviscid, irrotational, isentropic, and compressible flow are derived [10] but they are essentially restricted to finite domain problems. This is because the velocity and density functions increase indefinitely with distance from the origin.

Only the *Hasimoto transform* [13], in conjunction with LIA based flow theory of curved vortex filaments, is known to be consistent with the (1+1) and (0+2) cubic NLS equations. However, what has completely been absent in the literature is a rational theory demonstrating the derivation or origin of the cubic NLS equation itself using the classical Navier-Stokes equations for compressible flow. This paper will show, apparently for the first time, a derivation based on an aerodynamic application.

4.5. Development of Convected Wave Equation

4.5.1. Source as a Harmonic Oscillator

The source disturbance will be treated as a harmonic oscillator. The resultant 3D wave will vary harmonically in time. One can then write the perturbed velocity potential $\hat{\phi}$, with units L^2/T , as a function of a steady-perturbation velocity potential ϕ_H , with units L^2/T , and a harmonic component:

$$\hat{\phi}(\rho, \theta, z, t) = \phi_H(\rho, \theta, z) e^{-i \omega(t-t_0)} \quad (66)$$

Term ω , with units $1/T$, represents the spin angular speed of the harmonic oscillator source.

4.5.2. Including a New Conservative Body Force

A conservative body force F_{BF} is expressed as the negative gradient of a potential Λ_{BF} :

$$F_{BF} = - \text{grad } \Lambda_{BF} \quad (67)$$

Potential Λ_{BF} , with units L^2/T^2 , is related to a potential energy source that is proportional α_{BF} , with units $1/L^2$, to the *bending stiffness* of an elastic, curved filament vortex undergoing *self-induction*:

$$\Lambda_{BF}(\rho, \theta, z) = -\frac{1}{2} \alpha_{BF} \phi_H^2 \quad (68)$$

4.5.3. Localized Induction Approximation

Assuming the applicability of the *localized induction approximation* (LIA) for a curved filament or vortex, then

the velocity induced, \mathbf{u}_{LIA} , with units L/T , at position X along the filament centerline is given by the time derivative of the position vector [13] [14] [15] [16] [17] [18]:

$$\begin{aligned}\frac{\partial X}{\partial t} &= \mathbf{u}_{LIA} \\ &= G_{LIA} \kappa \mathbf{B}\end{aligned}\quad (69)$$

Term G_{LIA} , with units of L^2/T , is called the coefficient of local induction.

4.5.4. Frenet-Serret Formulas of Differential Geometry

Term κ in (69), with units of $1/L$, is the curvature of the filament curve and the dimensionless unit vector \mathbf{B} is the *binormal vector* from the *Frenet-Serret formulas of differential geometry* [19]. If the dimensionless unit vector \mathbf{T} is the tangent vector parallel to the centerline of the filament curve and dimensionless unit vector \mathbf{N} is the normal vector, then the partial derivative of the tangent vector with respect to arc-length s along the curve is:

$$\frac{\partial \mathbf{T}}{\partial s} = \kappa \mathbf{N} \quad (70)$$

Hence, the magnitude of the self-induced velocity \mathbf{u}_{LIA} is given as [15] [20] [21] [22] [23]:

$$|\mathbf{u}_{LIA}| = G_{LIA} \kappa \quad (71)$$

Term G_{LIA} in (69) and (71) is a function of the vortex strength, radius of the vortex core, and cut-off arclength [13].

4.5.5. Components of the Fluid Velocity Vector \mathbf{V}_f

At any point in the unbounded domain, the fluid velocity \mathbf{V}_f will be assumed to consist of freestream velocity \mathbf{V}_∞ ; the perturbation velocity $\mathbf{v}(\rho, \theta, z, t)$ due to the presence of the solid; slender body moving through the fluid; and the induced velocity component $\mathbf{v}_{LIA}(\rho, \theta, z, t)$ generated by the presence of the curved filament $|\mathbf{u}_{LIA}| = G_{LIA} \kappa$ vortex that trails downwind of the slender body that behaves as a harmonic source:

$$\begin{aligned}\mathbf{V}_f(\rho, \theta, z, t) &= \mathbf{V}_\infty + \mathbf{v}(\rho, \theta, z, t) \\ &\quad + \mathbf{v}_{LIA}(\rho, \theta, z, t) \\ &= \mathbf{V}_\infty + \mathbf{u}(\rho, \theta, z) e^{-i\omega(t-t_0)} \\ &\quad + \mathbf{u}_{LIA}(\rho, \theta, z) e^{-i\omega(t-t_0)}\end{aligned}\quad (72)$$

Velocity \mathbf{u} , with units L/T , is the steady-perturbation velocity component due to the presence of a slender solid body; and \mathbf{u}_{LIA} , with units L/T , is the steady-perturbation velocity component due to the self-induced velocity produced by the curved filament vortex (i.e., at point (ρ, θ, z) along the filament).

4.5.6. Gradient of the Steady-Perturbation Velocity Potential

The gradient of the steady-perturbation velocity potential

ϕ_H consists of two contributions:

$$\begin{aligned}\text{grad } \phi_H &= \frac{\partial \phi_H}{\partial \rho} \hat{\mathbf{e}}_\rho + \frac{\partial \phi_H}{\partial \theta} \frac{1}{\rho} \hat{\mathbf{e}}_\theta + \frac{\partial \phi_H}{\partial z} \hat{\mathbf{e}}_z \\ &= \mathbf{u} + \mathbf{u}_{LIA}\end{aligned}\quad (73)$$

This means induced velocity vector \mathbf{u}_{LIA} vanishes if no curved filament vortex forms or if the vortex does not remain attached to the solid, slender translating body.

Take the dot product of fluid velocity \mathbf{V}_f and the $\text{grad } \phi_H$ term from (73):

$$\begin{aligned}\mathbf{V}_f \cdot \text{grad } \phi_H &= (\mathbf{V}_\infty + \mathbf{u} e^{-i\omega(t-t_0)} + \mathbf{u}_{LIA} e^{-i\omega(t-t_0)}) \cdot (\mathbf{u} + \mathbf{u}_{LIA}) \\ &= \mathbf{V}_\infty \cdot \mathbf{u} + \mathbf{V}_\infty \cdot \mathbf{u}_{LIA} \\ &\quad + (\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}_{LIA} + \mathbf{u}_{LIA} \cdot \mathbf{u} + \mathbf{u}_{LIA} \cdot \mathbf{u}_{LIA}) e^{-i\omega(t-t_0)}\end{aligned}\quad (74)$$

Assume the following relative magnitudes between velocity components:

a) Steady-perturbation velocity is small compared to freestream velocity:

$$|\mathbf{u}/\mathbf{V}_\infty| \ll 1 \quad (75)$$

b) Steady-perturbation velocity is small compared to induced velocity:

$$|\mathbf{u}/\mathbf{u}_{LIA}| \ll 1 \quad (76)$$

c) Induced velocity is not parallel to freestream velocity:

$$|\mathbf{V}_\infty \cdot \mathbf{u}_{LIA}| \ll \text{Max}\{|\mathbf{V}_\infty|^2, |\mathbf{u}_{LIA}|^2\} \quad (77)$$

The dot product of fluid velocity \mathbf{V}_f and $\text{grad } \phi_H$ given in (74) reduces as follows upon substitution of the assumptions given in (75), (76), and (77):

$$\begin{aligned}\mathbf{V}_f \cdot \text{grad } \phi_H &= |\mathbf{u}_{LIA}|^2 e^{-i\omega(t-t_0)} \\ &= G_{LIA}^2 \kappa^2 e^{-i\omega(t-t_0)}\end{aligned}\quad (78)$$

The gradient of the body force potential Λ_{BF} given in (68) can now be evaluate as follows:

$$\text{grad } \Lambda_{BF} = -\alpha_{BF} \phi_H \text{grad } \phi_H \quad (79)$$

Take the dot product of fluid velocity \mathbf{V}_f with the gradient expression given in (78) and (79):

$$\begin{aligned}\mathbf{V}_f \cdot \text{grad } \Lambda_{BF} &= -\alpha_{BF} G_{LIA}^2 \kappa^2 \phi_H e^{-i\omega(t-t_0)} \\ &= -\alpha_{BF} G_{LIA}^2 \kappa^2 \hat{\phi}\end{aligned}\quad (80)$$

4.5.7. Material Derivative of Body Force Potential

The time derivative of body force potential Λ_{BF} vanishes since the steady-perturbation velocity potential ϕ_H is not a function of time:

$$\begin{aligned}\frac{\partial \Lambda_{BF}}{\partial t} &= -\alpha_{BF} \phi_H \frac{\partial \phi_H}{\partial t} \\ &= 0\end{aligned}\quad (81)$$

The material derivative of body force potential Λ_{BF} can now be completely evaluated using the results of (80) and (81)

$$\begin{aligned}\frac{D\Lambda_{BF}}{Dt} &= \frac{\partial \Lambda_{BF}}{\partial t} + \mathbf{V}_f \cdot \text{grad } \Lambda_{BF} \\ &= -\alpha_{BF} G_{LIA}^2 \kappa^2 \hat{\phi}\end{aligned}\quad (82)$$

4.5.8. Derivatives of the Perturbed Velocity Potential

Differentiate the perturbed velocity potential $\hat{\phi}$ from (66) with respect to time for the case when the source is a simple harmonic oscillator with constant amplitude ϕ_H and a constant angular speed ω that does not depend on the amplitude:

$$\begin{aligned}\frac{\partial \hat{\phi}}{\partial t} &= -i\omega \phi_H e^{-i\omega(t-t_0)} \\ \frac{\partial^2 \hat{\phi}}{\partial t^2} &= -\omega^2 \phi_H e^{-i\omega(t-t_0)}\end{aligned}\quad (83)$$

Substitute the time derivatives of the perturbed velocity potential $\hat{\phi}$ from (83), the material derivative formula for the body force potential from (82) into the linearized wave equation (65), and divide out the common term $e^{-i\omega(t-t_0)}$ when finished:

$$\begin{aligned}&\frac{\phi_{H\theta\theta}}{\rho^2} (1 - M_\theta^2) + \phi_{H\rho\rho} + \phi_{Hzz} (1 - M_z^2) \\ &= -\frac{\omega^2}{c_\infty^2} \phi_H - \frac{2i\omega}{c_\infty} \left(M_\theta \frac{\phi_{H\theta}}{\rho} + M_z \phi_{Hz} \right) \\ &\quad + 2M_\theta M_z \frac{\phi_{H\theta z}}{\rho} - \frac{\phi_{H\rho}}{\rho} (1 + M_\theta^2) \\ &\quad - \frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \kappa^2 \phi_H\end{aligned}\quad (84)$$

4.5.9. Introducing a New Perturbed Velocity Potential

Create a new perturbation velocity potential $\phi(\rho, \theta, z)$, with units L^{2-a}/T , to replace the steady-perturbed velocity potential ϕ_H , with units L^2/T , where the coefficient a is unknown:

$$\phi_H(\rho, \theta, z) = \phi(\rho, \theta, z) \rho^a \quad (85)$$

$$\begin{aligned}\frac{\partial \phi_H}{\partial \rho} &= \left(\frac{\partial \phi}{\partial \rho} + a \frac{\phi}{\rho} \right) \rho^a \\ \frac{\partial^2 \phi_H}{\partial \rho^2} &= \left(\frac{\partial^2 \phi}{\partial \rho^2} + \frac{2a}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{a(a-1)}{\rho^2} \phi \right) \rho^a\end{aligned}\quad (86)$$

Substitute the formulas (85) and (86) into the wave

equation (84); replace $1 - M_\theta^2$ with term β_θ^2 and $1 - M_z^2$ with term β_z^2 ; and divide out the common term ρ^a , such that:

$$\begin{aligned}&\frac{\phi_{\theta\theta}}{\rho^2} \beta_\theta^2 + \phi_{\rho\rho} + \phi_{zz} \beta_z^2 + \frac{\phi_\rho}{\rho} (2a+1+M_\theta^2) \\ &\quad + \frac{\phi}{\rho^2} \left(a(a-1) + a(1+M_\theta^2) \right) = \\ &\quad - \frac{2i\omega}{c_\infty} \left(M_\theta \frac{\phi_\theta}{\rho} + M_z \phi_z \right) \\ &\quad + 2M_\theta M_z \frac{\phi_{\theta z}}{\rho} - \phi \left(\frac{\omega^2}{c_\infty^2} + \frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \kappa^2 \right)\end{aligned}\quad (87)$$

The ϕ_ρ/ρ expression in (87) can be reduced to a standard radial coordinate form by setting coefficient a equal to the following:

$$a \equiv -\frac{M_\theta^2}{2} \quad (88)$$

Substitute coefficient a from (88) into (87):

$$\begin{aligned}&\frac{\phi_{\theta\theta}}{\rho^2} \beta_\theta^2 + \phi_{\rho\rho} + \phi_{zz} \beta_z^2 + \frac{\phi_\rho}{\rho} - \frac{M_\theta^4}{4} \frac{\phi}{\rho^2} \\ &= -\frac{2i\omega}{c_\infty} \left(M_\theta \frac{\phi_\theta}{\rho} + M_z \phi_z \right) \\ &\quad + 2M_\theta M_z \frac{\phi_{\theta z}}{\rho} - \phi \left(\frac{\omega^2}{c_\infty^2} + \frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \kappa^2 \right)\end{aligned}\quad (89)$$

4.5.10. Change in Spatial and Temporal Coordinates

Define the longitudinal spatial variable s , with units L , and the transverse spatial variable b , with units L , for subsonic velocity conditions, where:

$$s = \frac{(\theta - \theta_0)}{\beta_\theta} \rho \quad (90)$$

$$b = \frac{(z - z_0)}{\beta_z} \quad (91)$$

The s variable represents an estimate of the circumferential arclength travelled when the azimuth angle changes by the amount $\theta - \theta_0$ and then rescaled by β_θ . The b variable represents an estimate of the transvers arclength travelled when the Z coordinate changes by the distance $z - z_0$ and then rescaled by β_z .

Take the following partial derivatives in terms of the s variable (90) using the chain-rule of differentiation:

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial s} \frac{\rho}{\beta_\theta} \quad (92)$$

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial s^2} \frac{\rho^2}{\beta_\theta^2} \quad (93)$$

Take the following partial derivatives in terms of the b variable (91) using the chain-rule of differentiation:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial b} \frac{1}{\beta_z} \quad (94)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial b^2} \frac{1}{\beta_z^2} \quad (95)$$

Substitute the change of variables and derivatives from (90) to (95) into the wave equation of (89)

$$\begin{aligned} & \phi_{ss} + \phi_{bb} + \phi_{\rho\rho} + \frac{\phi_\rho}{\rho} - \frac{M_\theta^4}{4} \frac{\phi}{\rho^2} \\ &= -\frac{2i\omega}{c_\infty} \left(\frac{M_\theta}{\beta_\theta} \phi_s + \frac{M_z}{\beta_z} \phi_b \right) \\ &+ 2 \frac{M_\theta}{\beta_\theta} \frac{M_z}{\beta_z} \phi_{bs} - \phi \left(\frac{\omega^2}{c_\infty^2} + \frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \kappa^2 \right) \end{aligned} \quad (96)$$

4.6. Separation-of-Variables

Consider the following separation-of-variables solution $\phi = R\Psi$ to the compressible, convected wave equation (96), where only the R term varies with the polar radius ρ :

$$\phi(s, \rho, b) \equiv R(\rho) \Psi(s, b) \quad (97)$$

The separation-of-variables term $\Psi(s, b)$ in (97) is called the wave function of the compressible, convected wave equation given in (96). Substitute the new transform (97) into the wave equation (96) and divide the resultant expression by the term $R\Psi$:

$$\begin{aligned} & \frac{1}{\Psi} \left\{ \Psi_{ss} + \Psi_{bb} + \left(\frac{\omega^2}{c_\infty^2} + \frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \kappa^2 \right) \Psi \right. \\ &+ \frac{2i\omega}{c_\infty} \left(\frac{M_\theta}{\beta_\theta} \Psi_s + \frac{M_z}{\beta_z} \Psi_b \right) \\ &\left. - 2 \frac{M_\theta}{\beta_\theta} \frac{M_z}{\beta_z} \Psi_{bs} \right\} \\ &= -\frac{1}{R} \left(R_{\rho\rho} + \frac{R_\rho}{\rho} - \frac{M_\theta^4}{4} \frac{R}{\rho^2} \right) \\ &\equiv \lambda_0 \end{aligned} \quad (98)$$

Term λ_0 , with units $1/L^2$, is an unknown constant.

Introduce the following new dimensionless space variable η to replace the polar radius ρ in the R bracketed expression when the coefficient λ_0 is greater than zero:

$$\eta = \rho \sqrt{\lambda_0} \quad (99)$$

$$\begin{aligned} \frac{\partial R}{\partial \rho} &= \frac{\partial R}{\partial \eta} \sqrt{\lambda_0} \\ \frac{\partial^2 R}{\partial \rho^2} &= \frac{\partial^2 R}{\partial \eta^2} \lambda_0 \end{aligned} \quad (100)$$

Substitute the above relationships (99) to (100) into the R bracketed expression in (98) after setting it equal to λ_0 :

$$\frac{1}{R} \left\{ \lambda_0 \frac{\partial^2 R}{\partial \eta^2} + \frac{\lambda_0}{\eta} \frac{\partial R}{\partial \eta} + \lambda_0 R - \nu^2 \lambda_0 \frac{R}{\eta^2} \right\} \equiv 0 \quad (101)$$

Term ν is defined by the following expression:

$$\nu^2 = \frac{1}{4} M_\theta^4 \quad (102)$$

Multiply the bracketed expression in (101) by the term $R\eta^2/\lambda_0$, such that:

$$\eta^2 R_{\eta\eta} + \eta R_\eta + R(\eta^2 - \nu^2) = 0 \quad (103)$$

The ordinary differential equation in (103), which is not a function of time, is called the *wave equation for the Bessel Laplacian* of order ν . It should be obvious from (102) that the order ν is a non-integer with a value always greater than zero for subsonic flow velocities. The general solution to (103) is:

$$R(\eta) = C_{pv} J_\nu(\eta) + C_{nv} J_{-\nu}(\eta) \quad (104)$$

Bessel functions of the first kind with positive valued orders ν behave [24] with $J_\nu(\eta) \rightarrow (\frac{1}{2}\eta)^\nu / \Gamma(\nu+1)$ as the argument $|\eta| \rightarrow 0$ with a fixed value for order ν when $\nu \neq -1, -2, -3, \dots$; and where $\Gamma(\cdot)$ is the gamma function. With $J_\nu(\eta) \rightarrow \cos(\eta - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) \sqrt{2/\sqrt{\pi\eta}}$ as the argument $|\eta| \rightarrow \infty$. Bessel functions of the first kind with negative valued orders behave with $|J_{-\nu}(\eta)| \rightarrow \infty$ as the argument $|\eta| \rightarrow 0$. Hence, the constant C_{nv} must vanish for physically meaningful solutions for finite valued induced velocities along the curved filament vortex:

$$R(\eta) = C_{pv} J_\nu(\eta) \quad (105)$$

Now return to the Ψ bracketed expression in (98) that was set equal to λ_0 . Multiply all terms by the wave function Ψ and rearrange terms, such that:

$$\begin{aligned} & i \left(\frac{2\omega}{c_\infty} \frac{M_z}{\beta_z} \right) \Psi_b + \Psi_{ss} + i \left(\frac{2\omega}{c_\infty} \frac{M_\theta}{\beta_\theta} \right) \Psi_s \\ &+ \left(\frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \right) \kappa^2 \Psi - \left(\lambda_0 - \frac{\omega^2}{c_\infty^2} \right) \Psi \\ &= - \left(\Psi_{bb} - 2 \frac{M_\theta}{\beta_\theta} \frac{M_z}{\beta_z} \Psi_{bs} \right) \end{aligned} \quad (106)$$

The left side of the expression given in (106) is in the form of the (0+2) focusing cubic *nonlinear Schrödinger* (NLS) equation. It will be shortly shown how the remaining two terms on the right side of the expression in (106) are to be properly treated.

4.7. Solving for the Wave Function Ψ

In the previous section, we solved the separation-of-variable term R from the convected wave

equation as a function of the polar radius. The remaining separation-of-variable term Ψ , called the wave function, will now be evaluated.

4.7.1. Introducing the Hasimoto transform

Consider the Hasimoto transform [13], [23], [25] for the complex valued wave function Ψ when the torsion of the filament vortices is not constant with arclength s :

$$\Psi(s, b) = \frac{\kappa}{\kappa_0} e^{i \int_{s'=s_0}^s (\tau - \tau_0) ds'} \quad (107)$$

Term τ with units of $1/L$ is the torsion of the filament vortex and τ_0 is a reference torsion value. For the special case of constant torsion $\tau = \tau_0$, the Hasimoto transform reduces to the following form:

$$\Psi(s, b) = \frac{\kappa}{\kappa_0} e^{i \tau_0 (s - s_0)} \quad (108)$$

It should be noted that both curvature $\kappa(s, b)$ and torsion $\tau(s, b)$ will be considered general functions of arclength s along the filament and the transverse distance b from the filament's centerline.

Let $\bar{\Psi}$ represent the complex conjugate of the wave function Ψ , such that from (107):

$$\bar{\Psi}(s, b) = \frac{\kappa}{\kappa_0} e^{-i \int_{s'=s_0}^s (\tau - \tau_0) ds'} \quad (109)$$

It then follows from (107) and (109) that:

$$\bar{\Psi} \Psi = \left(\frac{\kappa}{\kappa_0} \right)^2 \quad (110)$$

4.7.2. Differential and Riemannian geometry

The (0+2) NLS equation is formulated in differential geometry using Riemannian geometry with s -line and b -line coordinate curves on the $\Omega_n \equiv 0$ congruence surface with Riemannian metric [26]:

$$I_{sb} = ds^2 + 2F_{(I)} ds db + G_{(I)} db^2 \quad (111)$$

The metric coefficients $F_{(I)}$ and $G_{(I)}$ are given as $F_{(I)} \equiv 0$ and $G_{(I)} \equiv \kappa^2 / \kappa_0^2$ on the $\Omega_n \equiv 0$ surface. Function Ω_n is called [26] the abnormality of the vector N-field.

Two of the constraints for torsion and curvature functions on the $\Omega_n \equiv 0$ manifold are called the *Da Rios-Betchov* equations [25]:

$$\kappa_0 \kappa_b + \kappa \tau_s + 2 \tau \kappa_s = 0 \quad (112)$$

$$\kappa_0 \int_{s'=s_0}^s \tau_b ds' - \left(\frac{1}{\kappa} \kappa_{ss} - \tau^2 + \frac{\kappa^2}{2} \right) = B_0 \quad (113)$$

The B_0 coefficient is defined as $B_0 = \tau_0^2 - \frac{1}{4} \kappa_0^2 A_A$. Terms A_A and B_0 are integration constants.

4.7.3. Solution for Torsion Function

A solution for the torsion function $\tau(s, b)$ that satisfies both the LIA assumption, the Hasimoto transform, and Da Rios-Betchov equations on the $\Omega_n \equiv 0$ manifold surface is:

$$\tau(s, b) = \frac{\kappa_0}{2} \left(C_1 + \frac{\kappa_0^2}{\kappa^2} C_2 \right) \quad (114)$$

Coefficient C_2 is defined in terms of LIA based coefficients C_1 , D_1 , & H_1 for the filament vortex:

$$C_2 = -C_1^3 + C_1 H_1^2 + C_1 D_1 - 2 H_1 \quad (115)$$

Terms C_1 , D_1 , & H_1 are integration and boundary constants.

4.7.4. Conservation Expressions for Torsion, Curvature, and Wave Function

Differentiate the torsion solution (114) with respect to the s -line and then the b -line coordinate curves:

$$\begin{aligned} \tau_s &= -C_2 \frac{\kappa_0^3}{\kappa^3} \kappa_s \\ \tau_b &= -C_2 \frac{\kappa_0^3}{\kappa^3} \kappa_b \end{aligned} \quad (116)$$

Or, upon rearranging (116)

$$\begin{aligned} \kappa_s &= -\frac{1}{C_2} \frac{\kappa^3}{\kappa_0^3} \tau_s \\ \kappa_b &= -\frac{1}{C_2} \frac{\kappa^3}{\kappa_0^3} \tau_b \end{aligned} \quad (117)$$

Differentiate the Hasimoto transform for wave function Ψ in (107) with respect to the s -line and the b -line coordinate curves:

$$\begin{aligned} \Psi_s &= \left(\frac{\kappa_s}{\kappa} + i(\tau - \tau_0) \right) \Psi \\ \Psi_b &= \left(\frac{\kappa_b}{\kappa} + i \int_{s'=s_0}^s \frac{\partial(\tau - \tau_0)}{\partial b} ds' \right) \Psi \end{aligned} \quad (118)$$

Substitute the formulas for torsion τ from (114) and derivative τ_s from (116) into the first Da Rios-Betchov equation (112) to obtain the conservation expression:

$$\kappa_b + C_1 \kappa_s = 0 \quad (119)$$

Substitute the formulas for torsion τ from (114), derivatives κ_s and κ_b from (117) into the first Da Rios-Betchov equation (112) to obtain the conservation expression:

$$\tau_b + C_1 \tau_s = 0 \quad (120)$$

Finally, substitute derivatives κ_b from (119) and τ_b from (120) into the Ψ_b derivative of the Hasimoto transform given in the second line of (118); rearrange and replace the resultant terms using the Ψ_s derivative of the Hasimoto transform given in the first line of (118), such that:

$$\Psi_b + C_1 \Psi_s = 0 \quad (121)$$

The derivation for all three *conservation form* expressions given in (119), (120), & (121) for the curvature κ , torsion τ and wave function Ψ is apparently new for the (0+2) NLS equation.

4.7.5. Wave Function with Multiple Derivatives

Differentiate the wave function Ψ conservation form expression in (121) with respect to the b -line and then the s -line coordinate curves:

$$\begin{aligned} \Psi_{bb} &= -C_1 \Psi_{sb} \\ \Psi_{bs} &= -C_1 \Psi_{ss} \end{aligned} \quad (122)$$

Substitute derivative Ψ_{bs} from the second line of (122) back into the first line of (122), such that:

$$\Psi_{bb} = C_1^2 \Psi_{ss} \quad (123)$$

The Ψ_b derivative from (121), the Ψ_{bs} derivative from the second line of (122), and the Ψ_{bb} derivative from (123) can now be used to replace terms in the (0+2) NLS equation (106), such that:

$$\begin{aligned} &\Psi_{ss} \left(1 + C_1 2 \frac{M_\theta}{\beta_\theta} \frac{M_z}{\beta_z} + C_1^2 \right) \\ &+ i \Psi_s \left(\frac{2\omega}{c_\infty} \frac{M_\theta}{\beta_\theta} - C_1 \frac{2\omega}{c_\infty} \frac{M_z}{\beta_z} \right) \\ &+ \Psi \left(\frac{\alpha_{BF}}{c_\infty^2} G_{LIA}^2 \right) \kappa^2 + \Psi \left(\frac{\omega^2}{c_\infty^2} - \lambda_0 \right) = 0 \end{aligned} \quad (124)$$

4.7.6. A New Compressibility Constant

Define a compressibility coefficient C_{comp} , such that:

$$C_{comp} = 1 + 2C_1 \frac{M_\theta}{\beta_\theta} \frac{M_z}{\beta_z} + C_1^2 \quad (125)$$

And define the proportionality coefficient α_{BF} from the body force term, such that:

$$\frac{\alpha_{BF}}{c_\infty^2} \frac{G_{LIA}^2}{C_{comp}} = \frac{1}{2} \quad (126)$$

Substitute the C_{comp} coefficient from (125) and the α_{BF} constant from (126) into the (0+2) NLS equation (124), such that:

$$\begin{aligned} &\Psi_{ss} + i \Psi_s \frac{\left(\frac{2\omega}{c_\infty} \frac{M_\theta}{\beta_\theta} - C_1 \frac{2\omega}{c_\infty} \frac{M_z}{\beta_z} \right)}{C_{comp}} \\ &+ \Psi \frac{\left(\frac{\omega^2}{c_\infty^2} - \lambda_0 \right)}{C_{comp}} = 0 \end{aligned} \quad (127)$$

4.7.7. Standard form of the (0+2) NLS

The standard form of the (0+2) NLS is as follows:

$$\begin{aligned} &i \kappa_0 \Psi_b + \Psi_{ss} + i \kappa_0 U_0 \Psi_s \\ &+ \frac{1}{2} \kappa^2 \Psi - \frac{1}{4} \kappa_0^2 A_A \Psi = 0 \end{aligned} \quad (128)$$

The dimensionless term U_0 is called the *pseudo-speed coefficient*. It is expressed in terms of the reference torsion τ_0 and reference curvature κ_0 values:

$$U_0 = \frac{2\tau_0}{\kappa_0} \quad (129)$$

4.7.8. Standard form of the (0+1) NLS

After eliminating the Ψ_b derivative in (128) using the conservation form expression (121), the standard form of the (0+1) NLS is written as follows:

$$\begin{aligned} &\Psi_{ss} + i \kappa_0 (U_0 - C_1) \Psi_s \\ &+ \frac{1}{2} \kappa^2 \Psi - \frac{1}{4} \kappa_0^2 A_A \Psi = 0 \end{aligned} \quad (130)$$

4.7.9. Association of Variables

Upon comparing the compressible (0+1) NLS in (124) with the standard form of the (0+1) NLS in (130), one can write the following three associations between variables:

$$2 \frac{\omega}{c_\infty} \frac{M_\theta}{\beta_\theta} \frac{1}{C_{comp}} \equiv \kappa_0 U_0 \quad (131)$$

$$2 \frac{\omega}{c_\infty} \frac{M_z}{\beta_z} \frac{1}{C_{comp}} \equiv \kappa_0 \quad (132)$$

$$\left(\frac{\omega^2}{c_\infty^2} - \lambda_0 \right) \frac{1}{C_{comp}} \equiv -\frac{1}{4} \kappa_0^2 A_A \quad (133)$$

Substitute the formula for pseudo-speed U_0 from (129) into the first association (131) and solve for the reference torsion τ_0 , such that:

$$\tau_0 = \frac{\omega}{c_\infty} \frac{M_\theta}{\beta_\theta} \frac{1}{C_{comp}} \quad (134)$$

Solve for the reference curvature κ_0 from the second association (132), such that:

$$\kappa_0 = 2 \frac{\omega}{c_\infty} \frac{M_z}{\beta_z} \frac{1}{C_{comp}} \quad (135)$$

The separation-of-variables coefficient λ_0 is evaluated from the third association expression (133), such that:

$$\lambda_0 = \frac{\omega^2}{c_\infty^2} + \frac{1}{4} \kappa_0^2 A_A C_{comp} \quad (136)$$

Take the ratio of τ_0/κ_0 from expressions (134) and (135), such that:

$$\begin{aligned} 2 \frac{\tau_0}{\kappa_0} &= \frac{M_\theta}{\beta_\theta} \frac{\beta_z}{M_z} \\ &= U_0 \end{aligned} \quad (137)$$

The vortex curve reference coefficients κ_0 , τ_0 , and separation-of-variables coefficient λ_0 have been expressed in terms of LIA based coefficients C_1 , D_1 , H_1 and aerodynamic source terms ω , M_θ , M_z , and c_∞ .

4.8. Solution to the NLS

4.8.1. Differential Equation for Curvature

Substitute for the derivative τ_b from the second conservation expression (120) into the integral term in the second Da Rios-Betchov equation (113) and then substitute in the LIA based torsion formula from (114), such that:

$$\begin{aligned} -C_1 \frac{\kappa_0^2}{2} \left(C_1 + C_2 \frac{\kappa_0^2}{\kappa^2} \right) + C_1 \kappa_0 \tau_0 - \frac{\kappa_{ss}}{\kappa} \\ + \frac{\kappa_0^2}{4} \left(C_1 + C_2 \frac{\kappa_0^2}{\kappa^2} \right)^2 - \frac{\kappa^2}{2} = B_0 \end{aligned} \quad (138)$$

Rearrange terms in (138) and multiply the resultant expression by $-\kappa$, such that:

$$\begin{aligned} \kappa_{ss} + \frac{\kappa^3}{2} - \kappa \left(C_1 \tau_0 \kappa_0 - C_1^2 \frac{\kappa_0^2}{4} - B_0 \right) \\ - C_2^2 \frac{\kappa_0^6}{4 \kappa^3} = 0 \end{aligned} \quad (139)$$

4.8.2. Cubic Polynomial Equation for the Auxiliary Function

Define the dimensionless auxiliary function F as $F = \kappa^2/\kappa_0^2$. The following cubic polynomial can be obtained by substitution of the auxiliary F into (139) and integration:

$$\begin{aligned} (F_s)^2 \\ + \kappa_0^2 \left\{ F^3 - F^2 (2H_1^2 - 3C_1^2 + 2D_1) + F B_3 + C_2^2 \right\} = 0 \end{aligned} \quad (140)$$

The integration constant B_3 in (140) is given by:

$$\begin{aligned} B_3 = H_1^4 - 4C_1^2 H_1^2 + 3C_1^4 + 2D_1 H_1^2 \\ - 4C_1^2 D_1 + D_1^2 + 4C_1 H_1 - 4 \end{aligned} \quad (141)$$

4.8.3. Elliptic function solution for the (0+2) cubic NLS

The solution to the nonlinear ordinary differential equation in (140) is the Jacobian elliptic sine function sn , such that:

$$\begin{aligned} F(s, b) &= \left(\frac{\kappa}{\kappa_0} \right)^2 \\ &= F_1 - (F_1 - F_2) sn^2[u_e - u_{e0}, k_e] \end{aligned} \quad (142)$$

Dimensionless term k_e is the Jacobi modulus; dimensionless term u_e is the Jacobian elliptic angle; and F_1 , F_2 , & F_3 are the three cubic roots when solving the nonlinear ODE given in (140) in terms of the coefficients κ_0 , τ_0 , C_1 , C_2 , and B_0 . Term C_1 is Kida's [27] sliding speed coefficient; term D_1 is a constant of integration for elevation of the resultant vortex filament; and H_1 is Kida's [27] translation speed coefficient.

An extensive discussion is given in Ch. 6 of [28] for additional constraints imposed on the C_1 , D_1 , & H_1 parameters for curve closure, periodicity, and knotting.

4.8.4. Final Solution to Case 2

The final solution of Case 2 for the velocity potential Φ_f of compressible flow in a fixed-to-body reference frame can be written in the following form:

$$\begin{aligned} \Phi_f(x, t) &= \Phi_\infty \\ &+ \frac{C_{cv}}{\rho_c^v} J_v \left(\rho_c \sqrt{\lambda_0} \right) \frac{\kappa(s_c, b)}{\kappa_0} e^{-i \int_{s_0}^{s_c} (\tau_c - \tau_{c0}) ds'_e - i \omega (t_c - t_{c0})} \end{aligned} \quad (143)$$

A subscript c is listed with coordinates ρ_c , s_c , t_c , and torsion τ_c to remind us that they are being used under the assumption of a reference frame subjected to compressible flow conditions. Term v in (143) is given by $v = \frac{1}{2} M_\theta^2$; $J_v(\cdot)$ is the Bessel function of the first kind, order v ; λ_0 is the separation-of-variables integration constant evaluated in (136); arclength coordinate s_c is given in (90); transverse arclength coordinate b is given in (91); torsion function τ is given in (114); curvature function κ is given in (142); reference torsion τ_0 is given in (134); and reference curvature κ_0 is given in (135).

4.9. Summary of Case 2

Authors since 1927 have solved the logarithmic density version of the (0+1) NLS using the Madelung transform and quantum hydrodynamics. However, a derivation of the (0+1) & (0+2) focusing cubic NLS equation from the Navier-Stokes equation using the Hasimoto transform and an aerodynamic body force is apparently new. The derivation presented here gives the exact expression for the

steady form of the cubic NLS that matches the curvature and torsion constraints derived from Riemann geometry for curved surfaces.

Case 2 involves the transient, 3D, nonlinear, convected wave equation (64) in a fixed-to-body reference frame for a compressible fluid with constant coefficients expressed in cylindrical-polar coordinates for a source that is both rotating and translating. There are three reasons for presenting this problem: the first is to show that the (0+2) cubic NLS is embedded within the PDE of the original wave equation (after converting the partial derivatives Ψ_{bs} and Ψ_{bb}); the second is to show that the (0+1) cubic NLS is embedded within the PDE of the wave equation; and the third is to show that there is an analytically solution (143) to the wave equation (64). The conversion of the partial derivatives Ψ_b , Ψ_{bs} , and Ψ_{bb} is based on the discovery of three new *conservation form* expressions (119), (120), & (121) for the curvature κ , torsion τ and wave function Ψ .

5. Discussion

This article examines the derivation and solution of unsteady convected waves for compressible fluids using an analogous problem from aerodynamics. The take-away conclusion is that one is able to scale from the smallest-to-largest and slowest-to-fastest processes in the universe using Newton's classical laws for fluids based on a system of absolute space and time coordinates. This is made possible by recognizing and retaining the effects of fluid compressibility that are intrinsically associated with the cross-derivative terms between space and time. The retention of the cross-derivatives makes the resultant convected nonlinear wave equations more difficult to solve. In compensation, it eliminates the artificial contradictions and mysticisms imposed with using elastic space-time coordinates when one insists on assuming incompressible flow conditions for in vacuo problems.

This paper and the 2017 paper [1] don't reject the scientific data obtained from 100 years of testing special relativity. What is presented here is a radically different physical interpretation of prior test results. Instead of interpreting the special relativity tests as proof-of-errors in ignoring the elasticity of relative space-time coordinates, it interprets prior tests as showing the error in dropping the nonlinear cross-derivatives from the convected wave equation that uses an absolute space and time coordinate system.

It might seem to be argumentative to reject the classical understanding on why speed affects the measurement of distance but it actual goes to the heart of science: progress is made by challenging theories that one takes for granted and replacing it with an improved version, with each iteration bringing humanity closer to the truth. The unmasking of the cubic NLS and special relativity relationships within the compressible, convected wave equation for laminar flow offers proof in the value of challenging what one thought was

totally understood for more than a century.

6. Conclusions

For the first time, an exact derivation of the (0+2) cubic NLS equation is obtained after combining the 3D Navier-Stokes equation, the Hasimoto transform, and an aerodynamic body force for induced velocity. Authors have previously derived a special logarithmic density version of the NLS using the Madelung transform and quantum hydrodynamics. However, the derivation given here results in an exact expression for the steady form of the (0+2) cubic NLS that matches the curvature and torsion constraints derived from Riemann geometry for curved surfaces. In addition, it is shown how special relativity expressions are obtainable within the steady-rotating source problem of the convected wave equation when written in cylindrical-polar coordinates and a non-inertial fixed-to-body reference frame.

Nomenclature

α_{BF}	Proportionality constant of body-force; $1/L^2$
β_θ	Circumferential Mach coefficient, $\beta_\theta = \sqrt{1 - M_\theta^2}$
β_z	Longitudinal Mach coefficient, $\beta_z = \sqrt{1 - M_z^2}$
ε	Transformed coordinate time, $\varepsilon = (t - t_0)V_{\infty\theta}$; L
κ	Curvature of vortex centerline; $1/L$
κ_0	Reference curvature for vortex centerline; $1/L$
λ_0	Separation-of-variables integration constant
Λ_{BF}	Potential of body force acting on fluid; L^2/T^2
ν	Order of Bessel function of the first kind
ϕ	Disturbed velocity potential $\{s, \rho, z, \varepsilon\}$ for Case 1; L^{2-a}/T
ϕ_H	Steady-disturbed velocity potential $\{\rho, \theta, z\}$ for Case 2; L^2/T
$\hat{\phi}$	Disturbed velocity potential $\{\rho, \theta, z, t\}$; L^2/T
Φ_∞	Velocity potential of undisturbed (freestream) fluid; L^2/T
Φ_f	Total velocity potential of perturbed fluid, $V_f = \text{grad } \Phi_f$; L^2/T
Ψ	Wave function $\{s, b\}$ solution for Case 2
$\bar{\Psi}$	Complex conjugate of wave function
ω	Angular rate of harmonic-oscillating source; radians/ T
ρ	Coordinate polar radius measured from central axis; L
τ	Torsion of vortex centerline; $1/L$
τ_0	Reference torsion for vortex centerline; $1/L$

a Polar-radius exponent coefficient
 b Transverse arclength from vortex centerline; L
 c_∞ Characteristic speed of undisturbed (freestream) fluid;
 L/T
 B_0 Integration constant used in PDE of cubic NLS; $1/L^2$
 B_3 Integration constant used in PDE of cubic NLS
 C_{nv} Integration coefficients for separation-of-variables
 term $R(\rho)$
 C_{comp} Compressibility coefficient used in Case 2
 C_1 Kida's [27] sliding speed coefficient
 C_2 Coefficient expressed in terms of coefficients C_1 ,
 D_1 , H_1
 D_1 A constant of integration for elevation of the resultant
 vortex filament
 H_1 Kida's [27] translation speed coefficient
 F Auxiliary function $\{s, b\}$ for NLS solution
 F_j Cubic roots to the auxiliary function F
 F_{BF} Body force per unit mass acting on fluid; L/T^2
 H_{BF} Bernoulli function of disturbed fluid; L^2/T^2
 G_{LIA} Coefficient of local induction; L^2/T
 $J_v(\cdot)$ Bessel function of the first kind, order v
 k_e Jacobi modulus
 M_f Total Mach number of a compressible fluid
 M_θ Circumferential Mach number component of fluid
 M_z Longitudinal Mach number component of fluid
 p_f Total pressure of disturbed fluid; $M/(LT^2)$
 R Separation-of-variable term $\{\rho\}$
 S Separation-of-variable term $\{s\}$ used in Case 1
 s Circumferential arclength about vortex centerline used
 in Case 1, $s = (\theta - \theta_0)\rho$; L
 s Circumferential arclength about vortex centerline used
 in Case 2; L
 T Separation-of-variable term $\{\varepsilon\}$ used in Case 1
 t_0 Reference coordinate time; T
 t Coordinate time elapsed since reference time t_0 ; T
 u_e Jacobian elliptic angles
 \mathbf{u} Steady-perturbation velocity $\{\rho, \theta, z\}$ due to presence
 of a source; L/T
 \mathbf{u}_{LIA} Steady-induced velocity $\{\rho, \theta, z\}$ generated by
 curved vortex; L/T
 U_0 Pseudo-speed coefficient used in Case 2
 \mathbf{v} Perturbation velocity $\{\rho, \theta, z, t\}$ due to presence of a
 source; L/T

\mathbf{v}_{LIA} Induced velocity $\{\rho, \theta, z, t\}$ generated by curved
 vortex; L/T
 \mathbf{V}_f Relative velocity between source and fluid initially at
 rest; L/T
 $V_{f\theta}$ Circumferential velocity component of
 compressible fluid; L/T
 V_{fz} Longitudinal velocity component of compressible
 fluid; L/T
 X Separation-of-variable term $\{s, z, \varepsilon\}$ used in Case 1
 Z Separation-of-variable term $\{z\}$ used in Case 1

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