

Dirac Equation and Electron Internal Structure

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Abstract Real waves presentation is applied to Dirac equation. It is then used to show a Klein Gordon wave functions solution. These equations resemble Maxwell's equations for the electromagnetic field. Assuming a non-point-like charged fermion, with an internal current distribution and an internal vector potential, the resulting Maxwell equations for the internal vector potentials of the charged fermion resembles Dirac equation in their Klein-Gordon form. This solution is to the best knowledge of the author, the first time that a connection is made between Dirac's fermion field and Maxwell's electromagnetic field.

Keywords Dirac equation, Real wave functions, Maxwell equation, Finite size Fermions, Internal Electromagnetic fields

1. Introduction

The relativistic equations by Dirac [1] and by Klein-Gordon (KG) [2] use complex wave functions. In a recent work [3], a formulation of quantum mechanics without use of complex numbers is described and was shown to be equivalent to the complex formulation.

Using real wave functions, the Dirac equation is presented as 4 real wave functions, solving 4 KG equations of massive point-like particles. These 4 solutions represent 4 states of a fermion (for instance an electron and a positron, with spin-up or spin down).

In this work, internal structure of the charged finite size fermion is examined.

Based on Maxwell's equations, a 4-vector *internal* electromagnetic field A^μ of the charged fermion is assumed. The solution gives 4 non-homogeneous KG equations with charge and current density.

Under certain assumptions on the relationship between the current density vector and the electromagnetic field vector, the Maxwell electromagnetic equations for the fermion internal vector field, are shown to have the same form as the Dirac equations for the fermion Dirac fields.

This solution assumes a mirror symmetry of the internal electromagnetic fields of the fermion.

2. Dirac Equation with Real Wave Functions

If one accepts the claim that a single non-relativistic Schrödinger equation is a description of a single massive particle, made up of two coupled strings, then one may project from this on the relativistic Dirac equation, describing a free Fermion of mass m :

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0 \quad (1)$$

One may assume, as before, separate the complex wave function Ψ into its real and imaginary parts

$$\Psi = \begin{pmatrix} \psi_r \\ i\psi_i \end{pmatrix} \quad (2)$$

and by using them in the Dirac equation, one will get two separate equations where imaginary and real parts have been separated.

This same procedure can be applied to any quantum field Lagrangian, resulting (at the cost of doubling the number of equations) in separation of the equations to real wave functions.

These wave functions may be interpreted as coupled strings.

In Dirac equation [43], the four γ matrices are:

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \quad (3)$$

$$\gamma^1 = \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix} \quad (4)$$

$$\gamma^2 = \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix} \quad (5)$$

$$\gamma^3 = \begin{bmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{bmatrix} \quad (6)$$

Where I_2 is the 2x2 unit matrix and $\sigma_x, \sigma_y, \sigma_z$ the Pauli 2x2 matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (7)$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (8)$$

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Published online at <http://journal.sapub.org/ijtmp>

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$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (9)$$

Of these matrices only σ_y , has imaginary components and one will replace it by writing

$$\sigma_y = i\sigma'_y$$

Where $\sigma'_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and therefore

$$\gamma^2 = i \begin{bmatrix} 0 & \sigma'_y \\ -\sigma'_y & 0 \end{bmatrix} \quad (10)$$

Decomposing Ψ into $\Psi_r + i\Psi_i$ where Ψ_r and Ψ_i are real, the Dirac equation becomes

$$(i\hbar\{\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z + i\gamma'^2\partial_y\} - mc)\begin{pmatrix} \Psi_r \\ i\Psi_i \end{pmatrix} = 0 \quad (11)$$

Namely, two separate real numbers equations:

$$(\hbar\gamma'^2\partial_y + mc)\Psi_r = 0 \quad (12)$$

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z)\Psi_i = 0 \quad (13)$$

One may notice that only the real part wave function depends on mass m and Planck constant \hbar . It is also time independent.

Using Pauli matrices in the gamma representation one obtains:

$$(\sigma_x\partial_x + \sigma_z\partial_z)\Psi_i = -\frac{mc}{\hbar}\Psi_r \quad (14)$$

$$(\sigma_x\partial_x + \sigma_z\partial_z)\Psi_r = -\frac{mc}{\hbar}\Psi_i \quad (15)$$

$$\partial_t\Psi_i + \sigma'_y\partial_y\Psi_r = 0 \quad (16)$$

$$\partial_t\Psi_r - \sigma'_y\partial_y\Psi_i = 0 \quad (17)$$

We see from these equations:

1. The real and imaginary components are coupled.
2. Both real and imaginary parts have 2 components each

Since both Ψ_r and Ψ_i are made of 2 real components each, there are 4 components altogether.

$$\Psi_r = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} \text{ and } \Psi_i = \begin{pmatrix} \Psi_C \\ \Psi_D \end{pmatrix}$$

Eqs. 23-26 can be then formulated as follows:

$$\partial_t\Psi_A + \partial_y\Psi_D = 0 \quad (18)$$

$$\partial_t\Psi_B - \partial_y\Psi_C = 0 \quad (19)$$

$$\partial_t\Psi_C - \partial_y\Psi_B = 0 \quad (20)$$

$$\partial_t\Psi_D + \partial_y\Psi_A = 0 \quad (21)$$

$$\partial_x\Psi_D - \partial_z\Psi_C = -\frac{mc}{\hbar}\Psi_A \quad (22)$$

$$\partial_x\Psi_C - \partial_z\Psi_D = -\frac{mc}{\hbar}\Psi_B \quad (23)$$

$$\partial_x\Psi_B + \partial_z\Psi_A = -\frac{mc}{\hbar}\Psi_C \quad (24)$$

$$\partial_x\Psi_A - \partial_z\Psi_B = -\frac{mc}{\hbar}\Psi_D \quad (25)$$

These equations hint to some connections between the possible states of the fermion.

Applying ∂_x to Eqs. 31-34 results in

$$\text{for } i = A, B, C, D \quad (\partial_x^2 + \partial_z^2)\Psi_i = \left(\frac{mc}{\hbar}\right)^2 \Psi_i \quad (26)$$

There are apparently 4 constituents of the Dirac particle. The above two groups of differential equations show the coupling that exists between the constituents.

There is a clear asymmetry in the Dirac equation, as there is a difference between the y coordinate and the x - z coordinates. This is in contradiction with the expected isotropic spherical symmetry of the Fermions and the need of the results to be independent of the coordinate system.

By applying second time derivative, Eqs. 28-31 can be written as

$$(\partial_t^2 + \partial_y^2)\Psi_i = 0 \text{ for } i = A, C \quad (27)$$

$$(\partial_t^2 - \partial_y^2)\Psi_i = 0 \text{ for } i = B, D \quad (28)$$

Eqs. 35, 36, 37 can be combined by subtraction/addition to give:

$$\left(\frac{\partial^2}{c^2\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\Psi_i = 0 \text{ for } i = A, C \quad (29)$$

$$\left(\frac{\partial^2}{c^2\partial t^2} + \nabla^2 - \left(\frac{mc}{\hbar}\right)^2\right)\Psi_i = 0 \text{ for } i = B, D \quad (30)$$

We thus see, that Dirac equation becomes 4 real waves equations. Each is a Klein-Gordon equation. Since the solutions to the Klein-Gordon [3] equations are well known, we can conclude and say, that the Dirac relativistic Fermion is actually a 4 Fermions equation. These 4 Fermions are of same mass m .

$$\frac{\partial^2\Psi_A}{c^2\partial t^2} - \nabla^2\Psi_A = -\left(\frac{mc}{\hbar}\right)^2\Psi_A \quad (31)$$

$$\frac{\partial^2\Psi_C}{c^2\partial t^2} - \nabla^2\Psi_C = -\left(\frac{mc}{\hbar}\right)^2\Psi_C \quad (32)$$

$$\frac{\partial^2\Psi_B}{c^2\partial t^2} + \nabla^2\Psi_B = +\left(\frac{mc}{\hbar}\right)^2\Psi_B \quad (33)$$

$$\frac{\partial^2\Psi_D}{c^2\partial t^2} + \nabla^2\Psi_D = +\left(\frac{mc}{\hbar}\right)^2\Psi_D \quad (34)$$

If now $\tilde{\Psi}_B$ and $\tilde{\Psi}_D$, are the mirror functions of Ψ_B and Ψ_D , respectively, then $\nabla^2\tilde{\Psi}_B = -\nabla^2\Psi_B$, and $\nabla^2\tilde{\Psi}_D = -\nabla^2\Psi_D$. Therefore

$$\frac{\partial^2\Psi_A}{c^2\partial t^2} - \nabla^2\Psi_A = -\left(\frac{mc}{\hbar}\right)^2\Psi_A \quad (35)$$

$$\frac{\partial^2\Psi_C}{c^2\partial t^2} - \nabla^2\Psi_C = -\left(\frac{mc}{\hbar}\right)^2\Psi_C \quad (36)$$

$$\frac{\partial^2\tilde{\Psi}_B}{c^2\partial t^2} - \nabla^2\tilde{\Psi}_B = +\left(\frac{mc}{\hbar}\right)^2\tilde{\Psi}_B \quad (37)$$

$$\frac{\partial^2\tilde{\Psi}_D}{c^2\partial t^2} - \nabla^2\tilde{\Psi}_D = +\left(\frac{mc}{\hbar}\right)^2\tilde{\Psi}_D \quad (38)$$

In other words, we have four possible KG solutions. Two solutions (A, C) with positive energy term and two solutions (B, D) with negative energy term. These equations show that the four states can exist independently of each other.

3. Maxwell's Equations

Maxwell's equations [4] and the Lorentz force law (see Appendix)) are extraordinarily successful at explaining and predicting a variety of phenomena; however they are not

exact, but a classical limit of quantum electrodynamics (QED) [5].

Some observed electromagnetic phenomena are incompatible with Maxwell's equations. These include photon-photon scattering [6] and many other phenomena related to photons or virtual photons, "nonclassical light" and quantum entanglement [7, 8] of electromagnetic fields.

4. Similarity of Dirac's with Maxwell's Equations

Consider the Dirac particle (Fermion) to be made of four constituents (fields), $\psi_A, \psi_B, \psi_C, \psi_D$. Each acts according to the KG equations, but they also show some mutual interactions, as can be seen by observing Eqs. 27-34.

Since the spatial derivative of a field is a force, one may consider a vector equilibrium interpretation for each of equations 32-35.

Suppose we define two 3 components vector fields

$$\vec{E}_1 = (\psi_C, \varphi_1, \psi_D) \quad (39)$$

$$\vec{B}_1 = \frac{mc}{h} \int dt (\varphi_2, \psi_A, \varphi_3) \quad (40)$$

With $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ some arbitrary vector field.

The fields \vec{E}_1 and \vec{B}_1 , satisfy:

$$\nabla \times \vec{E}_1 = -\frac{\partial}{\partial t} \vec{B}_1 \quad (41)$$

for the y component.

Same relationships exist for

$$\vec{E}_2 = (\psi_A, \varphi_1, \psi_B) \quad (42)$$

$$\vec{B}_2 = \frac{mc}{h} \int dt (\varphi_2, \psi_C, \varphi_3) \quad (43)$$

Leading to

$$\nabla \times \vec{E}_2 = -\frac{\partial}{\partial t} \vec{B}_2 \quad (44)$$

Again, for the y-component.

This may hint to some internal structure of a Dirac fermion, being made of some interacting electromagnetic fields.

Let us look at the electromagnetic field vector

$$A^\mu = (\phi, \vec{A})$$

It satisfies Maxwell's equations (see Appendix):

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0} \quad (45)$$

$$\frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J} \quad (46)$$

Assume the Dirac charged fermion has an internal current density j^μ and hence an internal vector potential A^μ .

This particle has an internal electric charge distribution ρ , and creates an internal current \vec{J} .

Because of current conservation and the definition of the vector potentials one has

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0 \quad (47)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \quad (48)$$

Suppose now that the current density is proportional to the vector potential:

$$\rho = k_1 \phi \quad (49)$$

$$\vec{J} = k_2 \vec{A} \quad (50)$$

(which, for example, is the case of the vector potential and current density for a current in a ring).

Then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = k_1 \frac{\partial \phi}{\partial t} + k_2 \nabla \cdot \vec{A} = k_2 \left(\frac{k_1}{k_2} \frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} \right) = 0 \quad (51)$$

Provided that $k_1/k_2 = 1/c^2$, then if Eq. 47 is true, then so is Eq. 48 and vice versa.

And the above conservation equations are always satisfied.

Then, from Eqs. 45, 46:

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi = k_1 \phi \quad (52)$$

$$\frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla^2 \vec{A} = k_2 \vec{A} \quad (53)$$

and $k_1/k_2 = \epsilon_0 \mu_0 = 1/c^2$

Assume next, that

$$\rho = \epsilon_0 \left(\frac{mc}{h} \right)^2 \phi \quad (54)$$

$$\vec{J} = \frac{1}{\mu_0} \left(\frac{mc}{h} \right)^2 \vec{A} \quad (55)$$

Then, when inserted in Eqs. 52, 53, they have the same form as the Dirac 4 entities Eqs. 35-38.

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi = \left(\frac{mc}{h} \right)^2 \phi \quad (56)$$

$$\frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla^2 \vec{A} = \left(\frac{mc}{h} \right)^2 \vec{A} \quad (57)$$

In other words, the four Dirac fields $\psi_A, \tilde{\psi}_B, \psi_C, \tilde{\psi}_D$ are the electromagnetic vector potential fields, ϕ, A_x, A_y, A_z provided that the current densities ρ, J_x, J_y, J_z are related to the electromagnetic fields ϕ, A_x, A_y, A_z by Eqs. 54, 55 through the constant $\left(\frac{mc}{h} \right)^2$.

So, if one identifies the four Dirac fields $(\psi_A, \tilde{\psi}_B, \psi_C, \tilde{\psi}_D)$ with the vector potential (ϕ, A_x, A_y, A_z) , Eqs. 35-38, become identical with Eqs. 56-57.

Note though, that in order to keep with the signs as required in Eqs 35 - 38, the currents \tilde{J}_x and \tilde{J}_z must be symmetric with J_x and J_z about their spatial coordinates, so that their spatial derivatives change signs accordingly.

There are no external electromagnetic fields, so these are confined electromagnetic fields which create the fermion and give it its mass m.

5. Conclusions

Assuming a charged fermion is not a point like particle, its internal charge dynamics should have a vector potential. Under certain assumptions on the proportionality of the vector potential to the current density, the Dirac equation wave functions are similar to the internal electromagnetic

vector potential.

The existence of a finite size particle with charge and mass may require some yet unexplained internal interaction force, to overcome the repulsion of the charge from the center of mass.

Appendix: Maxwell Equations

	Homogeneous equations	Inhomogeneous equations
Fields 3D Euclidean space + time	$\nabla \cdot \vec{B} = 0$ $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$	
Potentials (Lorenz gauge) 3D Euclidean space + time	$\vec{B} = \nabla \times \vec{A}$ $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$ $\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$	$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0}$ $\frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J}$
Relativistic representation		$\partial_\mu F^{\mu\nu} = J^\nu$ $F^{\mu\nu} \equiv \partial_\mu A^\nu - \partial_\nu A^\mu$

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