

# Fermions in d Dimensions

Doron Kwiatt

17/2 Erez Street, Mazkeret Batyia, Israel

**Abstract** Introducing d dimensional space, with NxN Lorentz invariant gamma matrices, the connection between N and d is  $N=2^{d/2}$  and since N is an integer, d must be an even number. Thus, the only candidates for a higher dimension universe for fermions may be SO(8), SO(16) etc. SO(7), SO(11) etc. are ruled out in case of fermions. However, it is shown that  $d>4$  is impossible since already for  $d=6$ , the representation is reducible. Thus, our universe, at least as far as Fermions goes, must be a (3+1) 4-dimensional.

**Keywords** Dirac Gamma Matrices, d dimensional universe, Clifford

## 1. Introduction

In spite of the fact that the universe is well described by four-dimensional spacetime, there are several reasons why physicists consider theories in other dimensions [5-15]. In some cases, by modeling spacetime in a different number of dimensions, a theory becomes more mathematically tractable, and one can perform calculations and gain general insights more easily. If there exist scenarios in which there could actually be more than four dimensions of spacetime, they have so far managed to escape detection.

One notable feature of string theories is that these theories require extra dimensions of spacetime for their mathematical consistency. In bosonic string theory, spacetime is 26-dimensional, while in superstring theory it is 10-dimensional, and in M-theory it is 11-dimensional. In order to describe real physical phenomena using string theory, one must therefore imagine scenarios in which these extra dimensions would not be observed in experiments.

When considering the Fermionic universe, which includes all the known massive particles, we rely on Dirac equation [1] and thereof on the so-called gamma matrices (denoted  $\gamma$ ).

Starting with the Dirac equation with N-dimensional spinors, we accept the fact that the gamma matrices, irrespective of the number of dimensions of our universe, they always must conform with the Clifford algebra [3, 4, 7] restriction.

Following this, it is proven in this work, that there must exist a connection between the dimensionality of these matrices (and hence of the spinors), and the number of dimensions d of our universe.

Unlike previous approaches, here there is no need to separate between odd and even dimensions. This leads us to the conclusion that  $N^2=2^d$ . Therefore, d must be even, and N must be an integer power of 2.

By a unique construction procedure, we use the gamma matrices to construct a relevant group of matrices. These groups presentations become reducible for any  $d > 4$ .

Thus, only  $d = 4$  will be allowed, and our fermionic world cannot be other than a 3+1 universe.

Physical problems cannot rely on imaginary numbers, except for the purpose of mathematical comfort. The introduction of imaginary and complex numbers is nothing but a mathematical convenience for shortening and unifying equations. Obviously, since complex entities can always be decomposed into real and imaginary parts and vice versa, using natural (real-entities only) approach does not contribute anything and makes things unjustifiably more complex. However, by using complex presentation one might be overlooking some hidden qualities [2].

For instance, the complex Schrodinger equation turns out to be a pair of coupled equations of the real and imaginary parts [2]:

The Schrödinger equation for a free particle ( $v(x)=0$ ) becomes:

$$\frac{\partial \varphi_1}{\partial t} = + \frac{\hbar}{2m} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (1)$$

$$\frac{\partial \varphi_2}{\partial t} = - \frac{\hbar}{2m} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (2)$$

These two coupled equations can be shown [1] to be equivalent to coupling between two strings having initial tension  $T_0$  and initial coupling  $k_{s0}$  between the two strings:

$$\frac{\partial \varphi_1}{\partial t} = + \frac{T_0}{K_{s0}m} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (3)$$

$$\frac{\partial \varphi_2}{\partial t} = - \frac{T_0}{K_{s0}m} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (4)$$

Provided that

\* Corresponding author:

doron.kwiat@gmail.com (Doron Kwiatt)

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$$\hbar = \frac{2T_0}{K_{s0}} \quad (5)$$

As  $T_s$  and  $K_s$  are inversely proportional to time, the ratio  $T_s/K_s = \frac{1}{2}\hbar$  is fixed in time.

This interpretation of a particle as made up of two coupled strings, whose tensions and interaction are inversely proportional to time, gives us a classical interpretation of the Schrödinger equation.

As  $T_s$  and  $K_s$  are inversely proportional to time, the ratio  $T_s/K_s$  is constant in time.

Therefore, without loss of generality

$$\frac{T_s}{K_s} = \frac{1}{2}\hbar \quad (6)$$

This shows a derivation of Planck's universal constant, as a result of string structure of elementary particles.

## 2. Spatial Asymmetry of Dirac Equation

One may project from this on the relativistic Dirac equation [1], describing a free Fermion of mass  $m$ :

$$(i\hbar\gamma^\mu\partial_\mu - mcI)\Psi = 0 \quad (7)$$

One may separate the complex wave function  $\Psi$  into its real and imaginary parts [2]

$$\Psi = \begin{pmatrix} \Psi_R \\ i\Psi_I \end{pmatrix} \quad (8)$$

and by using them in the Dirac equation, one will get two separate equations where imaginary and real parts have been separated.

This same procedure can be applied to any quantum field Lagrangian, resulting (at the cost of doubling the number of equations) in separation of the equations to real wave functions.

In Dirac equation, Eq. [7], the four  $\gamma$  matrices are:

$$\gamma^3 = \begin{bmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{bmatrix} \quad \gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \quad \gamma^1 = \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix} \quad (9)$$

$$\gamma^2 = \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix}$$

where  $I_2$  is the 2x2 unit matrix and  $\sigma_x, \sigma_y, \sigma_z$  the Pauli 2x2 matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (10)$$

Of these matrices, only  $\sigma_y$  has imaginary components and it can be replaced by writing

$$\sigma_y = i\sigma'_y \quad \text{where} \quad \sigma'_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and therefore}$$

$$\gamma^2 = i \begin{bmatrix} 0 & \sigma'_y \\ -\sigma'_y & 0 \end{bmatrix} = i\gamma^{2'} \quad (11)$$

$$\text{So that } \gamma^{2'} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Decomposing  $\Psi$  into  $\Psi_R + i\Psi_I$  where  $\Psi_R$  and  $\Psi_I$  are

real, the Dirac equation becomes two separate equations:

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z)\Psi_R - \gamma^{2'}\partial_y\Psi_I - \left(\frac{mc}{\hbar}\right)\Psi_I = 0 \quad (12)$$

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z)\Psi_I + \gamma^{2'}\partial_y\Psi_R + \left(\frac{mc}{\hbar}\right)\Psi_R = 0 \quad (13)$$

- There are now 8 real wave functions instead of 4 in the complex presentation.
- All the  $\gamma$  matrices are real.
- Notice the asymmetry of the y-component in Eqs. [12, 13] due to the complex nature of Pauli's  $\sigma_y$ .
- Notice the change of sign in the mass terms.
- Even if  $m=0$ , the asymmetry in y remains.
- Notice the coupling between  $\Psi_R$  and  $\Psi_I$ , which remains even when the mass term vanishes.

Eqs. [12-13] may be written

$$(\sigma_0\partial_t + \sigma_x\partial_x + \sigma_z\partial_z)\Psi_R + \sigma_y'\partial_y\Psi_I - \left(\frac{mc}{\hbar}\right)\Psi_I = 0 \quad (14)$$

$$(\sigma_0\partial_t + \sigma_x\partial_x + \sigma_z\partial_z)\Psi_I + \sigma_y'\partial_y\Psi_R + \left(\frac{mc}{\hbar}\right)\Psi_R = 0 \quad (15)$$

In these coupled differential equations, all the  $\sigma$  are real 2x2 matrices. Hence  $\Psi_R$  and  $\Psi_I$  are 2-vectors.

When combined into the complex  $\Psi$  may be united into

$$(\sigma_0\partial_t + \sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z)\Psi = \left(\frac{mc}{\hbar}\right)\Psi = 0 \quad (16)$$

Where only  $\sigma_y$  is complex, and  $\Psi$  is a complex 4-vector

Setting  $m=0$  gives the Weyl equation:

$$(\sigma_0\partial_t + \sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z)\Psi = 0 \quad (17)$$

$$\sigma^\mu\partial_\mu\Psi = 0 \quad (18)$$

We see that by use of real components 2-vectors instead of a complex single 4-vector, there are two equations instead of one, and some coupling mechanism between the real part 2-vector and imaginary part 2-vector.

Notice that this coupling interaction does not vanish with mass ( $m \rightarrow 0$ ). However, in the case of  $m=0$ , one cannot discern between the two 2-vectors  $\Psi_R$  and  $\Psi_I$ .

## 3. N -Dimensional Representation of Dirac Equation

Following Dirac's idea [1], of taking the square root of the wave operator, as suggested:

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right) = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad (19)$$

in order to force all cross-terms such as  $\partial_x\partial_y$  to vanish, one must assume

$$\{A, B\} = \{A, C\} = \{A, D\} = \{B, C\} = \{B, D\} = \{C, D\} = 0 \quad (20)$$

$$A^2 = B^2 = C^2 = 1 \quad (21)$$

$$D^2 = -1 \quad (22)$$

Or, if  $\chi^0 = A, \chi^1 = B, \chi^2 = C, \chi^3 = D$  represent the A, B, C, D terms, then, in a compact form

$$\{X^\mu, X^\nu\} = 2g^{\mu\nu} \quad (\mu = 0, 1, 2, 3) \quad (23)$$

These conditions are met if A, B, C and D are matrices, with the implication that the wave function has components, as the dimension of the matrices. This explained the appearance of two-component wave functions in Pauli's theory of spin. The minimum would be to have A, B, C and D as 4x4 matrices. This is done in combinations of 2x2 Pauli matrices, in such a way that will obey the  $\{X^\mu, X^\nu\} = 2g^{\mu\nu}$  constrains. This leads to the assertion, that  $\Psi$  is a 4-vector of complex wave functions.

In a 3+1 world,  $g^{\mu\nu}$  is a 4x4 matrix, so A, B, C and D are 4x4 matrices. However, in an N-dimensional world, one needs NxN matrices [4], with  $N \geq 4$ , to set up a system with the properties required. In Dirac's original work  $N=4$ , so the wave function had *four* components, not two, as in the Pauli theory, or one, as in the Schrödinger equation. When looking into the initial development strategy of Dirac, his initial demand was to create an equation with first derivatives:

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t - \frac{mc}{\hbar})\Psi = 0 \quad (24)$$

Since it originated in the Klein-Gordon [5] equation, which by itself originated from Schrödinger equation, it was assumed there that  $\Psi$  is complex and therefore can be presented as a 2-vector of two complex wave functions.

Instead of using the conventional 4x4  $\gamma$  matrices (A, B, C and D) as introduced by Dirac, a set of four  $N \times N$  matrices  $\chi^\mu$ , acting on a complex N-vector  $\Psi$  are introduced, which satisfy Eq. [18], and the constrains in Eq. [20].

These  $N \times N$   $\chi$  matrices replace the 4x4  $\gamma$  matrices in Dirac 4x4 equation to introduce the Dirac  $N \times N$  equation:

$$(i\hbar\chi^\mu\partial_\mu - mcI_N)\Psi = 0 \quad (25)$$

where  $\Psi$  is now a complex N-vector

and where  $\chi^\mu$ , are four **real**  $N \times N$  matrices, satisfying the Clifford requirement:

$$\chi^\mu\chi^\nu + \chi^\nu\chi^\mu = 2g^{\mu\nu}I_N \quad (26)$$

This Dirac equation representation, has now 2N real wave functions (N and N) instead of 8 (4 and 4).

In their work about the dimensionality of our universe, Rugh and Nielsen [8], have shown that it is a straight-forward by induction procedure to go, from a set of d-2 matrices in a d-2 universe to a d dimensional universe by adding 2 gamma matrices from any given set of d-2 matrices. This is done by way of constructing

$$\chi = \begin{bmatrix} & \gamma_\mu \\ \gamma_\mu & \end{bmatrix}$$

So, from a set of d  $N \times N$  matrices, one can have a set of d+2  $2N \times 2N$  matrices satisfying the extended Clifford condition.

This procedure will thus create presentations of  $SO(N)$  to fermions in d-dimensional universe, for any even or odd d.

However, as will be shown, the condition  $N \times N = 2^d$  requires d to be an even number and hence only  $SO(4)$ ,  $SO(8)$ ,  $SO(16)$ ,  $SO(32)$  etc., representations are allowed.

## 4. Group Limitation on $\chi^\mu$

The Dirac matrices must satisfy the canonical anti-commutation relation  $\{\chi^\mu, \chi^\nu\} = 2g^{\mu\nu}$

Other bases are possible, and are related to the chiral basis by rotations.

The Dirac matrices generate a Euclidean Clifford algebra [2, 3].

N-dimensional gamma matrices [5, 6, 12-16], are a generalization of the four-dimensional Gamma matrices of Dirac to arbitrary dimension N. They are utilized in relativistic invariant wave equations for fermions (such as spinors) in arbitrary space-time dimensions, notably in string theory and supergravity [12, 15, 16].

Consider a d-dimensional space-time, with a flat Minkowski metric  $g_{ab}$  where a, b = 0, 1, ..., d-1. Original Dirac matrices correspond to taking d = N = 4.

For a d-dimensional space (d-1 spatial + 1 temporal coordinates) dimension, there are d such matrices  $\chi^i$  (i = 0, ... d-1), of size  $N \times N$ , which satisfy the anticommutator relation from the Clifford algebra  $Cl_{1,d-1}(\mathbb{R})$ ,  $\{\chi_a, \chi_b\} = 2g_{ab}I_N$

where  $I_N$  is the N dimensional identity matrix. (The spinors acted on by these matrices have N components each).

For a Clifford algebra generated by a set of d  $N \times N$   $\chi^\mu$  matrices,

$\{\chi^\mu, \chi^\nu\} = 2g^{\mu\nu}$  with  $\mu, \nu = 0, 1, 2, \dots, d-1$  and where the metric signature  $g^{\mu\nu} = \text{diag}(+, -, -, \dots)$ .

Using the matrices  $I_N$  and  $\chi^\mu$  we can construct a set of  $2^d$   $N \times N$  matrices as follow

$$I_N, \chi^\mu, \chi^\mu\chi^\nu, \chi^\mu\chi^\nu\chi^\lambda, \dots, \chi^0\chi^1\chi^2 \dots \chi^{d-1}$$

Over all combinations of indices, where  $\mu < \nu < \lambda < \dots$  etc.

For a specific case denote d by D.

Since there are D different  $\chi^\mu$  matrices, picking up p different ones, in an increasing index order, out of D for all  $0 \leq p \leq D$ , there are

$$1 + \sum_{p=1}^D \binom{D}{p} = 2^D \quad (27)$$

such matrix combinations.

Denote these possible matrix combinations by  $\Gamma_x$ , where  $0 \leq x \leq 2^D - 1$ .

Thus creating a set  $\Gamma^+ =$

$\{\Gamma_0 = I_N, \Gamma_1 = \chi^0, \Gamma_2 = \chi^1, \dots, \Gamma_{2^D-1}\}$ , with  $2^D$  members.

Because of the anticommutation relation between the gammas, the product of any  $\Gamma_x$  by any  $\Gamma_y$  is, up to a sign, a third member  $\Gamma_z$  of the set.

Define a set  $\Gamma^- = -\Gamma^+$  whose elements are the same members of  $\Gamma^+$ , but with negative sign, namely:

$$\Gamma^- = \{-\Gamma_0, -\Gamma_1, -\Gamma_2, \dots, -\Gamma_{2^D-1}\}$$

Create now a union set  $G = \Gamma^- \cup \Gamma^+$ , so G becomes the set of matrices  $\{\pm I_N, \pm \Gamma_1, \pm \Gamma_2, \dots, \pm \Gamma_x, \dots, \pm \Gamma_{2^D-1}\}$

Re-assign notations for the  $\Gamma_i$  members of the set G as follows:

( $\Gamma_0 = I_N, \Gamma_1 = -I_N, \Gamma_2 = \chi^1, \Gamma_3 = -\chi^1 \dots, \Gamma_{2^{D+1}-1}$ ) with  $2^{D+1}$  terms.

It contains the unit element  $I_N$  and is closed under multiplication, as the product of any  $\Gamma_x \in \mathbf{G}$  by any  $\Gamma_y \in \mathbf{G}$  is a third member  $\Gamma_z \in \mathbf{G}$ .

Thus  $\mathbb{G}$ , is a group with  $|\mathbb{G}| = 2^{D+1}$  members.

Let  $\chi^\mu$  be  $N$ -dimensional ( $N \times N$ ) irreducible matrix representation.

In order to establish the relationship between  $D$  and  $N$ , define a matrix

$$S = \sum_{x=0}^{2^{D+1}-1} (\Gamma_x^{-1} Z \Gamma_x) \quad (28)$$

where  $Z$  is an arbitrary  $N \times N$  matrix (the summation runs over all  $\Gamma_x \in \mathbb{G}$ ).

It follows that

$$\begin{aligned} \Gamma_y^{-1} S \Gamma_y &= \sum_{x=0}^{2^{D+1}-1} (\Gamma_x \Gamma_y)^{-1} Z (\Gamma_x \Gamma_y) \\ &= \sum_{w=0}^{2^{D+1}-1} (\Gamma_w)^{-1} Z (\Gamma_w) = S \end{aligned}$$

Or,

$$S \Gamma_y = \Gamma_y S \quad (29)$$

Since  $S$  commutes with all the matrices in the set, by Schur's first lemma ("A non-zero matrix which commutes with all of the matrices of an irreducible representation, is a constant multiple of the unit matrix") we conclude that  $S$  must be proportional to the identity matrix thus one can write

$$S = \sum_{x=0}^{2^{D+1}-1} \Gamma_x^{-1} Z \Gamma_x = \alpha I_N \quad (30)$$

Taking the trace on both sides gives

$$\begin{aligned} \text{Tr}(S) &= \text{Tr} \left( \sum_{x=0}^{2^{D+1}-1} Z \Gamma_x^{-1} \Gamma_x \right) \\ &= \sum_{x=0}^{2^{D+1}-1} \text{Tr}(Z) = \alpha \text{Tr}(I_N) = \alpha N \end{aligned} \quad (31)$$

Hence

$$\begin{aligned} \alpha N &= \text{Tr}(Z) \sum_{x=0}^{2^{D+1}-1} 1 = \text{Tr}(Z) (2^{D+1} - 1 + 1) \\ \alpha &= \frac{2^{D+1}}{N} \text{Tr}(Z) \end{aligned}$$

and since

$$\text{Tr} \left( \sum_{x=0}^{2^{D+1}-1} Z \Gamma_x^{-1} \Gamma_x \right) = \text{Tr}(\alpha I_N)$$

one obtains:

$$\sum_{x=0}^{2^{D+1}-1} \Gamma_x^{-1} Z \Gamma_x = \alpha I_N = \frac{2^{D+1}}{N} \text{Tr}(Z) I_N \quad (32)$$

Taking the (i;j)<sup>th</sup> element of both sides:

$$\begin{aligned} S_{ij} &= \sum_{x=0}^{2^{D+1}-1} \sum_{l,m} (\Gamma_x^{-1})_{il} Z_{lm} (\Gamma_x)_{mj} \\ &= \frac{2^{D+1}}{N} \sum_{l,m} (\delta_{ml} Z_{lm}) \delta_{ij} \end{aligned} \quad (33)$$

Where use is made of the fact that  $\text{Tr}(Z) = \sum_{l,m} (Z_{lm} \delta_{ml})$  and  $(I_N)_{ij} = \delta_{ij}$

$$\sum_{l,m} Z_{lm} \left( \sum_{x=0}^{2^{D+1}-1} (\Gamma_x^{-1})_{il} (\Gamma_x)_{mj} - \frac{2^{D+1}}{N} \delta_{ij} \delta_{ml} \right) = 0 \quad (34)$$

and since  $Z$  is an arbitrary matrix:

$$\sum_{x=0}^{2^{D+1}-1} (\Gamma_x^{-1})_{il} (\Gamma_x)_{mj} - \frac{2^{D+1}}{N} \delta_{ij} \delta_{ml} = 0 \quad (35)$$

for any  $i, j, l, m$ .

Setting  $l=i, m=j$  and summing over  $i$  and  $j$ :

$$\sum_{ij} \sum_{x=0}^{2^{D+1}-1} (\Gamma_x^{-1})_{ii} (\Gamma_x)_{jj} - \frac{2^{D+1}}{N} \sum_{ij} \delta_{ij} \delta_{ij} = 0 \quad (36)$$

and so,

$$\sum_{x=0}^{2^{D+1}-1} \text{Tr}(\Gamma_x^{-1}) \text{Tr}(\Gamma_x) - \frac{2^{D+1}}{N} N = 0 \quad (37)$$

But

$$\text{Tr}(\Gamma_x) = 0 \quad (38)$$

for all  $\Gamma_x \neq \Gamma_0, \Gamma_1$ , and

$$\text{Tr}(\Gamma_0) = \text{Tr}(\Gamma_0^{-1}) = \text{Tr}(I_N) = N \quad (39)$$

$$\text{Tr}(\Gamma_1) = \text{Tr}(\Gamma_1^{-1}) = \text{Tr}(-I_N) = -N \quad (40)$$

Therefore:

$$2N^2 = 2^{D+1} \quad (41)$$

and finally,

$$N = 2^{D/2} \quad (42)$$

This result was reached based on the following assumptions:

1. There exist  $d$  gamma  $n \times n$  matrices where both  $d$  and  $N$  are integers.
2. These gammas are Lorentz covariant
3. The gammas must obey the Clifford algebra canonical anti-commutation relation  $\{\chi^a, \chi^b\} = 2g^{ab}$  with  $a, b = (0, \dots, d)$ .
4. The number  $d$  specifies the total number of world coordinates, of which there are the usual  $x, y, z$  spatial coordinate, at least one is a temporal coordinate and the rest are extended coordinates (may be temporal and may be spatial). By convention, in our 4 dimensional world we use  $d=3+1$ , namely  $x, y, z$  and  $t$ .
5. By separation of the wave function into real and imaginary components, the resulting equations are doubled and so instead of a single  $N$ -vector complex spinor, one obtains 2  $N$ -vector real spinors, with  $\chi^\mu$  are  $d = D$  real,  $N \times N$  matrices.

Proof of the  $N d$  relations based on induction [10] (starting at  $d=2$ ) did not take into account the Group limitations as shown in paragraph v. above.

Thus,  $\chi^\mu \in \mathcal{C}\ell_N(\mathbb{R})$ , and must satisfy the restriction of  $N = 2^{d/2}$ .

It is therefore, that only even  $d$  dimensional universe is acceptable for fermions.

The representations are confined to those where  $N$  is even only ( $N=4, 8, 16, \dots$  etc.).

Many theories [8-15] of (N,d) universe make use of odd d. But as was shown here, odd-d universe is not allowed for fermions.

World-sheet action [14] is given by an action in which fermions are separated from Bosonic fields.

Since all actions with fermions have terms of the form

$$\dots \Psi^\dagger \chi^\mu \partial_\mu \Psi \dots$$

It means that the Clifford condition is valid for the world action [13, 14] and the fermionic part of the action must obey the dimensionality requirement of  $N^2 = 2^d$ .

## 5. An Upper Limit on d

So far, it was shown that d must be an even number. But *is there an upper limit on d?*

For each dimension, there is a single NXN Dirac  $\gamma$  matrix  $\gamma^\mu$ .

Let us start with Clifford algebra generated by  $N \times N$   $\gamma^\mu$  matrices.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

with  $\mu, \nu = 0, 1, 2, \dots, D$  and where the metric signature  $g^{\mu\nu} = \text{diag}(+, -, -, \dots)$ .

For different d-dimensions, construct the following sets of  $\gamma$  matrices:

dimensions	Sets of Gamma matrices	Set size
$d = 1$	$S_1 = \{I, \gamma^0\}$	$ S_1  = 2^1$
$d = 2$	$S_2 = \{I, \gamma^0, \gamma^1\}$	$ S_2  = 2^2$
$d = 3$	$S_3 = \{I, \gamma^0, \gamma^1, \gamma^2\}$	$ S_3  = 2^3$
$d = 4$	$S_4 = \{I, \gamma^0, \gamma^1, \gamma^2, \gamma^3\}$	$ S_4  = 2^4$
$d = 6$	$S_6 = \{I, \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^a, \gamma^b\}$	$ S_6  = 2^6$
$d=D$	$S_D = \{I, \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^a, \gamma^b, \dots, \gamma^{(D-1)}\}$	$ S_D  = 2^D$

Notice that from now on it is sufficient to concentrate on an even dimensional Fermions universe, as it was shown above.

For a D dimensional universe, the matrices, use  $I_N$  and  $\gamma^\mu$  to construct the following set  $G_D$  of  $2^D N \times N$  matrices:

$$G_D = \{I_N, \gamma^\mu, \gamma^\mu \gamma^\nu, \gamma^\mu \gamma^\nu \gamma^\lambda, \dots, \gamma^0 \gamma^1 \gamma^2 \dots \gamma^D\}$$

Over all combinations of indices, where  $\mu < \nu < \lambda, \dots$  etc. Create next the set

$$-G_D = \{-I_N, -\gamma^\mu, -\gamma^\mu \gamma^\nu, -\gamma^\mu \gamma^\nu \gamma^\lambda, \dots, -\gamma^0 \gamma^1 \gamma^2 \dots \gamma^D\}$$

And define the group

$$\begin{aligned} \mathbb{K}_D &= G_D \cup (-G_D) = \\ &\{I_N, -I_N, \gamma^0, -\gamma^0, \gamma^1, -\gamma^1, \gamma^2, -\gamma^2, \dots, \gamma^{(D-1)}, \\ &\quad -\gamma^{(D-1)} \dots, \gamma^0 \gamma^1 \gamma^2 \dots \gamma^D, -\gamma^0 \gamma^1 \gamma^2 \dots \gamma^D\} \end{aligned}$$

Define next the matrices  $\Gamma_x \subset \mathbb{K}$ :

$$\Gamma_i = \prod \gamma^i, \quad \Gamma_{i+1} = -\Gamma_i$$

For all  $i = 0, 1, \dots, 2^{D+1} - 1$ .

With  $\Gamma_0 = I_N, \Gamma_1 = -I_N$ , etc. Thus,

$$\mathbb{K}_D = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \dots, \Gamma_{2D-1}, \Gamma_{2D}, \dots, \Gamma_{2^{D+1}-1}\}$$

with  $|\mathbb{K}_D| = 2|G_D| = 2^{D+1}$

It is obvious that  $\mathbb{K}_D$  is a group of order  $2^{D+1}$

It is obvious that  $\mathbb{K}_D$  contains a set of sub-groups:

$$\mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \mathbb{K}_3 \dots \subseteq \mathbb{K}_D$$

Which is described in the following:

$$\mathbb{K}_0 = \{\Gamma_0, \Gamma_1\}$$

$$\mathbb{K}_1 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$$

$$\mathbb{K}_2 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\}$$

$$\mathbb{K}_3 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \dots, \Gamma_{15}\}$$

$$\mathbb{K}_4 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \dots, \Gamma_{31}\}$$

etc.

All  $\Gamma_i$  (except for  $i=0,1$ ) are  $\gamma^\mu$  or products of  $\gamma^\mu$ 's ( $\mu=0,1,2,3$ ). Hence, when two new matrices  $\gamma^4$  and  $\gamma^5$  are introduced in a 6-dimensional universe, we define  $A = \gamma^4 \gamma^5$ , and it is a straight forward procedure to show that  $[\Gamma_i, A] = 0$  for all  $\Gamma_i \subset \mathbb{K}_4$ .

**if a matrix  $U$ , which is not a multiple of the unit matrix ( $U \neq \lambda I$ ), commutes  $[U, R_i] = 0$  with all matrices  $R_0, R_1, R_2, \dots$ , of a representation, then that representation is necessarily reducible. And, if a non-zero matrix commutes with all of the matrices of an irreducible representation, that matrix must be a multiple of the unit matrix.**

Therefore, the group  $\mathbb{K}_6$  must be a **reducible** representation.

This forces us to conclude, that no irreducible representation  $\mathbb{K}_d$  may be found for  $d > 4$  and therefore, the fermionic universe must have  $d=4$ .

## 6. Conclusions

Separation of the complex wave function into real and imaginary components splits the Dirac equation into two coupled equations, with real  $\gamma^\mu$  matrices. Dirac equation can be represented by real  $2^D \times 2^D$  matrices, where D is the number of matrices and the dimension of the universe in which the fermions exist.

There is no implication from this on Bosons universe and they may exist in another kind of universe.

Based on the canonical anti-commutation relation  $\{\chi^\mu, \chi^\nu\} = 2g^{\mu\nu}$ , it was shown that the  $\chi^\nu$  matrices can be used as generators of a group G, which dimension  $|G| = 2^d = N^2$  and since N must be an integer, so must be d.

Therefore, any fermionic theory, even supergravity Kaluza-Klein theory, cannot be an odd dimensional d, unless one gives up on the Clifford restriction Eq. [20]. An upper limit on the dimensions is  $d \leq 5$ .

**The final conclusion is then, that for a Fermionic universe,  $d = 4$ .**

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