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Dirac Equation in SO(6) Real Representation Leading to Four Triplets

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Abstract Introducing four 6x6 Lorentz invariant gamma matrices removes asymmetry in the three spatial coordinates of Dirac equation. Use of the SO(6) representation provides 12 real wave functions which are divided into 4 groups of triplets. These 4 groups interact with each other and each element of a triplet satisfies the Klein-Gordon equation.

Keywords Dirac Equation, 6x6 Gamma Matrices, Triplets

1. Introduction

Recently, it was shown [1] that a single non-relativistic Schrödinger equation is a description of a single massive particle, made up of two coupled strings. One may project from this on the relativistic Dirac equation [2], describing a free Fermion of mass m:

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - mcI\right)\Psi = 0 \qquad (1)$$

One may separate the complex wave function Ψ into its real and imaginary parts [3]

$$\Psi = \begin{pmatrix}\Psi_r \\ i\Psi_i\end{pmatrix} \qquad (2)$$

and by using them in the Dirac equation, one will get two separate equations where imaginary and real parts have been separated.

This same procedure can be applied to any quantum field Lagrangian, resulting (at the cost of doubling the number of equations) in separation of the equations to real wave functions.

These wave functions may be interpreted as coupled strings.

In Dirac equation (1), the four γ matrices are:

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \ \gamma^1 = \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix}$$

$$\gamma^2 = \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix} \ \gamma^3 = \begin{bmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{bmatrix} \qquad (3)$$

where I_2 is the 2x2 unit matrix and σ_x, σ_y, σ_z the Pauli 2x2 matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (4)$$

Of these matrices, only σ_y has imaginary components and it can be replaced by writing

$\sigma_y = i\sigma'_y$ where $\sigma'_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and therefore

$$\gamma^2 = i\begin{bmatrix} 0 & \sigma'_y \\ -\sigma'_y & 0 \end{bmatrix} \qquad (5)$$

Decomposing Ψ into $\Psi_r + i\Psi_i$ where Ψ_r and Ψ_i are real, the Dirac equation becomes two separate equations,

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z)\Psi_r - \gamma'^2\partial_y\Psi_i - \left(\frac{mc}{\hbar}\right)\Psi_i = 0 \qquad (6)$$

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z)\Psi_i + \gamma'^2\partial_y\Psi_r + \left(\frac{mc}{\hbar}\right)\Psi_r = 0 \qquad (7)$$

There are now 8 real wave functions instead of 4 in the complex presentation. Notice the asymmetry of the y-component in Eqs. (6,7) due to the complex nature of Pauli's σ_y.

2. Changing Dirac 4x4 γ Matrices to 6x6 Representation

To move from classical QM to relativistic QM, Dirac came with the idea of taking the square root of the wave operator thus, he suggested:

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right) = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad (8)$$

and in order to get all the cross-terms such as $\partial_x\partial_y$ to vanish, one must assume

$$\{A,B\} = \{A,C\} = \{A,D\} = \{B,C\} = \{B,D\} = \{C,D\} = 0$$
$$A^2 = B^2 = C^2 = 1$$
$$D^2 = -1 \qquad (9)$$

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Or, if X^μ ($\mu = 0,1,2,3$) represent the A, B, C, D terms, can be written in a compact form

$$\{X^\mu, X^\nu\} = 2g^{\mu\nu} \tag{10}$$

These conditions are met if A, B, C and D are matrices, with the implication that the wave function has multiple components. This immediately explained the appearance of two-component wave functions in Pauli's theory of *spin*. The most direct suggestion would be to have A, B, C and D as 4x4 matrices. This is done in combinations of 2x2 Pauli matrices, in such a way that will obey the $\{X^\mu, X^\nu\} = 2g^{\mu\nu}$ constrains. This leads to the assertion to be made, that Ψ is a 4-vector of complex wave functions.

In a 3+1 world, $g^{\mu\nu}$ is a 4x4 matrix, so the dimension of the A, B, C and D are 4x4 matrices. However in an N-dimensional world, one needs ***NxN*** matrices [4], with N>=4, to set up a system with the properties required. In Dirac's original work N=4, so the wave function had *four* components, not two, as in the Pauli theory, or one, as in the bare Schrödinger theory. When looking into the initial development strategy of Dirac, his initial demand was to create an equation with first derivatives:

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t - \frac{mc}{\hbar}\right)\Psi = 0 \tag{11}$$

Since it originated in the Klein-Gordon [5] equation, which by itself originated from Schrödinger equation, it was assumed there that Ψ is complex and therefore can be presented as a 2-vector of two complex wave functions.

In this work, instead of using the conventional 4x4 γ matrices (A, B, C and D) as introduced by Dirac, a set of four 6x6 matrices χ^μ, matrices acting on a complex 6-vector Ψ are introduced, which satisfy Eq. (11), and the constrains in Eq. (10).

This is equivalent to changing from complex representation of rotations in $\mathbb{C}_2$ to real rotations [7] in $\mathbb{R}_3$. Since SO(3) is homomorphic with SU(2), the rotations in $\mathbb{R}_3$ of a 3-component real wave vector is equivalent to a rotation in $\mathbb{C}_2$ of a 2-component complex wave vector.

Instead of using Pauli's 2x2 matrices, one may use the 3x3 matrix presentations of angular momentum generators L_x L_y and L_z, of the rotation group SO(3).

Using L_x L_y and L_z outer multiplications with $i\sigma_y$ and with σ_z: $\sigma_z \otimes I_4$ and $i\sigma_y \otimes L$ to define the 6x6 χ matrices, satisfying the requirements on A, B, C and D according to Eq. (11).

Thus:

$$\chi^0 = \sigma_z \otimes I_4 = \begin{bmatrix} I_3 & 0 \\ 0 & -I_3 \end{bmatrix} \tag{12}$$

$$\chi^1 = i\sigma_y \otimes L_x = \begin{bmatrix} 0 & [L_x] \\ [-L_x] & 0 \end{bmatrix} \tag{13}$$

$$\chi^2 = i\sigma_y \otimes L_y \begin{bmatrix} 0 & [L_y] \\ [-L_y] & 0 \end{bmatrix} \tag{14}$$

$$\chi^3 = i\sigma_z \otimes L_z = \begin{bmatrix} 0 & [L_z] \\ [-L_z] & 0 \end{bmatrix} \tag{15}$$

These 6x6 χ matrices replace the 4x4 γ matrices in Dirac 4x4 equation to introduce the Dirac 6x6 equation:

$$\left(i\hbar\chi^\mu\partial_\mu - mcI\right)\Psi = 0 \tag{16}$$

where Ψ is now a complex 6-vector

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix} \tag{17}$$

and where χ^μ, are four ***real*** 6x6 matrices, satisfying the Clifford requirement:

$$\chi^\mu\chi^\nu + \chi^\nu\chi^\mu = 2g^{\mu\nu} \tag{18}$$

It has now 12 real wave functions (6 and 6) instead of 8 (4 and 4).

3. Lorentz Covariance of the 6x6 χ Matrices

To prove Lorentz covariance of the Dirac equation, with the 4x4 gamma matrices, one requires the existence of a 4x4 transformation matrix S, such that $\Psi'(x') = S\Psi(x)$ under a Lorentz transformation.

Suppose that

$$S\gamma^\mu S^{-1}\Lambda^\mu{}_\nu = \gamma^\nu \text{ under } x^{\mu'} = \Lambda^\mu_\nu x^\nu$$

Obviously, if such transformation S exists, the 4x4 Dirac equation is Lorentz invariant.

By assuming an infinitesimal Lorentz transformation

$$\Lambda^\mu_\nu = g^\mu_\nu + \Delta\omega^\mu_\nu \tag{19}$$

($\Delta\omega^\mu_\nu$ are real numerical coefficients, independent of x), one can write for S

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu} \tag{20}$$

Where $\sigma_{\mu\nu}$ are 4x4 matrices that satisfy

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{21}$$

This derivation of the Lorentz group representation of 4x4 matrix generators is based on the relationship $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

Since χ^μ satisfies the same constraints as γ^μ we conclude that there exist 6x6 generators of the S transformation of the Dirac 6x6 representation, and hence Dirac's 6x6 representation, Eq. (16), is Lorentz covariant.

The Dirac matrices must satisfy the canonical anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

The above definition corresponds to the so-called "chiral basis," where Dirac matrices are block anti-diagonal.

Other bases are possible, and are related to the chiral basis by rotations. The Dirac matrices generate a Euclidean Clifford algebra [6].

N-dimensional gamma matrices are a generalization of the four-dimensional Gamma matrices of Dirac to arbitrary dimension N. They are utilized in relativistic invariant wave equations for fermions (such as spinors) in arbitrary space-time dimensions, notably in string theory and supergravity.

Consider a d-dimensional space-time, with a flat Minkowski metric g_{ab} where a, b = 0,1, ..., d−1. standard Dirac matrices correspond to taking d = N = 4.

The higher dimension (N>4) γ matrices are a d-long sequence of complex N×N matrices Γ_i (i =0, … d-1) which satisfy the anticommutator relation from the Clifford algebra $C\ell_{1,d-1}(\mathbb{R})$, generating a representation for this algebra

$$\{\Gamma_a, \Gamma_b\} = 2g_{ab} I_N$$

where I_N is the N dimensional identity matrix. (The spinors acted on by these matrices have N components in d dimensions.) Such a sequence exists for all values of d.

In this work, χ^μ matrices are real, with N=6 and d=4.

Thus, $\chi^\mu \subset C\ell_6(\mathbb{R})$, though without the restriction of $N = 2^{\lfloor d/2 \rfloor}$.

4. Solution Leading to Triplets

To solve Eq. (16) split the 6-vector into real and imaginary parts (both Ψ_R and Ψ_I are real 6-vectors)

$$\Psi = \Psi_R + i\Psi_I$$

Further split the 6-vectors into two triplets (3-vectors) each:

$$\Psi_R = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad (22)$$

$$\Psi_I = \begin{pmatrix} \psi_C \\ \psi_D \end{pmatrix} \quad (23)$$

So that ψ_A, ψ_B, ψ_C and ψ_D are 4 different ***real*** triplets.

Substituting Eqs. (22,23) in Dirac's 6x6 representation Eq. (16) gives 4 differential equations with 12 unknowns (4 triplets) parameters:

$$\dot{\psi}_A + (L\cdot\nabla)\,\psi_B = +\left(\frac{mc}{\hbar}\right)\psi_C \quad (24)$$

$$\dot{\psi}_B + (L\cdot\nabla)\,\psi_A = -\left(\frac{mc}{\hbar}\right)\psi_D \quad (25)$$

$$\dot{\psi}_C + (L\cdot\nabla)\,\psi_D = -\left(\frac{mc}{\hbar}\right)\psi_A \quad (26)$$

$$\dot{\psi}_D + (L\cdot\nabla)\,\psi_C = +\left(\frac{mc}{\hbar}\right)\psi_B \quad (27)$$

Where $L\cdot\nabla$ stands for $L_x\partial_x + L_y\partial_y + L_z\partial_z$, a 3x3 operator.

Applying time derivative to Eqs. (24-27) gives

$$\ddot{\psi}_A + (L\cdot\nabla)\,\dot{\psi}_B = +\left(\frac{mc}{\hbar}\right)\dot{\psi}_C \quad (28)$$

$$\ddot{\psi}_B + (L\cdot\nabla)\,\dot{\psi}_A = -\left(\frac{mc}{\hbar}\right)\dot{\psi}_D \quad (29)$$

$$\ddot{\psi}_C + (L\cdot\nabla)\,\dot{\psi}_D = -\left(\frac{mc}{\hbar}\right)\dot{\psi}_A \quad (30)$$

$$\ddot{\psi}_D + (L\cdot\nabla)\,\dot{\psi}_C = +\left(\frac{mc}{\hbar}\right)\dot{\psi}_B \quad (31)$$

Define two matrices:

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad and \quad \boldsymbol{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (32)$$

The differential equations (24-27) and (28-31) can be written as

$$\dot{\boldsymbol{\Psi}} + (L\cdot\nabla)\boldsymbol{Q}\,\boldsymbol{\Psi} = +\left(\frac{mc}{\hbar}\right)\boldsymbol{R}\,\boldsymbol{\Psi} \quad (33)$$

$$\ddot{\boldsymbol{\Psi}} + (L\cdot\nabla)\boldsymbol{Q}\,\dot{\boldsymbol{\Psi}} = +\left(\frac{mc}{\hbar}\right)\boldsymbol{R}\,\dot{\boldsymbol{\Psi}} \quad (34)$$

Which combines to:

$$\ddot{\boldsymbol{\Psi}} - (L\cdot\nabla)^2\boldsymbol{Q}^2\,\dot{\boldsymbol{\Psi}} = \left(\frac{mc}{\hbar}\right)^2\boldsymbol{R}^2\,\dot{\boldsymbol{\Psi}} \quad (35)$$

We notice that $\mathbf{Q}^2 = I_4$, $\mathbf{R}^2 = -I_4$ and $\{\mathbf{Q}, \mathbf{R}\} = 0$, and also $(L\cdot\nabla)^2 = \nabla^2$. Thus:

$$\frac{\partial^2\psi_A}{c^2\partial t^2} - \nabla^2\psi_A = -\left(\frac{mc}{\hbar}\right)^2\psi_A \quad (36)$$

$$\frac{\partial^2\psi_B}{c^2\partial t^2} - \nabla^2\psi_B = -\left(\frac{mc}{\hbar}\right)^2\psi_B \quad (37)$$

$$\frac{\partial^2\psi_C}{c^2\partial t^2} - \nabla^2\psi_C = -\left(\frac{mc}{\hbar}\right)^2\psi_C \quad (38)$$

$$\frac{\partial^2\psi_D}{c^2\partial t^2} - \nabla^2\psi_D = -\left(\frac{mc}{\hbar}\right)^2\psi_D \quad (39)$$

Dirac equation has become symmetrical in x, y and z. The spatial asymmetry has been removed.

Second, Dirac equation Eq. (16) turned into four real, apparently uncoupled (3-vectors) triplets. Each triplet satisfies Klein-Gordon [5] wave equation (Eqs. (36-39)), with well-known solutions.

However, when introducing in Eq. (16)

$$\Psi = \Psi_R + i\Psi_I$$

one can separate it into the real and imaginary parts and obtain:

$$\mathfrak{D}\Psi_R = \frac{mc}{\hbar}\,\Psi_I \quad (40)$$

$$\mathfrak{D}\Psi_I = -\frac{mc}{\hbar}\,\Psi_R \quad (41)$$

where by definition: $\mathfrak{D} \equiv \frac{1}{c}\chi^0\partial_t + \chi^1\partial_x + \chi^2\partial_y + \chi^3\partial_z$ is a 6x6 matrix operator, and

$$\Psi_R = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \quad \Psi_I = \begin{pmatrix} \psi_C \\ \psi_D \end{pmatrix}$$

Equations (40) and (41) are two differential equations of first order and show that the Ψ_R and Ψ_I sextets (6-vectors) are coupled to each other.

In fact, one can say that there are 12 real wave functions. Each satisfies the Klein-Gordon equation individually and independently. These 12 real wave functions are combined into two sextets, and each sextet is made of two triplets. Altogether there are four triplets.

Looking for instance at Eqs. (24-27), the time derivative of one triplet, depends on the spatial derivatives of the second triplet and on a mass-proportional term of the third triplet. This coupling mechanism exists amongst the other triplets combinations in turn (see Figure 1). The four triplets ψ_A, ψ_B, ψ_C and ψ_D, are denoted here by A B C and D. Each triplet's time derivative is coupled to the spatial derivative of two other triplets and proportional by a $\frac{mc}{\hbar}$ factor to the fourth triplet entity, at a time. This coupling is changing between the 4 entities so that all four combinations are allowed. Each component of the three components the triplets are made of, satisfies the Klein-Gordon wave equation.

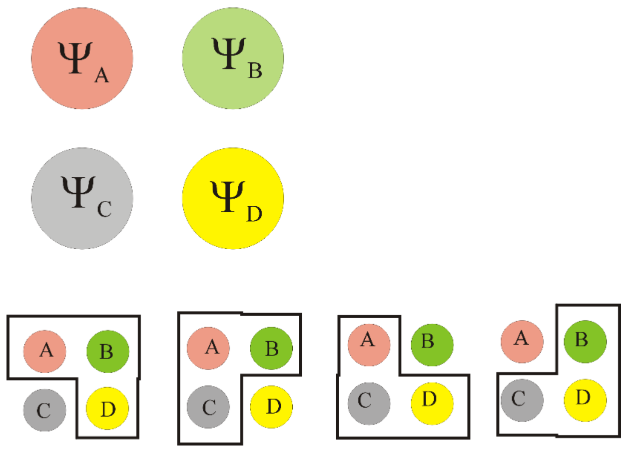

Figure 1. The quartet of ψ_A, ψ_B, ψ_C and ψ_D entities. Each entity (denoted here by A B C and D) is a 3-vector. Each single entity time derivative is coupled to the spatial derivative of two other entities and proportional by a $\frac{mc}{\hbar}$ factor to the fourth entity, at a time. This coupling is changing between the 4 entities so that all four combinations are allowed

The $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices are responsible for the exchanges between the triplets

$\boldsymbol{Q}$ exchanges $\psi_A \leftrightarrow \psi_B$ and $\boldsymbol{R}$ exchanges $\psi_C \leftrightarrow \psi_D$.

Therefore, the $\boldsymbol{R}$ term will affect the lower triplet pair (C and D) by exchange as well as by mass change, whereas the $\boldsymbol{Q}$ term will only cause an exchange in the upper triplet pair (A and B).

To summarize, the following notations are introduced:

$$\Psi_{R\uparrow} = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad \Psi_{R\downarrow} = \begin{pmatrix} \psi_B \\ \psi_A \end{pmatrix} \quad \Psi_{\widetilde{R\uparrow}} = \begin{pmatrix} \psi_B \\ -\psi_A \end{pmatrix}$$

$$\Psi_{I\uparrow} = \begin{pmatrix} \psi_C \\ \psi_D \end{pmatrix} \quad \Psi_{I\downarrow} = \begin{pmatrix} \psi_D \\ \mathrm{C} \end{pmatrix} \quad \Psi_{\widetilde{I\uparrow}} = \begin{pmatrix} \psi_C \\ -\psi_D \end{pmatrix}$$

Then, combining Eqs. (22-23,24-27,40-43) gives

$$\frac{\partial \Psi_{R\uparrow}}{\partial t} + L \cdot \nabla \Psi_{R\downarrow} = \mathrm{m}\Psi_{\widetilde{I\uparrow}}$$

$$\frac{\partial \Psi_{I\uparrow}}{\partial t} + L \cdot \nabla \Psi_{I\downarrow} = -\mathrm{m}\Psi_{\widetilde{R\uparrow}}$$

This form helps in the interpretation of how the real and imaginary parts of the Dirac fields interact.

5. Electromagnetic Field

Dirac 6x6 equation Eq. (16) in the presence of electromagnetic field $A_\mu = (\Phi/\mathrm{c}, -\vec{A})$ becomes

$$\left(i\hbar\chi^\mu(\partial_\mu - eA_\mu) - mcI\right)\Psi = 0 \qquad (42)$$

Hence, Eq. (66) becomes

$$\dot{\boldsymbol{\Psi}} + (L \cdot \nabla)\boldsymbol{Q}\,\boldsymbol{\Psi} = \left\{\left(\frac{mc}{\hbar}\right)\boldsymbol{R} + \left(L \cdot \vec{A}\boldsymbol{Q}\right) - \left(\frac{e\Phi}{c}I_4\right)\right\}\boldsymbol{\Psi} \qquad (43)$$

Eq. (43) is similar to Eq. (33) except for the term

$$\left(L \cdot \vec{A}\boldsymbol{Q} - \frac{\mathrm{e}\Phi}{c}I_4\right)$$

added to the l.h.s.

Thus, for the very weak field approximation, $(\vec{A} \to 0)$, there will be no exchange between the triplets more than in the case without EM field.

The solution of Eq. (43) will be approximately similar to that of Eq. (66), except for an interaction term

$$\left(L \cdot \vec{A}\boldsymbol{Q} - \frac{\mathrm{e}\Phi}{c}I_4\right)$$

6. A Vibrating Membrane [8, 9]

Assuming polar symmetry, Eqs. (36-39) become

$$\partial^2{}_t\Psi_i(r,\theta,t) = \frac{\partial^2\Psi_i(r,\theta,t)}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi_i(r,\theta,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi_i(r,\theta,t)}{\partial\theta^2} \qquad (44)$$

Where the y coordinate is the cylinder's main axis and x and z are in the plane perpendicular to the y-axis (see Figure 2).

The solution to Eq. (44) is

$$\Psi_i(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{J}_n(k_{nm}r)(A\sin(n\theta) + B\cos(n\theta))(A'\sin(k_{nm}t) + B'\cos(k_{nm}t)) \qquad (45)$$

where knm is the mth root of $\mathcal{J}_n(\mathrm{r}) = 0$.

Defining

$$\Theta(x) = A\sin(x) + B\cos(x) \qquad (46)$$

and

$$T(x) = A'\sin(x) + B'\cos(x) \qquad (47)$$

One can write

$$\Psi_i(r,\theta,t) = \sum_{mn}^{\infty}\mathcal{J}_n(k_{nm}r)\,\Theta(n\theta)\,T(k_{nm}t) \qquad (48)$$

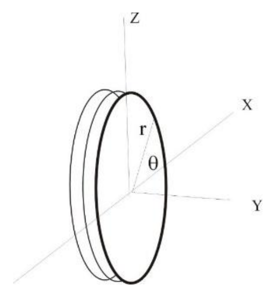

Figure 2. A membrane in x, y, z coordinate system

Assuming spherical symmetry, Dirac equation to spherical coordinates Eqs. (36-39), become

$$\partial^2{}_t\Psi_i(r,\theta,\varphi,t) = \frac{\partial^2\Psi_i(r,\theta,\varphi,t)}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi_i(r,\theta,\varphi,t)}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Psi_i(r,\theta,\varphi,t) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Psi_i(r,\theta,\varphi,t) \qquad (49)$$

The most general solution is then

$$\Psi_i(r,\theta,\varphi,t) = \sum_{k,\ell,m}^{\infty}\left(A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{(\ell+1)}}\right)Y_\ell^m(\theta,\varphi)\,T_k(t) \qquad (50)$$

Where A_{lm} and B_{lm}, are constants to be determined by the boundary conditions.

The time dependency is given by

$$T_k(t) = Ae^{+ikt} + Be^{-ikt}$$

The spherical dependency is given by the spherical harmonic functions, with the Legendre polynomials $P_l^m(cos\theta)$ and normalization constants $C_{\ell m}$.

Since by boundary conditions $\mathbf{R}(\mathbf{r})$=0 for $\mathbf{r} \to \infty$, one must have $\mathbf{A}_{\boldsymbol{\ell}\mathbf{m}} = \mathbf{0}$ except for $\boldsymbol{\ell} = \mathbf{0}$.

Since by boundary conditions $\mathbf{R}(\mathbf{r})$ is finite for $\mathbf{r} \to \mathbf{0}$, one must have $\mathbf{r} > 0$.

The solution is then

$$\Psi_i(r,\theta,\varphi,t) = \sum_{k,m}^{\infty}(A_{0m} + \frac{B_{\ell m}}{r^{(\ell+1)}})Y_\ell^m(\theta,\varphi)\, T_k(t) \quad (51)$$

One may interpret the wave functions in Dirac equation, as describing 4 independent membranes 1. Each two membranes are coupled into a two-folded membrane, closed at their common circular edges.

The membranes can be at different modes of vibrations (see Figure 3).

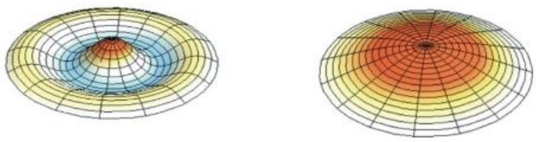

Figure 3. A Dirac membrane modes

7. Conclusions

Dirac equation can be represented in a real 6x6 matrix form. This is done by changing the wave function to a 6-vector instead of a 4-vector and replacing Dirac's 4x4 γ^μ matrices, by four 6x6 χ^μ matrices.

This results in two similar Klein-Gordon equations, acting on two independent real wave functions, represented by two 6-vectors.

Dirac's equation, represents then a system of four (a quartet) coupled triplets A, B, C and D. Each of this quartet has an internal structure (triplet) of 3 sub-entities. The 4 members of the quartet, are coupled and interact with each other. Each member of the triplets satisfies Klein-Gordon equation.

The solution to Dirac equation is made up of 12 real wave functions (four triplets).

The quartet members are coupled in pairs A and B, and C and D. The spatial derivative of A affects the time derivative of B and vice versa. The spatial derivative of C affects the time derivative of D and vice versa. Yet, when second order time derivatives are considered, the quartet members act as 4 independent entities obeying KG wave equations independent of each other.

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