

The Schrödinger Equation and Asymptotic Strings

Doron Kwiat

17/2 Erez Street, Mazkeret Batya, Israel

Abstract One can represent quantum mechanics with real wave functions. The use of real wave functions leads to the interpretation of a single particle as consisting of two entities represented as coupled wave functions. Probability distributions are shown to be preserved for coupled wave functions just as for a single complex function. Assuming a string model [12-17], a double coupled string description is suggested, whereby the Schrödinger equation emerges naturally. This double-string description assumes a time-dependent tension [18] in the strings, together with a time-dependent interaction between the two strings. If the time pattern is similar for both tension and interaction, their ratio is shown to be $\hbar/2$. This leads to the derivation of Planck's constant as a result of strings interactions and characteristics.

Keywords Quantum Mechanics, Strings, Planck Constant

1. Introduction

The Schrödinger equation is a mathematical equation that describes changes over time of a physical system in which quantum effects, such as wave-particle duality, are significant. These systems are referred to as quantum (mechanical) systems. The equation is considered a central result in the study of quantum systems, and its derivation was a significant landmark in the development of the theory of quantum mechanics. It was named after Erwin Schrödinger, who derived the equation in 1925 and published it in 1926 [1].

In 1952, Bohm [2] attempted to justify the Schrödinger equation with the existence of hidden variables. In 1966, Nelson [3] suggested a classical formulation of the equation, based on the hypothesis that every particle of mass m is subject to a Brownian motion with a diffusion coefficient $\hbar/2m$ and no friction.

Ever since, many solutions have been given to the Schrödinger equation, for various physical systems [4-10].

In this work, a formulation of quantum mechanics without use of complex numbers is described and is shown to be equivalent to the complex formulation. Moreover, dealing with the real and imaginary parts separately, a single particle shows a 2-entities behavior. As a result, a double-string description is suggested, whereby the Schrödinger equation emerges naturally. This double-string description assumes a time-dependent decaying tension in the strings, together with

a time-dependent interaction between the two strings. If the decay pattern is similar for both tension and interaction, their ratio is shown to be $\hbar/2$.

2. A Non-relativistic Schrödinger Equation

In non-relativistic quantum mechanics, a particle (such as an electron or proton) is described by a complex wave-function, $\psi(x, t)$, whose time evolution is governed by the Schrödinger equation [9]:

$$-i\hbar \frac{\partial}{\partial t} \psi(x, t) = \mathcal{H}\psi(x, t) \\ = -\frac{\hbar}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t) \quad (1)$$

Here, m is the particle's mass and $V(x)$ is the applied potential. Physical information about the behavior of the particle is extracted from the wave function by constructing expected values for various quantities; for example, the expected value of the particle's position is given by integrating $\psi^*(x) \times x \psi(x)$ over all of space, and the expected value of the particle's momentum is found by integrating $-i\hbar \psi^*(x) \partial \psi / \partial x$. The quantity $\psi^*(x)\psi(x)$ is itself interpreted as a probability density function. This treatment of quantum mechanics, where a particle's wave function evolves against a classical background potential $V(x)$, is sometimes called first quantization.

$\psi(x, t)$ may be written as a linear combination of the functions $\varphi_i(x, t)$, which are solutions of $\mathcal{H}\varphi_i(x, t) = i\hbar \frac{\partial \varphi_i(x, t)}{\partial t}$.

$\varphi_i(x, t)dx$ may be interpreted as the probability density of a single particle in the i^{th} energy state, to be found at an interval dx around position x .

* Corresponding author:

doron.kwiat@gmail.com (Doron Kwiat)

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3. Real Wave Functions

The introduction of imaginary and complex numbers is nothing but a mathematical convenience for shortening and unifying our equations. Physical problems cannot rely on imaginary numbers, except for the purpose of mathematical comfort. Obviously, since complex entities can always be decomposed into real and imaginary parts and vice versa, using natural (real-entities only) approach does not contribute anything and makes things unjustifiably more complex. But, by using complex presentation we might be overlooking some hidden qualities.

The basic equation of quantum mechanics is the famous one-particle time-dependent Schrödinger equation:

$$-i\hbar \frac{\partial}{\partial t} \psi(x, t) = \mathcal{H}\psi(x, t) \quad (2)$$

where i is the imaginary number $\sqrt{-1}$, \hbar is the reduced Planck constant which is $h/2\pi$, the symbol $\partial/\partial t$ indicates a partial derivative with respect to time t , $\psi(x, t)$ (the Greek letter psi) is the wave function of the quantum system, x is the position in a one-dimensional coordinate system, and t the time. \mathcal{H} is the Hamiltonian operator (which characterizes the total energy of the system under consideration).

Where

$$\mathcal{H} = -\frac{p^2}{2m} + v(x) \quad (3)$$

with $p = -i\hbar \frac{\partial}{\partial x}$ and where $v(x) = v_1(x) + iv_2(x)$ is the potential energy, which may also have real and imaginary parts v_1 and v_2 , respectively.

As described by Hatfield [16], One may separate the complex wave function into its real and imaginary components.

$$\psi(x, t) = \varphi_1 + i \varphi_2 \quad (4)$$

Using a vector notation may be useful:

$$|\Psi\rangle = \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} \quad (5)$$

and the Schrödinger equation may be written:

$$-i\hbar \frac{\partial}{\partial t} |\Psi\rangle = H|\Psi\rangle \quad (6)$$

or

$$-i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix}$$

When separated into real and imaginary components, it is equivalent to:

$$\frac{\partial \varphi_1}{\partial t} = +\frac{\hbar}{2m} \frac{\partial^2 \varphi_2}{\partial x^2} - v_2(x)\varphi_1 - v_1(x)\varphi_2 \quad (7)$$

$$\frac{\partial \varphi_2}{\partial t} = -\frac{\hbar}{2m} \frac{\partial^2 \varphi_1}{\partial x^2} + v_1(x)\varphi_1 - v_2(x)\varphi_2 \quad (8)$$

This form indicates the coupling that exists between the two constituent entities, which comprise the "single" particle.

We remember that $\psi(x, t)dx$ is interpreted as the probability density of finding a single particle at an interval dx around position x .

Likewise, since $\psi(x, t) = \varphi_1 + i \varphi_2$ and $\int \psi^*(x, t)\psi(x, t)dx = 1$ then

$$\begin{aligned} \int (\varphi_1 - i \varphi_2) * (\varphi_1 + i \varphi_2) dx \\ = \int \int (\varphi_1^2 + \varphi_2^2) dx = 1 \end{aligned} \quad (9)$$

Here, all integrals over $f(x)$ are definite integrals over an interval of length $L \rightarrow \infty$:

$$\int dx f(x) \equiv \frac{1}{L} \int_{-L/2}^{+L/2} f(x) dx \quad (10)$$

Therefore, one may interpret this as two particles interacting with each other, whose probability densities φ_1^2 and φ_2^2 are affected by each other, and yet, they sum up to 1. However, we cannot tell their probabilities apart.

The Schrödinger equation is now interpreted as two probability density functions coupled according to eqs. [7] and [8].

The complex wave equation of a single particle, as described by the Schrödinger equation, is actually a mathematical description of two real wave functions. A single particle may be interpreted as a coupled two-waves entity (particles?).

Surprising Results

Presentation of quantum mechanics with real wave-functions does not produce contradicting results to the complex presentation. This is obvious, as complex presentation is nothing but a mathematical manipulation of real variables.

However, when using real wave functions, some hidden characteristics emerge.

We shall see in the following, that the two real entities of which the particle comprises of, behave in an unexpected manner.

4. Linear Momentum

In terms of quantum mechanics, we have for the particle's momentum:

$$\hat{p}_x |\Psi\rangle = \hat{p}_x \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} = -i\hbar \frac{\partial}{\partial x} \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} \quad (11)$$

Looking at the real and imaginary parts separately, we see that this is equivalent to:

$$\hat{p}_x \varphi_1 = \text{Real} \left(-i\hbar \frac{\partial}{\partial x} \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} \right) = +\hbar \frac{\partial}{\partial x} \varphi_2 \quad (12)$$

$$\hat{p}_x \varphi_2 = \text{Im} \left(-i\hbar \frac{\partial}{\partial x} \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} \right) = -\hbar \frac{\partial}{\partial x} \varphi_1 \quad (13)$$

Using equations [7, 8], the Schrödinger equation for a free particle ($v(x)=0$) is

$$\frac{\partial}{\partial t} \varphi_1 = +\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \varphi_2 \quad (14)$$

$$\frac{\partial}{\partial t} \varphi_2 = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \varphi_1 \quad (15)$$

Multiply eq [14] and [15] by φ_1 and φ_2 respectively, one obtains:

$$\frac{1}{2} \frac{\partial[\varphi_1^2 + \varphi_2^2]}{\partial t} = \frac{\hbar}{2m} \left(\varphi_1 \frac{\partial^2 \varphi_2}{\partial x^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial x^2} \right) \quad (16)$$

Integrate over x, on the l.h.s

$$\int \frac{1}{2} \frac{\partial[\varphi_1^2 + \varphi_2^2]}{\partial t} dx = \frac{1}{2} \frac{\partial}{\partial t} \int (\varphi_1^2 + \varphi_2^2) dx = 0 \quad (17)$$

Therefore

$$\int \left(\varphi_1 \frac{\partial^2 \varphi_2}{\partial x^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial x^2} \right) dx = 0 \quad (18)$$

Also,

$$\varphi_1 \frac{\partial^2 \varphi_2}{\partial x^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial x^2} = \frac{\partial}{\partial x} \left(\varphi_1 \frac{\partial \varphi_2}{\partial x} - \varphi_2 \frac{\partial \varphi_1}{\partial x} \right) - \left[\frac{\partial \varphi_1}{\partial x}, \frac{\partial \varphi_2}{\partial x} \right] \quad (19)$$

and since $\left[\frac{\partial \varphi_1}{\partial x}, \frac{\partial \varphi_2}{\partial x} \right] = 0$ we obtain:

$$\begin{aligned} \int \left(\varphi_1 \frac{\partial^2 \varphi_2}{\partial x^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial x^2} \right) dx &= \\ \int \frac{\partial}{\partial x} \left(\varphi_1 \frac{\partial \varphi_2}{\partial x} - \varphi_2 \frac{\partial \varphi_1}{\partial x} \right) dx &= \varphi_1 \frac{\partial \varphi_2}{\partial x} - \varphi_2 \frac{\partial \varphi_1}{\partial x} + C_0 \end{aligned} \quad (20)$$

Hence, combining this with the result in [18] gives:

$$\varphi_1 \frac{\partial \varphi_2}{\partial x} - \varphi_2 \frac{\partial \varphi_1}{\partial x} + C_0 = \int \left(\varphi_1 \frac{\partial^2 \varphi_2}{\partial x^2} - \varphi_2 \frac{\partial^2 \varphi_1}{\partial x^2} \right) dx = 0 \quad (21)$$

and therefore

$$\varphi_1 \frac{\partial \varphi_2}{\partial x} = \varphi_2 \frac{\partial \varphi_1}{\partial x} - C_0 \quad (22)$$

(unless we put the integration limits at $\pm L/2$, we cannot omit the integration constant).

If $\langle p_{x1} \rangle \equiv \int \varphi_1 \hat{p}_x \varphi_1 dx$ represents the observable momentum p_{x1} and $\langle p_{x2} \rangle \equiv \int \varphi_2 \hat{p}_x \varphi_2 dx$ represents the observable momentum p_{x2} , we obtain:

$$\langle p_{x1} \rangle \equiv \int \varphi_1 \hat{p}_x \varphi_1 dx = -\hbar \int \varphi_1 \frac{\partial}{\partial x} \varphi_2 dx \quad (23)$$

$$\langle p_{x2} \rangle \equiv \int \varphi_2 \hat{p}_x \varphi_2 dx = +\hbar \int \varphi_2 \frac{\partial}{\partial x} \varphi_1 dx \quad (24)$$

Summing equations [23] and [24], and using [22] results in:

$$\langle p_{x1} \rangle = -\langle p_{x2} \rangle + C_0/\hbar \quad (25)$$

The total momentum is:

$$\langle p_x \rangle = \langle p_{x1} \rangle + \langle p_{x2} \rangle = C_0/\hbar \quad (26)$$

And if we designate the total momentum by p , then $p = C_0/\hbar$.

For zero total momentum, $p = 0$, we see that

$$\langle p_{x1} \rangle = -\langle p_{x2} \rangle \quad (27)$$

Meaning that the two entities have opposite linear momentum.

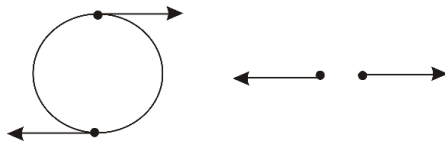


Figure 1. A combined particle of two opposite momenta, with either linear or circular motion

This can be resolved if the two constituents are spinning in opposite directions around a common center, or if the two

components are entangled and moving linearly away in opposite directions. But the latter solution will not allow us to observe them in one place simultaneously.

5. Angular Momentum

Assuming spherical symmetry, with polar angle θ , the z-component of its angular momentum in spherical coordinates is $\hat{L}_z = -i\hbar \frac{\partial}{\partial \theta}$.

To find the eigenvalues we write:

$$\hat{L}_z \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} = -i\hbar \frac{\partial}{\partial \theta} \begin{pmatrix} \varphi_1 \\ i\varphi_2 \end{pmatrix} \quad (28)$$

$$\hat{L}_z \varphi_1 = \hbar \frac{\partial \varphi_2}{\partial \theta} \quad (29)$$

$$\hat{L}_z \varphi_2 = -\hbar \frac{\partial \varphi_1}{\partial \theta} \quad (30)$$

Suppose m_1, m_2 , are the angular momentum eigenvalues for each entity of the particle:

$$\hat{L}_z \varphi_1 = m_1 \varphi_1 \quad (31)$$

$$\hat{L}_z \varphi_2 = m_2 \varphi_2 \quad (32)$$

$$\frac{\partial \varphi_1}{\partial \theta} = -\frac{m_2}{\hbar} \varphi_2 \quad (33)$$

$$\frac{\partial \varphi_2}{\partial \theta} = +\frac{m_1}{\hbar} \varphi_1 \quad (34)$$

Equations [34] and [35] can be immediately solved to give:

$$\varphi_1(\theta) = a_1 \cos\left(\frac{m_1 m_2}{\hbar^2} \theta\right) + b_1 \sin\left(\frac{m_1 m_2}{\hbar^2} \theta\right) \quad (35)$$

$$\varphi_2(\theta) = a_2 \cos\left(\frac{m_1 m_2}{\hbar^2} \theta\right) + b_2 \sin\left(\frac{m_1 m_2}{\hbar^2} \theta\right) \quad (36)$$

Assuming a boundary condition $\varphi_1(0) = \varphi_1(0) = 0$

We find

$$\frac{\varphi_1(\theta)}{\varphi_2(\theta)} = \frac{b_1}{b_2} \quad (37)$$

And therefore

$$\frac{\partial(\frac{\varphi_1}{\varphi_2})}{\partial \theta} = 0 \quad (38)$$

And therefore

$$\varphi_1 \frac{\partial \varphi_2}{\partial \theta} = \varphi_2 \frac{\partial \varphi_1}{\partial \theta} \quad (39)$$

$$\langle L_{z1} \rangle \equiv \int \varphi_1 \hat{L}_z \varphi_1 d\theta = \hbar \int \varphi_1 \frac{\partial \varphi_2}{\partial \theta} d\theta \quad (40)$$

$$\langle L_{z2} \rangle \equiv \int \varphi_2 \hat{L}_z \varphi_2 d\theta = -\hbar \int \varphi_2 \frac{\partial \varphi_1}{\partial \theta} d\theta \quad (41)$$

We conclude

$$\langle L_{z1} \rangle = -\langle L_{z1} \rangle \quad (42)$$

Thus, independently of the composite spin for this particle, its two components (entities) will have an opposite spin.

6. A 2-string Analog to the Schrödinger Equation

Assume a one-dimensional description in x, and time t.

Let a single particle of mass m be described as consisting of two classical real strings, $\varphi_1(x, t)$ and $\varphi_2(x, t)$. Here, the functions, $\varphi_1(x, t)$ and $\varphi_2(x, t)$ represent the amplitudes of the perturbation of the strings from the x -axis as a function of time and position.

Let the coupling between these two strings be given a constant K_s and described by the following coupled, differential equations:

$$\frac{\partial \varphi_1}{\partial t} = +k_s \frac{\partial^2 \varphi_2}{\partial x^2} \quad (43)$$

$$\frac{\partial \varphi_2}{\partial t} = -k_s \frac{\partial^2 \varphi_1}{\partial x^2} \quad (44)$$

Let us present a physical consideration of these two equations.

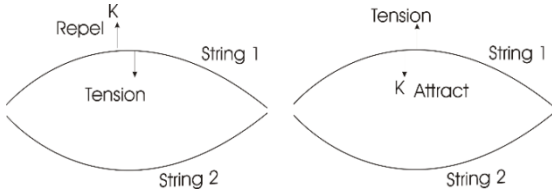


Figure 2. Closed-loop double-strings. Attraction, or repulsion and tension, will change directions according to the deviation from the steady-state

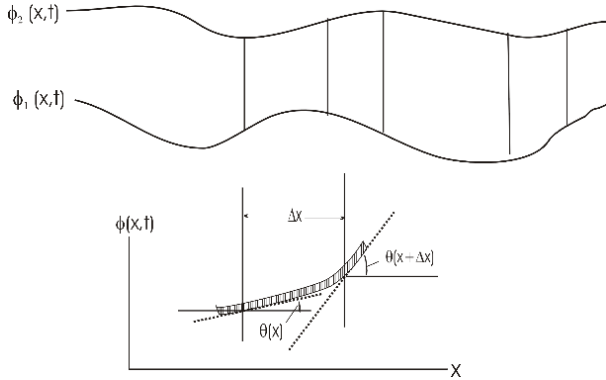


Figure 3. A double-string analog, as described in Eqs. [43, 44]. The angle θ describes the tangent, in the approximation of $\tan \theta \approx \sin \theta$, for $\theta \ll 1$

Consider two strings φ_1 and φ_2 .

Let $\varphi_1(x, t)$ represent the displacement in the string at time t and position x . Let τ be the tension in the string. Though we start in a classical model, this can be directed to modern string models too [18]. It is well known that the force exerted by this tension on a string element X (see figure 3) is described by:

$$\tau \frac{\partial^2 \varphi_1(x, t)}{\partial x^2} \quad (45)$$

Suppose now that this string is connected by an endless number of springs to a second string, which is described by $\varphi_2(x, t)$. Let the individual springs each have a spring constant K_s . Assume further that this spring constant is not fixed in time. Rather, it changes with time, starting at time t_0 , the moment in which the two strings interact. Suppose the two strings are bound together at their ends, thus creating a closed loop. Suppose further that the closed loop of strings is at an equilibrium state. The tension in both strings is assumed to be changing in time. There is an interaction force

exerted between the strings, pushing them apart or towards each other, against the tendency to collapse or to expand due to tension forces. This will happen when, for some reason, the two strings are perturbed from their steady-state equilibrium of tension and distance.

As the force tends to weaken or strengthen the interaction as it stretches apart or contracts from its steady state, it is therefore reasonable to assume that the spring constant changes as:

$$k_s = k_s(t) = \frac{K_{s0}}{t_0 + t} \quad (46)$$

We may then assume that the tension force in string 1 creates a disturbance from equilibrium $-\frac{\partial \varphi_2}{\partial t}$ in string 2, proportional to the spring constant K_s and to the string mass m_2 .

Since φ describes a displacement amplitude, it has the dimensions of $[m]$.

The force in Eq [45] has dimensions $[kg][m]/[sec^2]$.

We may then write

$$\tau_1 \frac{\partial^2 \varphi_1}{\partial x^2} = -k_s(t) m_2 \frac{\partial \varphi_2}{\partial t} = -\frac{K_{s0}}{t_0 + t} m_2 \frac{\partial \varphi_2}{\partial t} \quad (47)$$

Notice, that K_{s0} is dimensionless (or, has dimensions set according to the dimension of φ).

Rearranging [4] we obtain:

$$\frac{\partial \varphi_2}{\partial t} = -\frac{\tau_1}{K_{s0} m_2} \frac{\partial^2 \varphi_1}{\partial x^2} = -\frac{(t_0 + t) \tau_1}{K_{s0} m_2} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (48)$$

By the same argument,

$$\frac{\partial \varphi_1}{\partial t} = +\frac{\tau_2}{K_{s0} m_1} \frac{\partial^2 \varphi_2}{\partial x^2} = +\frac{(t_0 + t) \tau_2}{K_{s0} m_1} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (49)$$

For reasons of symmetry we assume $\tau_1 = \tau_2$, $m_1 = m_2$

Furthermore, we will assume also that the string tension τ , decays with time in the same manner as the interaction spring force k_s .

$$\text{Thus, } \tau = \tau(t) = \frac{\tau_0}{t + t_0}$$

The resulting two equations for the interacting strings are now:

$$\frac{\partial \varphi_1}{\partial t} = +\frac{\tau(t)(t_0 + t)}{K_{s0} m} \frac{\partial^2 \varphi_2}{\partial x^2} = +\frac{\tau_0}{K_{s0} m} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (50)$$

$$\frac{\partial \varphi_2}{\partial t} = -\tau \frac{\partial^2 \varphi_1}{\partial x^2} = -\frac{\tau_0}{K_{s0} m} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (51)$$

The resulting two equations

$$\frac{\partial \varphi_1}{\partial t} = +\frac{\tau_0}{K_{s0} m} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (52)$$

$$\frac{\partial \varphi_2}{\partial t} = -\frac{\tau_0}{K_{s0} m} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (53)$$

are similar to the coupled wave functions equations:

$$\frac{\partial \varphi_1}{\partial t} = +\frac{\hbar}{2m} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (54)$$

$$\frac{\partial \varphi_2}{\partial t} = -\frac{\hbar}{2m} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (55)$$

provided that

$$\hbar = \frac{2\tau_0}{K_{s0}} \quad (56)$$

This relates the initial tension τ_0 in the strings to the initial

coupling k_{s0} between the two strings. As τ_s and K_s are inversely proportional to time, the ratio $\tau_s/K_s = \frac{1}{2}\hbar$ is fixed in time. This interpretation of a particle as consisting of two coupled strings whose tensions and interaction are inversely proportional to time gives us a classic interpretation of the Schrödinger equation.

This relates the initial tension τ_0 in the strings to the initial coupling k_{s0} between the two strings.

As τ_s and K_s are inversely proportional to time, the ratio τ_s/K_s is constant in time.

We will assume now, without loss of generality

$$\frac{\tau_s}{K_s} = \frac{1}{2}\hbar \quad (57)$$

7. Asymptotic Correction

Since the denominators' ratio cancels out, the $\frac{\tau_s}{K_s}$ ratio remains constant in time, however each of them decays to 0 as $t \rightarrow \infty$.

In order to avoid the vanishing of tension and the mutual spring interaction, we may introduce a different time dependency:

$$k_s = k_s(t) = \frac{K_{s0}}{A+Bte^{-Ct}} \quad (58)$$

$$\tau = \tau(t) = \frac{\tau_0}{A+Bte^{-Ct}} \quad (59)$$

With A, a constant of dimension [sec].

The time behavior is now different, but the ratio $\frac{\tau_s}{K_s} = \frac{1}{2}\hbar$ remains constant in time as before. But now, the vanishing of the constants as $t \rightarrow \infty$ is avoided. We also notice that for both τ and K the parameters A, B, and C must be the same, otherwise the ratio will become time dependent.

Characteristics of the interaction

Define the function $f(t)$:

$$f(t) = \frac{1}{A+Bte^{-Ct}} \quad (60)$$

By investigating its behavior, we see that $f(t)$ is bounded and has a minimum at some t_{min} . After this minimum, it returns asymptotically to its initial value.

The upper bound on $f(t)$ is given by

$$\lim_{t \rightarrow \infty} f(t) = \frac{1}{A} \quad (61)$$

The minimum is obtained at $t_{min} = \frac{1}{C}$, and at minimum

$$f(t_{min}) = \frac{1}{A+\frac{B}{eC}} \quad (62)$$

And the following is also true:

$$f(\infty) = 1/A \quad (63)$$

$$f'(0) = B/A^2 \quad (64)$$

and

$$B = f'(0)/f(\infty)^2 \quad (65)$$

This may be interpreted as a double string initially at a steady state, with a constant tension τ . Then, for some reason, an interaction (disturbance) occurs and the tension

drops to minimum at time $t_{min} = 1/C$ and then gradually returns to its initial steady-state tension $\tau(0)$. A similar argument is true for the interaction constant K .

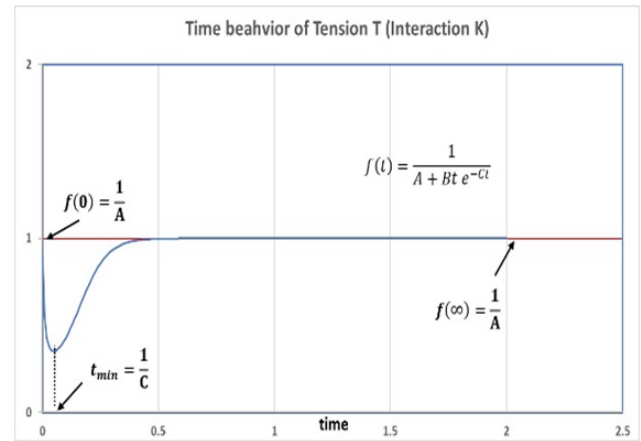


Figure 4. Time behavior of tension τ in a string and interaction constant K between two strings. Behavior is governed by the function $f(t)$ given in Eq [60]. Parameter A controls the level of stable-state equilibrium given by $1/A$. The amplitude drops for a while and rises back to equilibrium, with minimum obtained at time $t_{min} = 1/C$. (This figure used $A=1$, $B=100$, $C=20$)

This describes a single particle of mass m , whose interaction parameters at time $t < t_0$ are null: $B=C=0$.

At time t_0 an interaction occurs, whose intensity energy E_i may be estimated by $f(t_{min})^2$ or in other words:

$$E_i \sim f(t_{min})^2 = \frac{1}{(A+\frac{B}{eC})^2} \quad (66)$$

We see that the particle has a fixed amplitude for its component strings until the time of perturbation (collision?). The perturbation energy E_i creates a temporary displacement of amplitudes in the strings, which asymptotically return to their original equilibrium steady-state.

Complex wave-functions

Equations [54], [55] then give:

$$\frac{\partial \varphi_1}{\partial t} = + \frac{\hbar}{2m} \frac{\partial^2 \varphi_2}{\partial x^2} \quad (67)$$

$$\frac{\partial \varphi_2}{\partial t} = - \frac{\hbar}{2m} \frac{\partial^2 \varphi_1}{\partial x^2} \quad (68)$$

Defining the complex function $\psi(x, t) = \varphi_1 + i \varphi_2$

Eqs [67] and [68] may be combined together:

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \quad (69)$$

By multiplying [67] and [68] by φ_2 and φ_1 respectively, integrating over x , and imposing the mixed boundary conditions $\partial \psi / \partial x$, $\psi = 0$ on both functions at $L \rightarrow \pm \infty$, it is immediately shown that (up to a normalization factor):

$$\int (\varphi_1^2 + \varphi_2^2) dx = 1 \quad (70)$$

Hence,

$$\int \psi^*(x, t) \psi(x, t) dx = 1 \quad (71)$$

Thus, the classic coupled string system may be interpreted as a quantum mechanical single particle, described by a wave function $\psi(x, t)$ which is a probability distribution function. This function describes a free particle of momentum

$$p = -i\hbar \frac{\partial}{\partial x} \quad (72)$$

When substituted in

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

The result is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{p^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = \mathcal{H}\psi, \quad (73)$$

A free particle Hamiltonian.

8. Interpretation

This interpretation of a particle as made up of two real coupled strings whose tensions and interaction are inversely proportional to time is equivalent to the single-particle complex wave function described by the Schrödinger equation. When the original Schrödinger equation is used with a complex wave function, this internal string-like characteristic is not apparent because we treat the real and imaginary parts as a single entity. However, when the equation is separated into two parts, these two parts can be treated as independent strings interacting with each other, where both tension and interaction fall off abruptly and inversely proportional to time. This may be a result of weakening due to increased distance between the two strings, together with a drop in tension inside the strings.

Since each of the wave functions is now a real function of time and position, and since the sum of the squares of the two wave functions equals 1 (Eq [70]), these wave functions can describe the actual strings on one hand and be interpreted as probability distribution functions on the other hand. This unifies the two interpretations into one: A single-particle quantum mechanical description with a single complex wave function, which is equivalent to a classic description of two interacting strings. The quantum quality emerges when we understand that the ratio between the string tension and the interaction constant are time dependent and the ratio is constant in time, namely the Planck universal constant, \hbar .

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The result is the familiar Schrödinger equation.

9. Connection of Strings Structure to Gravity

Tension τ_s and the Planck constant \hbar , may be connected with the α' (alpha prime) parameter [19] and the

gravitational constant G .

Particles in a string theory are like the harmonic notes played on a string with a fixed tension

$$\tau_s = \frac{1}{2\pi\alpha'}$$

The parameter α' is called the string parameter and the square root of this number represents the approximate distance scale at which string effects should become observable.

Since $\tau_s = \frac{1}{2\pi\alpha'}$, and as the minimum observable length for a quantum string theory is given by

$$L_{min} = \sqrt{\alpha'} \quad (74)$$

then, if string theory is a theory of quantum gravity, this minimum length scale should be at least the size of the Planck length. Since it is the length scale made by the combination of Newton's gravitational constant G , the speed of light c and Planck's constant, \hbar

$$L_{min} = L_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \quad (75)$$

Thus

$$\tau_s = k_s \frac{4c^3}{\hbar G} \quad (76)$$

This indicates to the connection between the structure of the string structure to the three known constants of nature.

Expanding to other particles

If we accept the claim that a single, non-relativistic Schrödinger equation is a description of a single massive particle made up of two coupled strings, then we may project from this to the relativistic Dirac equation:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0 \quad (77)$$

Instead of a 4-spinor solution we may assume an 8-string solution, with each spinor made up of 2 coupled strings. To do this we separate the complex wave function Ψ into its real and imaginary parts, and by using them in the Dirac equation we will obtain two separate equations where the imaginary and real parts are separated. This same procedure can be applied to any quantum field Lagrangian, resulting in the doubling of the number of equations on one hand, but gaining a separation of the equations into real wave functions on the other. These wave functions may be interpreted as coupled strings.

10. A Balloon Model

In a 3D version, we can illustrate our concept with a balloon version. The spherical balloon has an internal pressure P that tends to inflate the balloon by exerting force perpendicular to the surface area. The surface area of the balloon has an internal tension force τ acting in the opposite direction, also perpendicular to the surface area, but opposing the force of internal pressure.

At equilibrium, the forces of inward tension and outward pressure are equal.

If at a given instance t_0 , a small disturbance occurs that moves the surface off its equilibrium, the force of tension will increase along with the pressure, which will bring the surface back to equilibrium. If, on the other hand, the balloon surface is pushed outwards, the pressure will decrease and the tension will increase in the opposite direction.

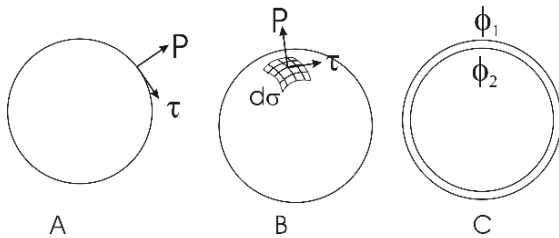


Figure 5. A closed string with "pressure" P and tension τ (A) and a 2-dimensional surface area of a balloon-like string, with forces acting on an infinitesimal area $d\sigma$ (B). For our model, we may assume two separate, closely packed and interacting (coupled) balloons (C), denoted by ϕ_1 and ϕ_2 .

Pressure force P and tension force τ are always perpendicular to the surface area and in opposite directions. Both P and τ will change in time due to some unsettling interaction. In its steady state, the balloon may be expanding or contracting at a steady rate but the ratio τ/P must be kept constant.

Instead of a coupled strings model, we may assume double sphere-like surfaces, one enclosed inside the other, with a very small gap between the two. Both have surface tension τ and they interact with each other with an interaction force K .

11. Conclusions

One can do quantum mechanics with real wave functions instead of complex ones.

Using real wave functions leads to the interpretation of a single particle as consisting of two interacting entities. This coupled-strings interaction model leads to the Schrödinger equation for a single particle.

Assuming Tension τ in a string and interaction force K between strings are time dependent, then the ratio τ/K asymptotically equals Planck's constant \hbar . This gives us a derivation of Planck's constant out of fundamental interaction physical model. Even though these interactions may become perturbed by some external interactions, it is shown here that these sudden perturbations affect the coupling only temporarily, and the system of coupled strings returns asymptotically to a steady-state equilibrium.

A single particle may consist of coupled 2D spherical sheets (membranes) of a given surface tension and mutual interaction force; both are time dependent and asymptotically return to equilibrium.

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