

# Riemannian Quantum Theory of a Particle in a Finite-Potential Well

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**Abstract** In this article, the golden Riemannian Laplacian operator was constructed using the golden metric tensor in spherical polar coordinate and was applied to the Schrodinger wave equation in order to obtain the golden Riemannian Schrodinger equation for a particle in a finite-potential well. The results are that the golden Riemannian Laplacian operator and golden Riemannian Schrodinger equation were augmented with additional correction terms; which are not found in the existing equations and can be applied to a finite-potential well problem, so as to obtain the expression for the allowed energy values.

**Keywords** Schrodinger wave equation, Spherical polar coordinate, Golden Riemannian, Laplacian operator, Golden metric tensor

## 1. Introduction

Quantum Physics is the study of Physics in the realm of atoms, nuclei and elementary particles [1]. This statement does not mean that quantum physics has nothing to do with the macroscopic world. In fact, all of physics is quantum physics and classical physics is a constrictive case of quantum physics.

The origin of quantum physics occupies a time period in history that covers a quarter of a century. Classical or Newtonian mechanics was available in the powerful formulations of Lagrange and Hamilton by the year 1900. Thus, classical electromagnetic theory was embodied in the differential equations of Maxwell. Defects were, however, made clear by the failure of the classical theories to explain some experimental results, notably, the frequency dependence of the intensity of radiation emitted by a blackbody, the photoelectric effect and the stability and size of atoms [2].

Quantum Physics came to existence in 1900 when a famous pronouncement was put forward by Planck to unfold and illustrate the meaning of the observed properties of the radiation ejected by a blackbody [3]. This phenomenon posed an unsolved problem to theoretical physicists for several decades.

Principles of thermodynamics and electromagnetism had been applied to the problem but, these classical methods had failed to give a sensible explanation of the experimental results [11, 1].

The quantum hypothesis of Planck and the subsequent interpretation of the idea by Einstein in 1905 gave electromagnetic radiation discrete properties; somewhat similar to those of a particle. The quantum theory made provision for radiation to have both wave and particle aspects in a complementary form of coexistences. The theory was extended when matter was found to have wave characteristics as well as particle properties by de Broglie in 1923 [9]. These notions continued to evolve until 1925 when the formal apparatus of quantum theory came into being.

The discovery of the wave like behavior of an electron created the need for a wave theory describing the behavior of a particle on the atomic scale. This theory was proposed by Schrodinger in the year 1926, two years after De Broglie formulated the idea of a particle wave nature [8]. Schrodinger reasoned that if an electron behaves as a wave, then it should be possible to mathematically describe the electrons behavior in space time coordinate as a wave.

The Schrodinger proposed theory; yielded the fundamental equation of quantum mechanics known as the Schrodinger wave equation. This equation has the same central importance to quantum mechanics as Newton's law of motion has for classical mechanics [10].

Howusu (2003) stated that, since 1926, there has been the need to derive one theory of quantum mechanics from or corresponding to Einstein's theory of special relativity for the description of the wave behavior of all entities in electromagnetic fields; since Schrodinger's Dynamical

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theory of quantum mechanics is related to Newton's Dynamical theory of classical mechanics.

On this basis, the quantum theory of the finite-potential well was constructed using the Riemannian golden metric tensor.

## 2. Theoretical Analysis

### 2.1. Derivation of Riemannian Laplacian Operator in Spherical Polar Coordinate Based upon the Golden Metric Tensor

Consider a particle of mass,  $m$  in a finite-potential well of width,  $a$  and depth,  $V_0$ .

The Riemannian Laplacian operator [12, 6] is given by

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left\{ \sqrt{g} \cdot g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right\} \quad (1)$$

where  $g_{\mu\nu} \equiv$  metric and  $g =$  determinant of  $g_{\mu\nu}$

The Golden Riemannian metric tensors in spherical polar coordinate [6, 7] are given by

$$g_{11} = \left(1 + \frac{2}{c^2} f\right)^{-1} \quad (2)$$

$$g_{22} = r^2 \left(1 + \frac{2}{c^2} f\right)^{-1} \quad (3)$$

$$g_{33} = r^2 \sin^2 \theta \left(1 + \frac{2}{c^2} f\right)^{-1} \quad (4)$$

$$g_{00} = - \left(1 + \frac{2}{c^2} f\right) \quad (5)$$

$$g_{\mu\nu} = 0; \text{ otherwise} \quad (6)$$

and

$$g = r^4 \sin^2 \theta \left(1 + \frac{2}{c^2} f\right)^{-2} \quad (7)$$

$$\sqrt{g} = r^2 \sin \theta \left(1 + \frac{2}{c^2} f\right)^{-1} \quad (8)$$

From equation (1) we have:

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{ \sqrt{g} \cdot g^{11} \frac{\partial}{\partial x^1} \right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{ \sqrt{g} \cdot g^{22} \frac{\partial}{\partial x^2} \right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{ \sqrt{g} \cdot g^{33} \frac{\partial}{\partial x^3} \right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} \left\{ \sqrt{g} \cdot g^{00} \frac{\partial}{\partial x^0} \right\} \quad (9)$$

If we let

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{ \sqrt{g} \cdot g^{11} \frac{\partial}{\partial x^1} \right\}, \\ \beta &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{ \sqrt{g} \cdot g^{22} \frac{\partial}{\partial x^2} \right\}, \\ \gamma &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{ \sqrt{g} \cdot g^{33} \frac{\partial}{\partial x^3} \right\} \text{ and} \\ \xi &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} \left\{ \sqrt{g} \cdot g^{00} \frac{\partial}{\partial x^0} \right\} \end{aligned}$$

Equation (9) reduces to

$$\nabla_R^2 = \alpha + \beta + \gamma + \xi \quad (10)$$

$$\text{For } \alpha = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{ \sqrt{g} \cdot g^{11} \frac{\partial}{\partial x^1} \right\} \quad (11)$$

To obtain  $\alpha$  in spherical polar coordinate, we substitute equations (2) and (7) into equation (11) as follows:

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{ \sqrt{g} \cdot g^{11} \frac{\partial}{\partial x^1} \right\} \\ &= \frac{1}{r^2 \sin \theta \left(1 + \frac{2}{c^2} f\right)^{-1}} \frac{\partial}{\partial r} \left\{ r^2 \sin \theta \left(1 + \frac{2}{c^2} f\right)^{-1} \cdot \left(1 + \frac{2}{c^2} f\right) \frac{\partial}{\partial r} \right\} \\ &= \frac{1}{r^2 \sin \theta \left(1 + \frac{2}{c^2} f\right)^{-1}} \frac{\partial}{\partial r} \left\{ r^2 \sin \theta \frac{\partial}{\partial r} \right\} \\ &= \frac{1}{r^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial}{\partial r} \right\} \\ \alpha &= \frac{1}{r^2} \left(1 + \frac{2}{c^2} f\right) \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial}{\partial r} \right\} \end{aligned} \quad (12)$$

$$\text{For } \beta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{ \sqrt{g} \cdot g^{22} \frac{\partial}{\partial x^2} \right\} \quad (13)$$

To obtain  $\beta$  in spherical polar coordinate, we substitute equations (3) and (7) into equation (13) as follows:

$$\begin{aligned} \beta &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{ \sqrt{g} \cdot g^{22} \frac{\partial}{\partial x^2} \right\} \\ &= \frac{1}{r^2 \sin \theta \left(1 + \frac{2}{c^2} f\right)^{-1}} \frac{\partial}{\partial \theta} \left\{ r^2 \sin \theta \left(1 + \frac{2}{c^2} f\right)^{-1} \cdot \left(1 + \frac{2}{c^2} f\right) \frac{1}{r^2} \frac{\partial}{\partial \theta} \right\} \\ \beta &= \frac{1}{r^2 \sin \theta} \left(1 + \frac{2}{c^2} f\right) \frac{\partial}{\partial \theta} \left\{ \sin \theta \frac{\partial}{\partial \theta} \right\} \end{aligned} \quad (14)$$

$$\text{For } \gamma = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{ \sqrt{g} \cdot g^{33} \frac{\partial}{\partial x^3} \right\} \quad (15)$$

To obtain  $\gamma$  in spherical polar coordinate, we substitute equations (4) and (7) into equation (15) as follows:

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{ \sqrt{g} \cdot g^{33} \frac{\partial}{\partial x^3} \right\} \\
&= \frac{1}{r^2 \sin \theta} \left( 1 + \frac{2}{c^2} f \right) \frac{\partial}{\partial \theta} \left\{ r^2 \sin \theta \left( 1 + \frac{2}{c^2} f \right)^{-1} \cdot \left( 1 + \frac{2}{c^2} f \right) \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \right\} \\
\gamma &= \frac{1}{r^2 \sin^2 \theta} \left( 1 + \frac{2}{c^2} f \right) \frac{\partial}{\partial \theta} \left\{ \frac{\partial}{\partial \theta} \right\}
\end{aligned} \tag{16}$$

$$\text{For } \xi = \frac{1}{\sqrt{g}} \frac{1}{\partial x^0} \left\{ \sqrt{g} \cdot g^{00} \frac{\partial}{\partial x^0} \right\} \tag{17}$$

To obtain  $\gamma$  in spherical polar coordinate, we substitute equations (5) and (7) into equation (17) as follows:

$$\begin{aligned}
\xi &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} \left\{ \sqrt{g} \cdot g^{00} \frac{\partial}{\partial x^0} \right\} \\
&= -\frac{1}{r^2 \sin \theta} \left( 1 + \frac{2}{c^2} f \right) \frac{\partial}{\partial x^0} \left\{ r^2 \sin \theta \left( 1 + \frac{2}{c^2} f \right)^{-1} \cdot \left( 1 + \frac{2}{c^2} f \right)^{-1} \frac{\partial}{\partial x^0} \right\} \\
\xi &= -\left( 1 + \frac{2}{c^2} f \right)^{-1} \frac{\partial}{\partial x^0} \left\{ \frac{\partial}{\partial x^0} \right\}
\end{aligned} \tag{18}$$

Substituting equations (12), (14), (16) and (18) into equation (10), we have thus:

$$\begin{aligned}
\nabla_R^2 &= \frac{1}{r^2} \left( 1 + \frac{2}{c^2} f \right) \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial}{\partial r} \right\} + \frac{1}{r^2 \sin \theta} \left( 1 + \frac{2}{c^2} f \right) \frac{\partial}{\partial \theta} \left\{ \sin \theta \frac{\partial}{\partial \theta} \right\} \\
&\quad + \frac{1}{r^2 \sin^2 \theta} \left( 1 + \frac{2}{c^2} f \right) \frac{\partial}{\partial \theta} \left\{ \frac{\partial}{\partial \theta} \right\} - \left( 1 + \frac{2}{c^2} f \right)^{-1} \frac{\partial}{\partial x^0} \left\{ \frac{\partial}{\partial x^0} \right\}
\end{aligned} \tag{19}$$

Equation (19) is the golden Riemannian Laplacian operator in spherical polar coordinate. The well-known Laplacian operator is derived based on Euclidean geometry while equation (19) is derived based on the Riemannian geometry using the golden metric tensor. This equation is further applied to the Schrodinger equation in order to obtain the golden Riemannian Schrodinger equation.

## 2.2. Derivation of Golden Riemannian Schrodinger Equation in Spherical Polar Coordinate

Consider the well-known Schrodinger equation [4, 5] given by

$$E\psi = H\psi = \frac{-\hbar^2 \nabla^2}{2m} \psi + V(r)\psi \tag{20}$$

where  $E$  is energy of the particle,  $H$  is Hamiltonian of the system,  $m$  is mass of the particle,  $\hbar$  is normalized Planck's constant,  $\nabla^2$  is Euclidean Laplacian of the system,  $V$  is particle potential and  $\psi$  is wave function.

We replace the Euclidean Laplacian operator with the golden Riemannian Laplacian operator in equation (19); that is:

$$E\psi = H\psi = \frac{-\hbar^2 \nabla_R^2}{2m} \psi + V(r)\psi \tag{21}$$

Substituting the expression for the Riemannian Laplacian operator,  $\nabla_R^2$  into equation (21), we obtain

$$\begin{aligned}
H\psi &= \\
&= -\frac{\hbar^2}{2m} \left\{ \frac{1}{r^2} \left( 1 + \frac{2f}{c^2} \right) \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left( 1 + \frac{2f}{c^2} \right) \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \left( 1 + \frac{2f}{c^2} \right) \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) - \left( 1 + \frac{2}{c^2} f \right)^{-1} \frac{\partial}{\partial x^0} \left( \frac{\partial}{\partial x^0} \right) \right\} \\
&\quad \psi(r, t) + V \psi(r, t)
\end{aligned} \tag{22}$$

Expanding equation (22) and considering that  $V = V_0$  which is the depth of the potential well, we obtain

$$\begin{aligned}
i\hbar \left( \frac{\partial}{\partial \theta} \psi(r, \theta, \phi, x^0) \right) &= -\frac{\hbar^2 \eta}{mr} \left( \frac{\partial}{\partial r} \psi(r, \theta, \phi, x^0) \right) - \frac{\hbar^2 \eta}{2m} \left( \frac{\partial^2}{\partial r^2} \psi(r, \theta, \phi, x^0) \right) - \frac{\hbar^2 \eta \cos \theta}{2mr^2 \sin \theta} \left( \frac{\partial}{\partial \theta} \psi(r, \theta, \phi, x^0) \right) \\
&\quad - \frac{\hbar^2 \eta}{2mr^2} \left( \frac{\partial^2}{\partial \theta^2} \psi(r, \theta, \phi, x^0) \right) - \frac{\hbar^2 \eta}{2mr^2 \sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} \psi(r, \theta, \phi, x^0) \right) - \frac{\hbar^2}{2m\eta} \left( \frac{\partial^2}{\partial (x^0)^2} \psi(r, \theta, \phi, x^0) \right) + V_0 \psi(r, t)
\end{aligned} \tag{23}$$

$$\text{where } \eta = \left( 1 + \frac{2}{c^2} f \right) \tag{24}$$

Equation (23) is the golden Riemannian Schrodinger equation in spherical polar coordinates.

### 3. Discussion

Equation (4.23) is the golden Riemannian Schrodinger equation in spherical polar coordinates.

Equation (4.23) contains  $\left(1 + \frac{2}{c^2} f\right)$  which is not found in the well-known Schrodinger equation [9]. This term is the golden Riemannian term.

The terms in braces are the additional correction terms which are not found in the existing Schrodinger equation. This additional correction terms give better explanation of the energy of a particle in a finite potential well.

This can also be applied to all entities of non-zero rest mass such as: infinite potential well, rectangular potential well, simple harmonic oscillator etc.

This equation can be analytically solved in order to obtain the particle energies in a finite potential well; unlike the existing Schrodinger equation that leads to transcendental equations which require either the graphical or numerical solutions when applied to the finite potential well.

### 4. Remarks and Conclusions

We have in this article, shown how to formulate and construct the Riemannian Laplacian operator and the golden Riemannian Schrodinger equation. This equation can be solved analytically to obtain the Riemannian energy levels of a particle in a finite potential well.

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