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Double Laplace Transform Method in Mathematical Physics

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Abstract Double Laplace transform method has not received much attention unlike other methods. This article presents its effectiveness while finding the solutions of wide classes of equations of mathematical physics.

Keywords Double Laplace transform, Single Laplace transform, Partial differential equations

1. Introduction

In recent years, Eltayeb and Kilicman [1-3] applied double Laplace transform (DLT) to solve wave, Laplace's and heat equations with convolution terms, general linear telegraph and partial integro-differential equations. In 2016, L. Debnath [4] discussed the properties and convolution theorem of DLT, and applied it to functional, integral and partial differential equations. Further, Ranjit Dhunde and G. L. Waghmare in [5] applied double Laplace transform technique for solving linear partial integro-differential equations with a convolution kernel.

Analogous to [6], we consider linear, one-dimensional, time-dependent partial differential equation (PDE) of the form

$$\sum_{n=0}^{N} a_n \frac{\partial^n u(x,t)}{\partial t^n} = \sum_{m=1}^{M} b_m \frac{\partial^m u(x,t)}{\partial x^m} + f(x,t), (x,t) \in \mathbb{R}_+^2, \tag{1.1}$$

where $a_n, 0 \le n \le N;\ b_m, 1 \le m \le M$ are given coefficients and N, M are positive integers and $f(x,t)$ is the source term. Associated with (1.1), we can consider the initial conditions

$$\frac{\partial^n u(x,0)}{\partial t^n} = g_n(x), n = 0, 1, \dots\dots\dots, N-1, x \in \mathbb{R}_+, \tag{1.2}$$

and boundary conditions

$$\frac{\partial^m u(0,t)}{\partial x^m} = f_m(t), m = 0, 1, \dots\dots\dots, M-1, t \in \mathbb{R}_+. \tag{1.3}$$

Further, we assume that the functions f, g_n, $n = 0, 1, \dots\dots\dots, N-1$ *and* $f_m, m = 0, 1, \dots\dots\dots, M-1$ are such that problems (1.1), (1.2) and (1.3) have a solution.

The main objective of this paper is to develop new applications of the double Laplace transform for solving linear PDE's of the type (1.1) subject to the initial conditions (1.2) and boundary conditions (1.3).

A wide range of linear PDE's are considered which include the advection-diffusion equation (Section 4.1), the reaction-diffusion equation (Section 4.2), the telegraph equation (Section 4.3), the Klein-Gordon equation (Section 4.4), the dissipative wave equation (Section 4.5), the Korteweg-de Vries (KdV) equation (Section 4.6) and the Euler-Bernoulli equation (Section 4.7).

2. A Brief Introduction to Double Laplace Transforms

Let $f(x,t)$ be a function of two variables x and t defined in the positive quadrant of the xt-plane. The double Laplace transform of the function $f(x,t)$ as given by Ian N. Sneddon [7] is defined by

$$L_x L_t\{f(x,t)\} = \bar{f}(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st}\ f(x,t)dtdx, \tag{2.1}$$

whenever that integral exist. Here p and s are complex numbers.

From this definition we deduce

$$L_x L_t[f(x)g(t)] = \bar{f}(p)\bar{g}(s) = L_x[f(x)]L_t[g(t)]. \tag{2.2}$$

The double Laplace transform formulas for the partial derivatives of an arbitrary integer order are

$$L_x L_t\left\{\frac{\partial^n f(x,t)}{\partial t^n}\right\} = s^n \bar{f}(p,s) - \sum_{k=0}^{n-1} s^{n-1-k} L_x\left\{\frac{\partial^k f(x,0)}{\partial t^k}\right\}, \tag{2.3}$$

$$L_x L_t\left\{\frac{\partial^m f(x,t)}{\partial x^m}\right\} = p^m \bar{f}(p,s) - \sum_{j=0}^{m-1} p^{m-1-j} L_t\left\{\frac{\partial^j f(0,t)}{\partial x^j}\right\}. \tag{2.4}$$

The inverse double Laplace transform $L_x^{-1} L_t^{-1}\{\bar{f}(p,s)\} = f(x,t)$ is defined as in [4] by the complex double integral formula

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$$L_x{}^{-1}L_t{}^{-1}\{\bar{f}(p,s)\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\, dp\, \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\, \bar{f}(p,s) ds, \tag{2.5}$$

where $\bar{f}(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $Re\ p \geq c\ and\ Re\ s \geq d$, where c and d are real constants to be chosen suitably.

3. Double Laplace Transforms Method

Applying the double Laplace transform on both sides of (1.1), we get

$$\sum_{n=0}^{N} a_n \left[s^n \bar{u}(p,s) - \sum_{k=0}^{n-1} s^{n-1-k} L_x\left\{\frac{\partial^k u(x,0)}{\partial t^k}\right\}\right] = \sum_{m=1}^{M} b_m \left[p^m \bar{u}(p,s) - \sum_{j=0}^{m-1} p^{m-1-j} L_t\left\{\frac{\partial^j u(0,t)}{\partial x^j}\right\}\right] + \bar{f}(p,s). \tag{3.1}$$

Further, applying single Laplace transform to initial (1.2) and boundary conditions (1.3), we get

$$L_x\left\{\frac{\partial^n u}{\partial t^n}(x,0)\right\} = \overline{g_n}(p), L_t\left\{\frac{\partial^m u}{\partial x^m}(0,t)\right\} = \overline{f_m}(s),\ \ n = 0,1,\dots\dots, N-1\ and\ m = 0,1,\dots\dots M-1. \tag{3.2}$$

By substituting (3.2) in (3.1) and simplifying, we obtain

$$\sum_{n=0}^{N} a_n \left[s^n \bar{u}(p,s) - \sum_{k=0}^{n-1} s^{n-1-k}\, \overline{g_k}(p)\right] = \sum_{m=1}^{M} b_m \left[p^m \bar{u}(p,s) - \sum_{j=0}^{m-1} p^{m-1-j}\, \overline{f_j}(s)\right] + \bar{f}(p,s). \tag{3.3}$$

Equation (3.3) is an algebraic equation in $\bar{u}(p,s)$. Solving this algebraic equation and taking in verse double Laplace transform of $\bar{u}(p,s)$, we get an exact solution $u(x,t)$ of (1.1).

4. Applications

In this section, we apply double Laplace transform (DLT) method to linear partial differential equations.

4.1. The Advection-Diffusion Equation

Taking $N = 1, M = 2, a_0 = f = 0,\ a_1 = 1$ in (1.1), we obtain the advection-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2} + b_1 \frac{\partial u(x,t)}{\partial x}, (x,t) \in \mathbb{R}_+^2. \tag{4.1}$$

It governs the release of hormones from secretory cells in response to a stimulus in a medium, flowing past the cells and through a diffusion column, also the dispersion of pollutants in rivers [6].

If (4.1) is solved subject to the initial condition

$$u(x,0) = g_0(x), x \in \mathbb{R}_+, \tag{4.2}$$

and the boundary conditions

$$u(0,t) = f_0(t), \frac{\partial u(0,t)}{\partial x} = f_1(t), t \in \mathbb{R}_+, \tag{4.3}$$

then (3.3) gives the solution of (4.1),

$$u(x,t) = L_x{}^{-1}L_t{}^{-1}\left[\frac{\overline{g_0}(p) - b_2 p\, \overline{f_0}(s) - b_2 \overline{f_1}(s) - b_1 \overline{f_0}(s)}{(s - b_2 p^2 - b_1 p)}\right]. \tag{4.4}$$

Example 4.1: Taking $b_1 = -1, b_2 = 1$ then (4.1) becomes

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x}, (x,t) \in \mathbb{R}_+^2, \tag{4.5}$$

and consider the initial and boundary conditions

$$u(x,0) = e^x - x = g_0(x), x \in \mathbb{R}_+, \tag{4.6}$$

$$u(0,t) = 1 + t = f_0(t), \frac{\partial u(0,t)}{\partial x} = 0 = f_1(t), t \in \mathbb{R}_+. \tag{4.7}$$

Substituting

$$\overline{g_0}(p) = \frac{1}{p-1} - \frac{1}{p^2}, \overline{f_0}(s) = \frac{1}{s} + \frac{1}{s^2}, \overline{f_1}(s) = 0, \tag{4.8}$$

in (4.4), we get solution of (4.5)

$$u(x,t) = L_x{}^{-1}L_t{}^{-1}\left[\frac{1}{(s-p^2+p)}\left[\left(\frac{1}{p-1} - \frac{1}{p^2}\right) - p\left(\frac{1}{s} + \frac{1}{s^2}\right) + \left(\frac{1}{s} + \frac{1}{s^2}\right)\right]\right]. \tag{4.9}$$

Simplifying, we obtain

$$u(x,t) = L_x{}^{-1}L_t{}^{-1}\left[\frac{1}{(p-1)s} - \frac{1}{p^2 s} + \frac{1}{p s^2}\right], \tag{4.10}$$

$$u(x,t) = e^x - x + t. \tag{4.11}$$

4.2. The Reaction-Diffusion Equation

Taking $N = 1, M = 2, f = 0, a_1 = 1, b_1 = 0 \; and \; b_2 > 0$ in (1.1), we get the reaction-diffusion equation

$$a_0 u(x,t) + \frac{\partial u(x,t)}{\partial t} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2}, (x,t) \in \mathbb{R}_+^2. \tag{4.12}$$

If (4.12) is solved subject to the initial condition (4.2) and boundary conditions (4.3) then (3.3) gives the solution of (4.12),

$$u(x,t) = L_x{}^{-1} L_t{}^{-1}\left[\frac{\overline{g_0}(p) - b_2 p\, \overline{f_0}(s) - b_2 \overline{f_1}(s)}{(a_0 + s - b_2 p^2)}\right]. \tag{4.13}$$

4.2.1. The Heat (Diffusion) Equation

Taking $a_0 = 0$ in (4.12), we obtain the linear heat equation

$$\frac{\partial u(x,t)}{\partial t} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2}, (x,t) \in \mathbb{R}_+^2, \tag{4.14}$$

where $b_2 > 0$ is the constant coefficient of diffusion.

If (4.14) is solved subject to the initial condition (4.2) and boundary conditions (4.3) then (4.13) gives the solution of (4.14),

$$u(x,t) = L_x{}^{-1} L_t{}^{-1}\left[\frac{\overline{g_0}(p) - b_2 p\, \overline{f_0}(s) - b_2 \overline{f_1}(s)}{(s - b_2 p^2)}\right]. \tag{4.15}$$

4.3. The Telegraph Equation

Taking $N = M = 2, a_0 = b_1 = 0, a_2 = 1$ in (1.1), we obtain the linear telegraph equation

$$a_1 \frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial t^2} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), (x,t) \in \mathbb{R}_+^2. \tag{4.16}$$

The telegraph equation is used in signal analysis for transmission and propagation of electrical signal and also modelling reaction diffusion.

If (4.16) is solved subject to the initial conditions

$$u(x,0) = g_0(x), \frac{\partial u(x,0)}{\partial t} = g_1(x), x \in \mathbb{R}_+, \tag{4.17}$$

and boundary conditions (4.3) then (3.3) gives the solution of (4.16),

$$u(x,t) = L_x{}^{-1} L_t{}^{-1}\left[\frac{a_1 \overline{g_0}(p) + s\overline{g_0}(p) + \overline{g_1}(p) - b_2 p\, \overline{f_0}(s) - b_2 \overline{f_1}(s) + \bar{f}(p,s)}{(a_1 s + s^2 - b_2 p^2)}\right]. \tag{4.18}$$

Example 4.2: Take $b_2 = 1, a_1 = 3, f(x,t) = 3(x^2 + t^2 + 1)$ in (4.16) to yield

$$3\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + 3(x^2 + t^2 + 1), (x,t) \in \mathbb{R}_+^2, \tag{4.19}$$

and consider the initial and boundary conditions

$$u(x,0) = x = g_0(x), \frac{\partial u(x,0)}{\partial t} = 1 + x^2 = g_1(x), x \in \mathbb{R}_+, \tag{4.20}$$

$$u(0,t) = t + \frac{t^3}{3} = f_0(t), \frac{\partial u(0,t)}{\partial x} = t = f_1(t), t \in \mathbb{R}_+. \tag{4.21}$$

Substituting

$$\overline{g_0}(p) = \frac{1}{p^2}, \overline{g_1}(p) = \frac{1}{p} + \frac{2}{p^3}, \bar{f_0}(s) = \frac{1}{s^2} + \frac{2}{s^4}, \;\; \overline{f_1}(s) = \frac{1}{s^2}\; , \bar{f}(p,s) = 3\left[\frac{2}{p^3 s} + \frac{2}{ps^3} + \frac{1}{ps}\right], \tag{4.22}$$

in (4.18), we get

$$u(x,t) = L_x{}^{-1} L_t{}^{-1}\left[\frac{1}{(3s + s^2 - p^2)}\left[\frac{3}{p^2} + \frac{s}{p^2} + \left(\frac{1}{p} + \frac{2}{p^3}\right) - p\left(\frac{1}{s^2} + \frac{2}{s^4}\right) - \frac{1}{s^2} + 3\left(\frac{2}{p^3 s} + \frac{2}{ps^3} + \frac{1}{ps}\right)\right]\right]. \tag{4.23}$$

Simplifying, we obtain

$$u(x,t) = L_x{}^{-1} L_t{}^{-1}\left[\frac{1}{p^2 s} + \frac{1}{ps^2} + \frac{2}{p^3 s^2} + \frac{2}{ps^4}\right], \tag{4.24}$$

$$u(x,t) = x + t + x^2 t + \frac{t^3}{3}. \tag{4.25}$$

4.3.1. The Wave Equation

Substituting $a_1 = 0, b_2 > 0 \; and \; f = 0$ in (4.16), we obtain wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2}, (x,t) \in \mathbb{R}_+^2. \tag{4.26}$$

If (4.26) is solved subject to the initial conditions (4.17) and boundary conditions (4.3) then (4.18) gives the solution of (4.26),

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{s\overline{g_0}(p) + \overline{g_1}(p) - b_2 p\, \overline{f_0}(s) - b_2 \overline{f_1}(s)}{(s^2 - b_2 p^2)}\right]. \tag{4.27}$$

4.4. The Klein-Gordon Equation

Taking $N = M = 2, a_1 = b_1 = 0, a_2 = 1$ in (1.1), we obtain the Klein-Gordon equation

$$a_0 u(x,t) + \frac{\partial^2 u(x,t)}{\partial t^2} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), (x,t) \in \mathbb{R}_+^2. \tag{4.28}$$

The Klein-Gordon equation plays an important role in the study of solutions in condensed matter physics, quantum mechanics and relativistic physics.

If (4.28) is solved subject to the initial conditions (4.17) and boundary conditions (4.3) then (3.3) gives the solution of (4.28),

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{s\overline{g_0}(p) + \overline{g_1}(p) - b_2 p\, \overline{f_0}(s) - b_2 \overline{f_1}(s) + \bar{f}(p,s)}{(a_0 + s^2 - b_2 p^2)}\right]. \tag{4.29}$$

Example 4.3: Take $b_2 = 1, a_0 = -1, f = 0$ in (4.28) to yield

$$\frac{\partial^2 u(x,t)}{\partial t^2} - u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, (x,t) \in \mathbb{R}_+^2, \tag{4.30}$$

and consider the initial and boundary conditions

$$u(x,0) = 1 + \sin x = g_0(x), \frac{\partial u\,(x,0)}{\partial t} = 0 = g_1(x), x \in \mathbb{R}_+, \tag{4.31}$$

$$u(0,t) = \cosh t = f_0(t), \frac{\partial u\,(0,t)}{\partial x} = 1 = f_1(t), t \in \mathbb{R}_+. \tag{4.32}$$

Substituting

$$\overline{g_0}(p) = \frac{1}{p} + \frac{1}{p^2+1}, \overline{f_0}(s) = \frac{s}{s^2-1}, \overline{f_1}(s) = \frac{1}{s},\ \ \overline{g_1}(p) = \bar{f}(p,s) = 0, \tag{4.33}$$

in (4.29), we get solution of (4.30)

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{1}{(s^2-1-p^2)}\left[s\left(\frac{1}{p} + \frac{1}{p^2+1}\right) - p\frac{s}{s^2-1} - \frac{1}{s}\right]\right]. \tag{4.34}$$

Simplifying, we obtain

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{1}{s(p^2+1)} + \frac{s}{p(s^2-1)}\right], \tag{4.35}$$

$$u(x,t) = \sin x + \cosh t. \tag{4.36}$$

Example 4.4: Take $b_2 = 1, a_0 = -2, f(x,t) = -2\sin x \sin t$ in (4.28) to yield

$$\frac{\partial^2 u(x,t)}{\partial t^2} - 2u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - 2\sin x \sin t\,, (x,t) \in \mathbb{R}_+^2, \tag{4.37}$$

and consider the initial and boundary conditions

$$u(x,0) = 0 = g_0(x), \frac{\partial u\,(x,0)}{\partial t} = \sin x = g_1(x), x \in \mathbb{R}_+, \tag{4.38}$$

$$u(0,t) = 0 = f_0(t), \frac{\partial u\,(0,t)}{\partial x} = \sin t = f_1(t), t \in \mathbb{R}_+. \tag{4.39}$$

Substituting

$$\overline{g_0}(p) = 0, \overline{g_1}(p) = \frac{1}{p^2+1}, \overline{f_0}(s) = 0,\ \ \overline{f_1}(s) = \frac{1}{s^2+1}, \bar{f}(p,s) = -2\frac{1}{(p^2+1)(s^2+1)}, \tag{4.40}$$

in (4.29), we get solution of (4.37)

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{1}{(s^2-2-p^2)}\left[\frac{1}{p^2+1} - \frac{1}{s^2+1} - 2\frac{1}{(p^2+1)(s^2+1)}\right]\right]. \tag{4.41}$$

Simplifying, we obtain

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{1}{(p^2+1)(s^2+1)}\right], \tag{4.42}$$

$$u(x,t) = \sin x \sin t. \tag{4.43}$$

4.5. The Linear Dissipative Wave Equation

Substituting $N = M = 2, a_0 = 0, a_2 = 1$ in (1.1), we obtain the linear dissipative wave equation

$$a_1 \frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial t^2} = b_2 \frac{\partial^2 u(x,t)}{\partial x^2} + b_1 \frac{\partial u(x,t)}{\partial x} + f(x,t), (x,t) \in \mathbb{R}_+^2. \quad (4.45)$$

If (4.45) is solved subject to the initial conditions (4.17) and boundary conditions (4.3) then (3.3) gives the solution of (4.45),

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{a_1 \overline{g_0}(p) + s\overline{g_0}(p) + \overline{g_1}(p) - b_2 p \overline{f_0}(s) - b_2 \overline{f_1}(s) - b_1 \overline{f_0}(s) + \bar{f}(p,s)}{(a_1 s + s^2 - b_2 p^2 - b_1 p)}\right]. \quad (4.46)$$

Example 4.5: Take $a_1 = b_1 = b_2 = 1, f(x,t) = 2(t-x)$ in (4.45) to yield

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} + 2(t-x), (x,t) \in \mathbb{R}_+^2, \quad (4.47)$$

and consider the initial and boundary conditions

$$u(x,0) = x^2 = g_0(x), \frac{\partial u(x,0)}{\partial t} = 0 = g_1(x), x \in \mathbb{R}_+, \quad (4.48)$$

$$u(0,t) = t^2 = f_0(t), \frac{\partial u(0,t)}{\partial x} = 0 = f_1(t), t \in \mathbb{R}_+, \quad (4.49)$$

Substituting

$$\overline{g_0}(p) = \frac{2}{p^3}, \overline{f_0}(s) = \frac{2}{s^3}, \overline{g_1}(p) = \overline{f_1}(s) = 0, \quad \bar{f}(p,s) = 2\left(\frac{1}{ps^2} - \frac{1}{p^2 s}\right), \quad (4.50)$$

in (4.46), we get solution of (4.47)

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{1}{(s+s^2-p^2-p)}\left[\frac{2}{p^3} + s\frac{2}{p^3} - p\frac{2}{s^3} - \frac{2}{s^3} + 2\left(\frac{1}{ps^2} - \frac{1}{p^2 s}\right)\right]\right]. \quad (4.51)$$

Simplifying, we obtain

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{2}{p^3 s} + \frac{2}{ps^3}\right], \quad (4.52)$$

$$u(x,t) = x^2 + t^2. \quad (4.53)$$

4.6. The Korteweg-de Vries (KdV) Equation

Substituting $N = 1, M = 3, a_0 = b_2 = 0, a_1 = 1$ in (1.1), we obtain the linear Korteweg-de Vries (KdV) equation

$$\frac{\partial u(x,t)}{\partial t} = b_3 \frac{\partial^3 u(x,t)}{\partial x^3} + b_1 \frac{\partial u(x,t)}{\partial x} + f(x,t), (x,t) \in \mathbb{R}_+^2. \quad (4.54)$$

It governs long water waves, in water of relatively shallow, for very small amplitudes [6]. When $b_1 = 0$, (4.54) represents a third-order dispersive equation.

If (4.54) is solved subject to the initial condition (4.2) and boundary conditions

$$u(0,t) = f_0(t), \frac{\partial u(0,t)}{\partial x} = f_1(t), \frac{\partial^2 u(0,t)}{\partial x^2} = f_2(t), t \in \mathbb{R}_+, \quad (4.55)$$

then (3.3) gives the solution of (4.54),

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{\overline{g_0}(p) - b_3 p^2 \overline{f_0}(s) - b_3 p \overline{f_1}(s) - b_3 \overline{f_2}(s) - b_1 \overline{f_0}(s) + \bar{f}(p,s)}{(s - b_3 p^3 - b_1 p)}\right]. \quad (4.56)$$

Example 4.6: Taking $b_3 = b_1 = -1, f = 0$ then (4.54) becomes

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^3 u(x,t)}{\partial x^3} - \frac{\partial u(x,t)}{\partial x}, (x,t) \in \mathbb{R}_+^2, \quad (4.57)$$

and consider the initial and boundary conditions

$$u(x,0) = e^{-x} = g_0(x), x \in \mathbb{R}_+, \quad (4.58)$$

$$u(0,t) = e^{2t} = f_0(t), \frac{\partial u(0,t)}{\partial x} = -e^{2t} = f_1(t), \frac{\partial^2 u(0,t)}{\partial x^2} = e^{2t} = f_2(t), t \in \mathbb{R}_+. \quad (4.59)$$

Substituting

$$\overline{g_0}(p) = \frac{1}{p+1}, \overline{f_1}(s) = \frac{-1}{s-2}, \overline{f_0}(s) = \overline{f_2}(s) = \frac{1}{s-2}, \bar{f}(p,s) = 0, \quad (4.60)$$

in (4.56), we get solution of (4.57)

$$u(x,t) = L_x^{-1} L_t^{-1} \left[\frac{1}{(s+p^3+p)}\left[\frac{1}{p+1} + p^2\frac{1}{s-2} - p\frac{1}{s-2} + \frac{1}{s-2} + \frac{1}{s-2}\right]\right]. \quad (4.61)$$

Simplifying, we obtain

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}\left[\frac{1}{(p+1)(s-2)}\right], \tag{4.62}$$

$$u(x,t) = e^{2t-x}. \tag{4.63}$$

Example 4.7: Taking $b_1 = 0, b_3 = 1, f(x,t) = 2e^{t-x}$ in (4.54) to obtain the linear third-order dispersive, inhomogeneous equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^3 u(x,t)}{\partial x^3} + 2e^{t-x}, (x,t) \in \mathbb{R}_+^2, \tag{4.64}$$

and consider the initial and boundary conditions

$$u(x,0) = 1 + e^{-x} = g_0(x), x \in \mathbb{R}_+, \tag{4.65}$$

$$u(0,t) = 1 + e^t = f_0(t), \frac{\partial u(0,t)}{\partial x} = -e^t = f_1(t), \frac{\partial^2 u(0,t)}{\partial x^2} = e^t = f_2(t), t \in \mathbb{R}_+. \tag{4.66}$$

Substituting

$$\overline{g_0}(p) = \frac{1}{p} + \frac{1}{p+1}, \bar{f_0}(s) = \frac{1}{s} + \frac{1}{s-1}, \bar{f_1}(s) = \frac{-1}{s-1}, \ \overline{f_2}(s) = \frac{1}{s-1}, \bar{f}(p,s) = \frac{2}{(s-1)(p+1)}, \tag{4.67}$$

in (4.56), we get solution of (4.64)

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}\left[\frac{1}{(s-p^3)}\left[\frac{1}{p} + \frac{1}{p+1} - p^2\left(\frac{1}{s} + \frac{1}{s-1}\right) + p\frac{1}{s-1} - \frac{1}{s-1} + \frac{2}{(s-1)(p+1)}\right]\right]. \tag{4.68}$$

Simplifying, we obtain

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}\left[\frac{1}{ps} + \frac{1}{(p+1)(s-1)}\right], \tag{4.69}$$

$$u(x,t) = 1 + e^{t-x}. \tag{4.70}$$

Example 4.8: Taking $b_1 = b_3 = -1 \ and \ f(x,t) = 1 + (1+t)e^x + e^{2x}$ in (4.54) to obtain

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^3 u(x,t)}{\partial x^3} - \frac{\partial u(x,t)}{\partial x} + 1 + (1+t)e^x + e^{2x}, (x,t) \in \mathbb{R}_+^2, \tag{4.71}$$

and consider the initial and boundary conditions

$$u(x,0) = \frac{e^x}{4} + \frac{e^{2x}}{10} = g_0(x), x \in \mathbb{R}_+, \tag{4.72}$$

$$u(0,t) = \frac{3t}{2} + \frac{7}{20} = f_0(t), \frac{\partial u(0,t)}{\partial x} = \frac{t}{2} + \frac{9}{20} = f_1(t), \ \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{t}{2} + \frac{13}{20} = f_2(t), t \in \mathbb{R}_+. \tag{4.73}$$

Substituting

$$\overline{g_0}(p) = \frac{1}{4(p-1)} + \frac{1}{10(p-2)}, \bar{f_0}(s) = \frac{3}{2s^2} + \frac{7}{20s}, \bar{f_1}(s) = \frac{1}{2s^2} + \frac{9}{20s}, \bar{f_2}(s) = \frac{1}{2s^2} + \frac{13}{20s},$$

$$\bar{f}(p,s) = \frac{1}{ps} + \frac{1}{s(p-1)} + \frac{1}{s^2(p-1)} + \frac{1}{s(p-2)}, \tag{4.74}$$

in (4.56), we get solution of (4.71)

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}$$

$$\left[\frac{1}{(s+p^3+p)}\left[\left[\frac{1}{4(p-1)} + \frac{1}{10(p-2)}\right] + p^2\left[\frac{3}{2s^2} + \frac{7}{20s}\right] + p\left[\frac{1}{2s^2} + \frac{9}{20s}\right] + \left[\frac{1}{2s^2} + \frac{13}{20s}\right] + \left[\frac{3}{2s^2} + \frac{7}{20s}\right] + \frac{1}{ps} + \frac{1}{s(p-1)} + \frac{1}{s^2(p-1)} + \frac{1}{s(p-2)}\right]\right]. \tag{4.75}$$

Simplifying, we obtain

$$u(x,t) = {L_x}^{-1}{L_t}^{-1}\left[\frac{1}{ps^2} + \frac{1}{2s^2(p-1)} + \frac{1}{4s(p-1)} + \frac{1}{10s(p-2)}\right], \tag{4.76}$$

$$u(x,t) = t + \frac{1}{2}te^x + \frac{1}{4}e^x + \frac{1}{10}e^{2x}. \tag{4.77}$$

4.7. The Euler-Bernoulli Equation

Taking $N = 2, M = 4, a_0 = a_1 = b_1 = b_2 = b_3 = 0, a_2 = -1$ in (1.1), we obtain the Euler-Bernoulli equation

$$-\frac{\partial^2 u(x,t)}{\partial t^2} = b_4\frac{\partial^4 u(x,t)}{\partial x^4} + f(x,t), (x,t) \in \mathbb{R}_+^2. \tag{4.78}$$

It governs the deflection of an elastic beam under the action of a load (x,t). In (4.78), the solution $u(x,t)$ represents the deflection of the beam and $b_4 > 0$ is its flexural rigidity.

If (4.78) is solved subject to the initial condition (4.17) and boundary conditions

$$u(0,t) = f_0(t), \frac{\partial u(0,t)}{\partial x} = f_1(t), \frac{\partial^2 u(0,t)}{\partial x^2} = f_2(t),\ \frac{\partial^3 u(0,t)}{\partial x^3} = f_3(t), t \in \mathbb{R}_+, \tag{4.79}$$

then (3.3) gives the solution of (4.78),

$$u(x,t) = L_x{}^{-1} L_t{}^{-1} \left[\frac{s\,\overline{g_0}(p) + \overline{g_1}(p) + b_4\{p^3\,\overline{f_0}(s) + p^2 \overline{f_1}(s) + p\overline{f_2}(s) + \overline{f_3}(s)\} - \bar{f}(p,s)}{(s^2 + b_4 p^4)}\right]. \tag{4.80}$$

Example 4.9: Take $b_4 = 1\ and\ f(x,t) = -xt - t^2$ in (4.71) to yield

$$-\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^4 u(x,t)}{\partial x^4} - xt - t^2, (x,t) \in \mathbb{R}_+^2, \tag{4.81}$$

and consider the initial and boundary conditions

$$u(x,0) = 0 = g_0(x), \frac{\partial u(x,0)}{\partial t} = \frac{x^5}{5!} = g_1(x), x \in \mathbb{R}_+, \tag{4.82}$$

$$u(0,t) = \frac{t^4}{12} = f_0(t), \frac{\partial u(0,t)}{\partial x} = 0 = f_1(t), \frac{\partial^2 u(0,t)}{\partial x^2} = 0 = f_2(t), \frac{\partial^3 u(0,t)}{\partial x^3} = 0 = f_3(t), t \in \mathbb{R}_+. \tag{4.83}$$

Substituting

$$\overline{g_0}(p) = 0, \overline{g_1}(p) = \frac{1}{p^6}, \overline{f_0}(s) = \frac{2}{s^5}, \overline{f_1}(s) = \overline{f_2}(s) = \overline{f_3}(s) = 0, \bar{f}(p,s) = -\frac{1}{p^2 s^2} - \frac{2}{p s^3}, \tag{4.84}$$

in (4.80), we get solution of (4.81)

$$u(x,t) = L_x{}^{-1} L_t{}^{-1} \left[\frac{1}{(p^4+s^2)} \left[\frac{1}{p^6} + p^3 \frac{2}{s^5} + \frac{1}{p^2 s^2} + \frac{2}{p s^3}\right]\right]. \tag{4.85}$$

Simplifying, we obtain

$$u(x,t) = L_x{}^{-1} L_t{}^{-1} \left[\frac{1}{p^6 s^2} + \frac{2}{p s^5}\right], \tag{4.86}$$

$$u(x,t) = \frac{t x^5}{5!} + \frac{t^4}{12}. \tag{4.87}$$

5. Conclusions

The examples show that double Laplace transform method is a best alternative for handling many equations of Mathematical Physics.

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