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Solution of Newton's Gravitational Field Equation of a Static Homogeneous Oblate Spheroidal Massive Body

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Abstract It is well known that Newton's dynamical gravitational field equation and Einstein's geometrical gravitational field equation were derived based upon the Euclidean geometry. In this paper, we used a golden metric tensor for all gravitational fields in nature to develop Riemannian Laplacian field equation which is assumed to be more general than the Euclidean geometry in order to derive Riemann's dynamical gravitational field equation for a static homogeneous oblate spheroidal massive bodies due correction terms of all order c^{-2} that is not found in Newton's gravitational field equation and also in this paper we derived the solution of Newton's gravitational field equation for a static homogeneous oblate spheroidal massive body.

Keywords Golden metric tensor, Riemannian Laplacian Equation, Spherical polar coordinates, Oblate spheroidal coordinates

1. Introduction

In the last six decades, the theoretical study of gravitational fields in Newton's dynamical gravitational field equation was treated under the assumption that massive bodies are perfectly spherical in nature. In the same way, Einstein's Geometrical gravitational field equation was treated under the assumption that the sun is exclusively a perfect sphere as seen in Schwarzschild's solution to Einstein's Geometrical gravitational field equation. It is well known that Newton's dynamical gravitational field equation and Einstein's geometrical gravitational field equation were derived based upon the Euclidean Laplacian operator. Recently, research has shown that all rotating bodies example the planets, sun, stars and the galaxies in the universe are either oblate spheroidal or prolate in shape [1, 2, 9, 10, 13].

In Newton's mechanics, the motion of particles (such as projectile, satellites and pendulli) in earth's atmosphere is treated under the assumption that the earth is a perfect sphere [14]. Similarly, in the solar system the motion of bodies (such as planets comets and asteroids) is treated under the assumption that the Sun is a perfect sphere [3, 12].

Also in Einstein's theory, the motion of bodies (such as planets) and particles (such as photons) are treated under the assumption that the Sun is a perfect sphere (Schwarzschild spacetime). But it is well known that the only reason for these restrictions is mathematical convenience and simplicity. The real fact of nature is that all rotating planets stars and galaxies in the universe are spheroidal [5, 6, 11]. And it is obvious that their spheroidal geometry will have corresponding consequences and effects in the motion of all particles in their gravitational fields. These effects will exist in both Newtonian mechanics and in Einstein's theory.

2. Theoretical Analysis

Consider a static homogeneous oblate spheroidal massive body of rest mass M and radius η_0. The Riemannian dynamical gravitational field equation is given explicitly by [7, 8, 10]

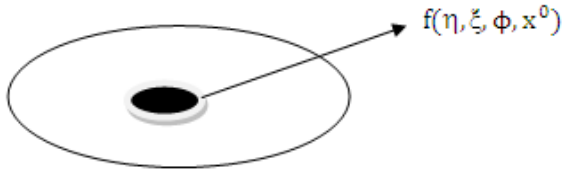

Figure 1

Density ρ_0
Radius η_0
Mass M

$$\nabla_R^2 f(\eta, \xi, \phi, x^0) = 4\pi G \rho_0(\eta, \xi, \phi, x^0) \qquad (1.0)$$

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Where ∇_R^2 is Riemannian Laplacian
Given

$$\nabla_R^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left\{\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right\} \tag{1.1}$$

and g_{uv} is the metric tensor (either schwarztchild or great metric tensor or golden metric tensor). Here we use golden metric tensor of oblate spheroidal coordinates. In spherical polar coordinates, Golden metric tensor is given by:

$$g_{11} = \left\{1 + \frac{2}{c^2}f(r,\theta,\phi,x^0)\right\}^{-1} \tag{1.2}$$

$$g_{22} = r^2\left\{1 + \frac{2}{c^2}f(r,\theta,\phi,x^0)\right\}^{-1} \tag{1.3}$$

$$g_{33} = r^2\sin^2\theta\left\{1 + \frac{2}{c^2}f(r,\theta,\phi,x^0)\right\}^{-1} \tag{1.4}$$

$$g_{00} = -\left\{1 + \frac{2}{c^2}f(r,\theta,\phi,x^0)\right\} \tag{1.5}$$

$$g_{uv} = 0, \text{otherwise} \tag{1.6}$$

To express Golden metric tensor in oblate spheroidal coordinates, we need to transform as we shall see later.

$$(r,\theta,\phi,x^0) \to (\eta,\xi,\phi,x^0)$$

Note that:

$$g_{11} = h_1^2\left(1 + \frac{2}{c^2}f\right)^{-1} \tag{1.7}$$

$$g_{22} = h_2^2\left(1 + \frac{2}{c^2}f\right)^{-1} \tag{1.8}$$

$$g_{33} = h_3^2\left(1 + \frac{2}{c^2}f\right)^{-1} \tag{1.9}$$

$$g_{00} = -\left(1 + \frac{2}{c^2}f\right) \tag{1.10}$$

$$g_{uv} = 0, \text{otherwise} \tag{1.11}$$

Where h_1, h_2, h_3 are the scale factors in oblate spheroidal coordinates to be evaluated.

Now consider Riemannian Laplacian

$$\nabla_R^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{1\beta}\frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{2\beta}\frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{3\beta}\frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{0\beta}\frac{\partial}{\partial x^\beta}\right\} \tag{1.12}$$

$$\begin{aligned}\nabla_R^2 = {} & \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{11}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{12}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{13}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{10}\frac{\partial}{\partial x^0}\right\} \\ & + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{21}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{22}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{23}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{20}\frac{\partial}{\partial x^0}\right\} \\ & + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{31}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{32}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{33}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{30}\frac{\partial}{\partial x^0}\right\} \\ & + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{01}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{02}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{03}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{00}\frac{\partial}{\partial x^0}\right\} \\ = {} & 4\pi G\rho_0(r,\theta,\phi,x^0)\end{aligned} \tag{1.13}$$

The above Riemannian field equation reduced to:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{11}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{22}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{33}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{00}\frac{\partial}{\partial x^0}\right\} = 4\pi G\rho_0(r,\theta,\phi,x^0) \tag{1.14}$$

By definition g is the determinant of the Golden metric tensor

$$g = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{00} \end{pmatrix} \tag{1.15}$$

$$\begin{aligned} g &= g_{11}\cdot\ g_{22}\ \cdot\ g_{33}\cdot\ g_{00} \\ &= -h_1^2\ h_2^2\ h_3^2\left(1 + \frac{2}{c^2}f\right)^{-3}\left(1 + \frac{2}{c^2}f\right)^{1} = -h_1^2\ h_2^2\ h_3^2\left(1 + \frac{2}{c^2}f\right)^{-2}\end{aligned} \tag{1.16}$$

$$\sqrt{g} = h_1\ h_2\ h_3\left(1 + \frac{2}{c^2}f\right)^{-1} \tag{1.17}$$

For oblate spheroidal coordinate

$$g^{11} = \frac{1}{g_{11}} = \frac{1}{h_1^2\left(1+\frac{2}{c^2}f\right)^{-1}} = \frac{\left(1+\frac{2}{c^2}f\right)}{h_1^2} \tag{1.18}$$

$$g^{22} = \frac{1}{g_{22}} = \frac{1}{h_2^2\left(1+\frac{2}{c^2}f\right)^{-1}} = \frac{\left(1+\frac{2}{c^2}f\right)}{h_2^2} \tag{1.19}$$

$$g^{33} = \frac{1}{g_{33}} = \frac{1}{h_3^2\left(1+\frac{2}{c^2}f\right)^{-1}} = \frac{\left(1+\frac{2}{c^2}f\right)}{h_3^2} \tag{1.20}$$

$$g^{00} = \frac{1}{g_{00}} = -\frac{1}{\left(1+\frac{2}{c^2}f\right)} = -\left(1+\frac{2}{c^2}f\right) \tag{1.21}$$

$$g^{uv} = 0, \quad \text{otherwise} \tag{1.22}$$

Note also the scale factors for oblate spheroidal coordinates are:

$$h_1 = a\left(\frac{\eta^2+\xi^2}{1+\eta^2}\right)^{1/2} \tag{1.23}$$

$$h_2 = a\left(\frac{\eta^2+\xi^2}{1+\xi^2}\right)^{1/2} \tag{1.24}$$

$$h_3 = a[(1+\eta^2)(1+\xi^2)]^{1/2} \tag{1.25}$$

By making use of equation (1.17) above and substituting h_1, h_2 and h_3. We get:

$$\sqrt{g} = a\left(\frac{\eta^2+\xi^2}{1-\eta^2}\right)^{1/2} \cdot a\left(\frac{\eta^2+\xi^2}{1+\xi^2}\right)^{1/2} \cdot a[(1-\eta^2)(1+\xi^2)]^{1/2} \times \left(1+\frac{2}{c^2}f\right)^{-1}$$

$$= [a^6(\eta^2+\xi^2)^2]^{1/2}\left(1+\frac{2}{c^2}f\right)^{-1} \tag{1.26}$$

$$\sqrt{g} = a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1} \tag{1.27}$$

Putting $\sqrt{g}$, g^{11}, g^{22}, g^{33} and g^{00} into the field equation we get

$$\frac{1}{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}}\frac{\partial}{\partial\eta}\left\{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\left(1+\frac{2}{c^2}f\right)}{\frac{a^2(\eta^2+\xi^2)}{(1-\eta^2)}}\frac{\partial}{\partial\eta}f\right\}$$

$$+\frac{1}{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}}\frac{\partial}{\partial\xi}\left\{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\left(1+\frac{2}{c^2}f\right)}{\frac{a^2(\eta^2+\xi^2)}{(1+\xi^2)}}\frac{\partial}{\partial\xi}f\right\}$$

$$+\frac{1}{a^3(\eta^2+\xi^2)(1+\frac{2}{c^2}f)^{-1}}\frac{\partial}{\partial\phi}\left\{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\left(1+\frac{2}{c^2}f\right)}{a^2(1-\eta^2)(1+\xi^2)}\frac{\partial}{\partial\phi}f\right\}$$

$$-\frac{1}{c^2(1+\frac{2}{c^2}f)^{-1}}\frac{\partial}{\partial t}\left\{\left(1+\frac{2}{c^2}f\right)^{-1}\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\partial}{\partial t}f\right\}$$

$$= 4\pi G\rho(\eta,\xi,\phi,x^0) \tag{1.28}$$

$$\frac{\left(1+\frac{2}{c^2}f\right)}{(\eta^2+\xi^2)}\frac{\partial}{\partial\eta}\left\{\frac{(\eta^2+\xi^2)(1-\eta^2)}{a^2(n^2+\xi^2)}\frac{\partial}{\partial\eta}f\right\}+\frac{\left(1+\frac{2}{c^2}f\right)}{(\eta^2+\xi^2)}\frac{\partial}{\partial\xi}\left\{\frac{(\eta^2+\xi^2)(1+\xi^2)}{a^2(n^2+\xi^2)}\frac{\partial}{\partial\xi}f\right\}$$

$$+(1+\frac{2}{c^2}f)\frac{\partial}{\partial\phi}\left\{\frac{1}{a^2(1-\eta^2)(1+\xi^2)}\frac{\partial}{\partial\phi}f\right\}-\frac{\left(1+\frac{2}{c^2}f\right)}{c^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}f\right\}$$

$$= 4\pi G\rho(\eta,\xi,\phi,x^0) \tag{1.29}$$

$$\left(1+\frac{2}{c^2}f\right)\left[\frac{1}{a^2(\eta^2+\xi^2)}\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\}+\frac{1}{a^2(\eta^2+\xi^2)}\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}+\frac{1}{a^2(1-\eta^2)(1+\xi^2)}\frac{\partial^2}{\partial\phi^2}f-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}f\right]$$

$$= 4\pi G\rho_0(\eta,\xi,\phi,x^0) \tag{1.30}$$

Static homogeneous oblate spheroidal distribution of massive body

Note: (1) No Time variation (static)

(2) No azimuthal angle ϕ (symmetry)

Equation (1.30) reduced to

$$\left(1+\frac{2}{c^2}f\right)\left[\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\}+\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}\right] = 4\pi G\rho_0(\eta,\xi,)\,a^2(\eta^2+\xi^2) \tag{1.31}$$

Expand: equation (1.31)

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\}+\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}+\frac{2}{c^2}f\left[\frac{\partial}{\partial\eta}\{(1-\eta^2)\}\frac{\partial}{\partial\eta}f\right]+\frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\{1+\xi^2\}\frac{\partial}{\partial\xi}f\right]$$

$$= a^2(\eta^2+\xi^2)\cdot 4\pi G\rho_0(\eta,\xi,) \tag{1.32}$$

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\}+\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}+\frac{2}{c^2}f\left[\frac{\partial}{\partial\eta}\{(1-\eta^2)\}\frac{\partial}{\partial\eta}f\right]+\frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\{1+\xi^2\}\frac{\partial}{\partial\xi}f\right]=0 \tag{1.33}$$

Note that:

- The first two terms of equation (1.32) are Newton's gravitational field equations interior to the gravitational scalar potential f due to a distribution of mass density $\rho_0(\eta,\xi,)$ and that of equation (1.33) are Newton's gravitational field equations exterior with no distribution of mass density.
- While the last two terms are the added terms or our contribution due to Riemann and is referred to as Riemann's gravitational field equation interior and exterior to a static homogeneous oblate spheroid.

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f(\eta,\xi,)\right\}+\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f(\eta,\xi,)\right\}=4\pi G\rho_0a^2(\eta^2+\xi^2) \tag{1.34}$$

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f(\eta,\xi,)\right\}+\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f(\eta,\xi,)\right\}=0 \tag{1.35}$$

Next, let us seek a separable complementary solution of the interior equation

$$\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}\Omega^-(\eta)+\lambda\Omega^-(\eta)=0 \tag{1.36}$$

$$\frac{\partial}{\partial\xi}(1+\xi^2)\frac{\partial}{\partial\xi}T^-(\xi)-\lambda T^-(\xi)=0 \tag{1.37}$$

where λ is the separation constants. Consequently, for the choice

$$\lambda=l(l+1);\, l=0,1,2\ldots \tag{1.38}$$

equation (1.34) becomes the Legendre's equation and hence has solution of the form

$$\Omega^-(\eta)=\begin{cases}P_l(\eta)\\ Q_l(\eta)\end{cases}\qquad l=0,1,2\ldots \tag{1.39}$$

where P_l and Q_l are the linearly independent solutions of the Legendre's differential equation.

Also by the transformation equation

$$\xi=i\tau \tag{1.40}$$

equation (1.37) becomes

$$\frac{\partial}{\partial\xi}(1+\tau^2)\frac{\partial}{\partial\xi}T^-(\tau)-l(l+1)T^-(\tau)=0 \tag{1.41}$$

This is the Legendre's differential equation and hence has the solution of the form

$$\Omega(\xi)=\begin{cases}P_l(-i\xi)\\ Q_l(-i\xi)\end{cases}\qquad l=0,1,2\ldots \tag{1.42}$$

Consequently

$$f^-(\eta,\xi)=\sum_{l=0}^{\infty}[A_l^-P_l(-i\xi)+B_l^-Q_l(-i\xi)][C_l^-P_l(\eta)+D_l^-Q_l(\eta)] \tag{1.43}$$

where $A_l^-, B_l^-, C_l^-,$ and P_l are arbitrary constants. Next let us seek a particular solution of the interior equation f as

$$f^-=k(\eta^2-\xi^2) \tag{1.44}$$

substituting equation (1.44) into equation (1.34) it follows that

$$4\pi G\rho_0a^2=6k \tag{1.45}$$

$$k=\frac{2}{3}\pi G\rho_0a^2 \tag{1.46}$$

Hence the general solution of the interior equation f^- given by:

$$f^-(\eta,\xi)=\frac{2\pi G\rho_0a^2(\eta^2+\xi^2)}{3}+\sum_{l=0}^{\infty}[A_l^-P_l(-i\xi)+B_l^-Q_l(-i\xi)][C_l^-P_l(\eta)+D_l^-Q_l(\eta)] \tag{1.47}$$

Where $A_l^-, B_l^-, C_l^-, D_l^-$ are constants, and P_l and Q_l are the Legendre's functions order l. Similarly, since the exterior equation is homogeneous its general solution f^+ is given by

$$f^+(\eta,\xi)=\sum_{l=0}^{\infty}[A_l^+P_l(-i\xi)+B_l^+Q_l(-i\xi)][C_l^+P_l(\eta)+D_l^+Q_l(\eta)] \tag{1.48}$$

Where $A_l^+, B_l^{\mp}, C_l^+, D_l^+$ are arbitrary constants. Now since the interior and exterior regions both contain the coordinate $\eta = 0$ which is a singularity of Q_l

$$D_l^- \equiv D_l^+ \equiv 0;\ l = 0, 1, 2, \ldots \tag{1.49}$$

Also since $Q_l(-i\xi)$ is not defined at the centre of the body $\xi = 0$ in the interior region

$$B_l^+ \equiv 0;\ l = 0, 1, 2, \ldots \tag{1.50}$$

Also, since $P_l(-i\xi)$ is not defined for $\xi \to \infty$ in the exterior region

$$A_l^+ \equiv 0;\ l = 0, 1, 2, \ldots \tag{1.51}$$

Hence from the continuity of the gravitational scalar potentials equations (1.47) and (1.48) across the boundary $\xi = \xi_0$ and the orthogonality of the Legendre functions P_l it follows that, the resulting equations are

$$A_0^- + \frac{2\pi G\rho_0 a^2}{3}\left(\xi_0^2 - \frac{1}{3}\right) = B_0^+ Q_0(-i\xi) \tag{1.52}$$

$$A_l^- P_l(-i\xi_0) = B_0^+ Q_0(-i\xi) \tag{1.53}$$

$$A_2^- P_2(-i\xi_0) - \frac{2\pi G\rho_0 a^2}{9} B_2^+ Q_2(-i\xi) \tag{1.54}$$

From equations (1.52) and (1.54) we get

$$\frac{4\pi G\rho_0 a^2}{3} = B_0^+ \left[\frac{d}{d\xi} Q_0(i\xi)\right]_{\xi=\xi_0} \tag{1.55}$$

and

$$A_0^+ \left[\frac{d}{d\xi} P_l(-i\xi)\right]_{\xi=\xi_0} = B_0^+ \left[\frac{d}{d\xi} Q_l(-i\xi)\right]_{\xi=\xi_0} \tag{1.56}$$

Thus solving (1.52)-(1.56) simultaneously

$$B_0^+ = \frac{4\pi G\rho_0 a^2 \xi_0}{3\left[\frac{d}{d\xi} Q_0(-i\xi_0)\right]_{\xi=\xi_0}} \tag{1.57}$$

and

$$A_0^l = \frac{4\pi G\rho_0 a^2 \xi_0 Q_0(-i\xi_0)}{3\left[\frac{d}{d\xi} Q_0(-i\xi_0)\right]_{\xi=\xi_0}} - \frac{2\pi\rho_0 a}{3}\left(\xi^3 - \frac{1}{3}\right) \tag{1.58}$$

$$A_0^- = 0 \tag{1.59}$$

$$B_l^+ = 0 \tag{1.60}$$

Next, let us consider equation (1.54) above i.e.

$$A_2^- P_2(-i\xi) - \frac{2\pi G\rho_0 a^2}{9} B_2^+ Q_2(-i\xi_0)$$

Now differentiate both sides of this equation with respect to ξ, it follow that

$$A_2^- \left[\frac{d}{d\xi} P_l(-i\xi_0)\right]_{\xi=\xi_0} = B_2^+ \left[\frac{d}{d\xi} Q_l(-i\xi_0)\right]_{\xi=\xi_0} \tag{1.61}$$

By multiplying both sides of equation (1.54) with

$$\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0}$$

It follows that

$$A_2^- P_2(-i\xi)\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0} - \frac{4\pi G\rho_0 a^2}{9}\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0} = B_2^+ Q_2(-i\xi_0)\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0} \tag{1.62}$$

This follows by multiplying both side of equation (1.62) with $P_2(-i\xi)$. Thus

$$B_2^+ = -\frac{4\pi G\rho_0 a^2\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0}}{9\Delta_2} \tag{1.63}$$

where

$$\Delta_2 = Q_2(-i\xi)\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0} - P_2(-i\xi)\left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0} \tag{1.64}$$

And

$$A_2^- = -\frac{4\pi G\rho_0 a^2\left[\frac{d}{d\xi} Q_2(-i\xi)\right]_{\xi=\xi_0}}{9\Delta_2} \tag{1.65}$$

Hence the general solution of the interior and exterior equation in equation (1.47) and (1.48) reduce to

$$f^-(\eta, \xi) = \left[\frac{2\pi G\rho_0 a^2(\eta^2+\xi^2)}{3} + A_2^- P_2(-i\xi)P_2(\eta) + A_0^- P_0(-i\xi)P_0(\eta)\right] \tag{1.66}$$

and

$$f^+(\eta, \xi) = B_0^+ Q_0(-i\xi)P_0(\eta) + B_2^+ Q_2(-i\xi)P_2(\eta) \tag{1.67}$$

These are the gravitational scalar potentials of the body in terms of rest mass density. It follows that gravitational scalar potential (1.66) and (1.67) may be expressed in terms of the total rest mass of the body by the relation

$$M_0 = \frac{4}{9} a^3 \rho_0 \pi \xi_0 (3 + \xi^2) \tag{1.68}$$

It may be noted that results from mathematical tables gives

$$Q_0(t) = \frac{1}{2}\ln\left(\frac{t+1}{t-1}\right) \tag{1.69}$$

$$Q_2(t) = \frac{1}{4}(3t^2 - 1)\ln\left(\frac{t+1}{t-1}\right) + \frac{3}{2}t \tag{1.70}$$

Consequently

$$Q_0(-i\xi) = i\left[\xi^{-1} + \frac{1}{3}\xi^{-3} + \frac{1}{5}\xi^{-5} + \cdots\right] \tag{1.71}$$

and

$$Q_2(-i\xi) = \frac{1}{2}\left[2\xi^{-1} + \left(\frac{1}{3} + \frac{1}{5}\right)\xi^{-3} + \left(\frac{1}{5} + \frac{1}{7}\right)\xi^{-5} + \cdots\right] \tag{1.72}$$

3. Summary and Conclusions

In this paper we showed how to formulate the Riemannian Gravitational Field Equation for static homogeneous oblate spheroidal massive bodies using the Riemannian Laplacian operator and golden metric tensor. Also, this paper Newton's gravitational scalar potentials for a homogeneous oblate spheroidal body was formulated and solved with the exact and complete results given by (1.71) and (1.72). Consequently, these potentials are now available for application in physics.

A profound philosophical inference is suggested by the results obtained in this paper that Newton's gravitational field equations for a spheroidal body are linear and separable, and hence solvable in terms of the well known special functions of mathematical physics, the Legendre, function.

This fact suggests that Einstein's geometrical theory of gravitation, GRT, whose gravitational field equation are non linear may not be the most natural generalization of Newton's dynamical theory of gravitation.

It is now known that most bodies in the universe are spheroidal in nature. As an example, it is now well known that satellite orbits around the earth are not governed by just the simple inverse distance squared gravitational fields of perfect spherical geometry. They are also governed by second harmonics (pole of order 3), as well as fourth harmonics (pole of order 5) of gravitational scalar potential due to imperfect spherical geometry.

In the first place this paper opens the door for the physical interpretation of all the solutions obtained in this work and hence experimental investigation in the motion of all bodies in the Earth's atmosphere and solar system as well as all other gravitating systems in the universe.

In the second place the door in henceforth opened for the mathematical study of all the unsolved equations of motion for all particles in all types of motions in all systems in the universe, such as more accurate calculation of (i) missile and satellite and space craft trajectories in the earth's atmosphere (ii) motions of moons around their planets, (iii) motion of planets (iv) comets and asteroids around the sun and (v) motions of stars around their galactic nuclei in the universe.

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