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The Electromagnetic Radiation and Gravity

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Abstract Is already known that a non-inertial reference frame is equivalent to a certain gravitational field. In this paper, a non-uniformly accelerated linear motion of the reference frame is analyzed. We will try to prove that this motion generates a non-uniform and variable gravitational field, described by a Finsler metric. The attaching to the non-inertial reference frame of an electric charge, leads to changes in the structure of this gravitational field. The changes are produced by the EM radiation emitted, and can be easily recognized in the mathematical expression of the metric of space-time. The motion of the electric charge is analyzed also from a quantum perspective. The connection of the Schrodinger equation with the metric of space-time can be realized by means of a function of coordinates, which defines some Lorentz non-linear transformations. At the basis of these theoretical approaches is lay a variational principle in which the velocity of the particle is considered as a function of coordinates

Keywords Damped quantum oscillator, Lorentz non-linear transformations, a different type of variational principle, Finsler space

1. Introduction

Let us consider that we have a positive charge (q) located in the center of a inertial reference frame (O; x, y, z, t), and a negative charge $(-q)$ located in the center of a non-inertial reference frame (O'; x', y', z', t'). Then, let us imagine that the non-inertial reference frame is moving with the acceleration (a_x) in the (x) direction with respect to the inertial reference frame. Also, let us assume that the accelerated charge is moving such that the bilinear form

$$x^2 - c^2t^2 \tag{1.1}$$

is an invariant with respect to the following non-linear transformations

$$\begin{cases} x = a_{11}\, x^{'} + a_{12}\, ct' \\ ct = a_{21}\, x^{'} + a_{22}\, ct' \end{cases} \tag{1.2}$$

where, $a_{ij} = a_{ij}(x^{'}, t')$ (i,j=1,2). So, we try to find the functions $a_{ij}(x^{'}, t')$ such that

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \tag{1.3}$$

First we replace the coefficients $a_{ij} = a_{ij}(x^{'}, t')$ as follows

$$\begin{cases} a_{11} = \cosh\beta(x^{'}, t') \\ a_{22} = \cosh\vartheta(x^{'}, t') \end{cases} \begin{cases} a_{21} = \sinh\beta(x^{'}, t') \\ a_{12} = \sinh\vartheta(x^{'}, t') \end{cases} \tag{1.4}$$

Then, substituting the transformations (1.2) into equation (1.3), we get

$$\sinh(\beta - \vartheta) = 0 \tag{1.5}$$

So, we must have $\beta(x^{'}, t') = \vartheta(x^{'}, t')$, and the transformations (1.2) become

$$\begin{cases} x = x^{'}\cosh\beta + ct^{'}\sinh\beta \\ ct = x^{'}\sinh\beta + ct^{'}\cosh\beta \end{cases} \tag{1.6}$$

Also, we can admit the following inverse transformations

$$\begin{cases} x^{'} = x\cosh\alpha - ct\sinh\alpha \\ ct^{'} = ct\cosh\alpha - x\sinh\alpha \end{cases} \tag{1.7}$$

where $\alpha = \alpha(x, t)$.

Differentiating now the direct transformations (1.6), we obtain

$$dx = \cosh\beta\ dx^{'} + \sinh\beta\ c\,dt^{'} + \frac{\partial x}{\partial\beta}\,d\beta \tag{1.8}$$

$$c\,dt = \cosh\beta\ c\,dt^{'} + \sinh\beta\ dx^{'} + c\frac{\partial t}{\partial\beta}\,d\beta \tag{1.9}$$

Differentiating also the inverse transformations (1.7), we obtain

$$dx^{'} = \cosh\alpha\ dx - \sinh\alpha\ c\,dt + \frac{\partial x'}{\partial\alpha}d\alpha \tag{1.10}$$

$$dt^{'} = \cosh\alpha\ dt - \frac{1}{c}\sinh\alpha\ dx + \frac{\partial t'}{\partial\alpha}d\alpha \tag{1.11}$$

In order to find the physical semnification of the function (x, t), we observe that, for the origin $x^{'} = 0$ of the non-inertial reference frame, the equation (1.10) becomes

$$\cosh\alpha\ dx - \sinh\alpha\ c\,dt = 0 \tag{1.12}$$

From this, we can deduce the derivative of the position x with respect to time t

$$\dot{x} = \frac{dx}{dt}\bigg|_{x^{'}=0} = c\tanh\alpha\,(x, t) \tag{1.13}$$

which signifies the velocity of non-inertial reference frame with respect to the inertial reference frame. Also, in order to find the physical semnification of the function $\beta(x', t')$, we observe that, for the origin $x = 0$ of the inertial reference

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frame, the equation (1.8) becomes

$$\cosh\beta \; dx^{'} + \sinh\beta \; c\, dt^{'} = 0 \tag{1.14}$$

From this, we can deduce also the derivative of the position x' with respect to time t'

$$\dot{x'} = \frac{dx^{'}}{dt^{'}}\bigg|_{x=0} = -c\tanh\beta(x^{'},t') \tag{1.15}$$

which signifies the velocity of the inertial reference frame with respect to the non-inertial reference frame. But, according to the motion relativity principle, we must have

$$\dot{x'} = -\dot{x} \tag{1.16}$$

Therefore, we can admit the following identity

$$\beta(x^{'},t') = \alpha(x,t) \tag{1.17}$$

2. The Wave Function

Further on, we can associate a de Broglie wave with the accelerated charge. The wave must be stationary with respect to the non-inertial frame. Therefore, we can introduce the following wave function for the accelerated charge

$$\Psi(t') = Ae^{-i\omega^{'}t^{'}} \tag{2.1}$$

where t' is given by the equation (1.7). Thus, we can rewrite this function with respect to the inertial reference frame as follows

$$\Psi(x,t) = Ae^{-i\omega'\left(t\cosh\ \alpha(x,t) - \frac{1}{c}x\sinh\ \alpha(x,t)\right)} \tag{2.2}$$

Taking the partial time derivative of this function, we get

$$\frac{\partial}{\partial t}\Psi(x,t) = -i\omega'\left(\cosh\alpha - \frac{1}{c}\, x'\frac{\partial\alpha}{\partial t}\right)\Psi(x,t) \tag{2.3}$$

Now, we can introduce by definition the Hamiltonian operator

$$\widehat{H}\Psi(x,t) = i\tilde{h}\frac{\partial}{\partial t}\Psi(x,t)\bigg|_{x^{'}=0} \tag{2.4}$$

According to the equation (2.3), for the point $x^{'} = 0$ we get

$$\widehat{H}\Psi(x,t) = \tilde{h}\omega^{'}\cosh\alpha(x,t)\,\Psi(x,t) \tag{2.5}$$

Therefore, we can deduce from here the following Hamiltonian function

$$H(x,t) = H_0\cosh\alpha(x,t) \tag{2.6}$$

where H_0 is the rest energy of the accelerated charge

$$H_0 = \tilde{h}\omega^{'} = m_0c^2 \tag{2.7}$$

and m_0 is the charge rest mass. Taking now the partial x derivative of the function $\Psi(x,t)$, we get

$$\frac{\partial}{\partial x}\Psi(x,t) = -i\omega^{'}\left(-\frac{1}{c}\sinh\alpha - \frac{1}{c}\, x^{'}\frac{\partial\alpha}{\partial x}\right)\Psi(x,t) \tag{2.8}$$

Further on, we can introduce by definition the linear momentum

$$\hat{p}_x\,\Psi(x,t) = -i\tilde{h}\frac{\partial}{\partial x}\Psi(x,t)\bigg|_{x^{'}=0} \tag{2.9}$$

So, according to the equation (2.8), for the point $x^{'} = 0$ we can write

$$\hat{p}_x\,\Psi(x,t) = \frac{1}{c}\tilde{h}\omega^{'}\sinh\alpha(x,t)\,\Psi(x,t) \tag{2.10}$$

Thus, we obtain the following expression for the linear momentum of the accelerated charge

$$p_x = \frac{1}{c}H_0\sinh\alpha(x,t) \tag{2.11}$$

Substituting now the equation (1.13) into the equation (2.6), we get the well known formula

$$H(p_x) = mc^2 = c\sqrt{p_x^2 + m_0^2c^2} \tag{2.12}$$

where m is the charge moving mass

$$m = m_0\cosh\alpha = \frac{m_0}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} \tag{2.13}$$

Also, substituting the equation (1.13) into the equation (2.11), we get the well known formula

$$p_x = m\dot{x} \tag{2.14}$$

3. The Hamilton's Equations

According to the principles of analytical mechanics, we can introduce now the Lagrange function

$$L(\dot{x},x,t) = \dot{x}\,p_x - H(p_x,x,t) \tag{3.1}$$

and the linear momentum p_x as the derivative of lagrange function with respect to $\dot{x}$

$$p_x = \frac{\partial L}{\partial\dot{x}} \tag{3.2}$$

where, according to the equation (1.13), we can consider

$$\dot{x} = v_x(x,t) \tag{3.3}$$

Let us consider also the action functional

$$S[x] = \int_{t_1}^{t_2} L[t,x(t),\dot{x}(x,t)]dt \tag{3.4}$$

where t_1, t_2 are constants. We assume that the action $S[x]$ attains a local minimum at $f(t)$, and $g(t)$ is an arbitrary function that has at least one derivative and vanishes at the endpoints t_1, t_2 . So, for any number ε close to zero, we can write

$$S[f] \le S[f+\varepsilon g] = G(\varepsilon) \tag{3.5}$$

Thus, if the functional $S[x]$ has a minimum for $x = f$, the function $G(\varepsilon)$ has a minimum at $\varepsilon = 0$

$$G^{'}(0) \equiv \frac{dG}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_{t_1}^{t_2}\frac{dL}{d\varepsilon}\bigg|_{\varepsilon=0}dt = 0 \tag{3.6}$$

where, according to the equation (3.3), $x(t)$ and $\dot{x}(x,t)$ are functions of ε, of the form

$$x(t) = f(t) + \varepsilon g(t) \tag{3.7}$$

$$\dot{x}(x,t) = v_x[f(t)+\varepsilon g(t),t] + \varepsilon\dot{g}(t) \tag{3.8}$$

Taking the total derivative of $L[t,x(t),\dot{x}(x,t)]$, we get

$$\frac{dL}{d\varepsilon} = \frac{\partial L}{\partial x}\frac{dx}{d\varepsilon} + \frac{\partial L}{\partial\dot{x}}\frac{d\dot{x}}{d\varepsilon} \tag{3.9}$$

But, according to the equations (3.7) and (3.8), we have

$$\frac{dx}{d\varepsilon} = g(t) \tag{3.10}$$

$$\frac{d\dot{x}}{d\varepsilon} = \frac{\partial v_x}{\partial x}g(t) + \dot{g}(t) \tag{3.11}$$

Therefore, we get

$$G'(0) = \int_{t_1}^{t_2} \frac{dL}{d\varepsilon}\bigg|_{\varepsilon=0} dt$$

$$= \int_{t_1}^{t_2} \left\{ \left[\frac{\partial L}{\partial x} + \frac{\partial L}{\partial \dot{x}} \frac{\partial v_x}{\partial x} \right] g(t) + \frac{\partial L}{\partial \dot{x}} \dot{g}(t) \right\} dt$$

$$= \int_{t_1}^{t_2} \left\{ \left[\frac{\partial L}{\partial x} + \frac{\partial L}{\partial \dot{x}} \frac{\partial v_x}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \right] g(t) \right\} dt + \frac{\partial L}{\partial \dot{x}} g(t) \bigg|_{t_1}^{t_2} \quad (3.12)$$

According to the fundamental lemma of calculus of variations, we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial \dot{x}} \frac{\partial v_x}{\partial x} - \frac{\partial L}{\partial x} = 0 \quad (3.13)$$

In consequence, we can introduce now the Hamiltonian function as

$$H = \dot{x}\, p_x - L(\dot{x}, x, t) = H(p_x, x, t) \quad (3.14)$$

Differentiating now this function, according to the equation (3.13), we get the following equation

$$dH = \dot{x}\, dp_x - \left(\dot{p}_x - \frac{\partial v_x}{\partial x} p_x \right) dx - \frac{\partial L}{\partial t} dt \quad (3.15)$$

Comparing this expression with the mathematical expression

$$dH = \frac{\partial H}{\partial p_x} dp_x + \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial t} dt \quad (3.16)$$

We get the following Hamilton equations

$$\dot{p}_x - \frac{\partial v_x}{\partial x} p_x = - \frac{\partial H}{\partial x} \quad (3.17)$$

$$\frac{\partial H}{\partial p_x} = \dot{x} = v_x(x, t) \quad (3.18)$$

4. The Schrodinger Equation

Further on, we consider a function $F(p_x, x, t)$ which describes a physical quantity of the accelerated charge. Taking the total time derivative of the function $F(p_x, x, t)$, we obtain

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial p_x} \dot{p}_x + \frac{\partial F}{\partial t} \quad (4.1)$$

According to the equations (3.17) and (3.18), we can rewrite this formula as follows

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial v_x}{\partial x} \frac{\partial F}{\partial p_x} p_x + \frac{\partial F}{\partial t} \quad (4.2)$$

where $\{F, H\}$ is the classical Poisson bracket

$$\{F, H\} = \left(\frac{\partial F}{\partial x} \frac{\partial H}{\partial p_x} - \frac{\partial F}{\partial p_x} \frac{\partial H}{\partial x} \right) \quad (4.3)$$

Let us write the equation (4.2) under the following form

$$\frac{dF}{dt} = \frac{DF}{dt} + \{F, H\} \quad (4.4)$$

where D_t is an operator whose expression is given by

$$D_t = \frac{D}{dt} = \frac{\partial}{\partial t} + \frac{\partial v_x}{\partial x} p_x \frac{\partial}{\partial p_x} \quad (4.5)$$

Taking into consideration the analogy between quantum mechanics and analytical mechanics, we can introduce an operator $\hat{F}$ corresponding to the physical quantity F. According to the equation (4.4), the corresponding equation for this operator must be

$$\frac{d\hat{F}}{dt} = \frac{D\hat{F}}{dt} + \{\hat{F}, \hat{H}\} \quad (4.6)$$

where $\{\hat{F}, \hat{H}\}$ is the quantum Poisson bracket

$$\{\hat{F}, \hat{H}\} = \frac{i}{\tilde{\hbar}} \left(\hat{H}\, \hat{F} - \hat{F}\, \hat{H} \right) \quad (4.7)$$

According to the equation (4.6), we can define the time derivative of the expectation value of the observable F as follows

$$\langle \frac{dF}{dt} \rangle = D_t \langle F \rangle \quad (4.8)$$

where $\langle F \rangle$ is the mean value (expectation value) of the observable F. Also, the law that describes the time evolution of a particle must be of the form

$$\hat{H} \Phi(p_x, t) = i\tilde{\hbar} D_t \Phi(p_x, t) \quad (4.9)$$

In this equation, $\Phi(p_x, t)$ is the wave function of a quantum system, in the p_x representation. According to this representation, the operators $\hat{x}$ and $\hat{p}_x$ are given by the expressions

$$\begin{cases} \hat{x} = i\tilde{\hbar} \frac{\partial}{\partial p_x} \\ \hat{p}_x = p_x \end{cases} \quad (4.10)$$

Therefore, in the expression (4.5), we can make the replacing

$$\frac{\partial}{\partial p_x} = - \frac{i}{\tilde{\hbar}} \hat{x} \quad (4.11)$$

which leads us to the following expression of the operator D_t in the p_x representation

$$D_t = \frac{D}{dt} = \frac{\partial}{\partial t} - \frac{i}{\tilde{\hbar}} \frac{\partial v_x}{\partial x} \hat{p}_x \hat{x} \quad (4.12)$$

But we seek the Schrodinger equation (4.9) in the x representation. For this, we must introduce the operators

$$\begin{cases} \hat{p}_x = -i\tilde{\hbar} \frac{\partial}{\partial x} \\ \hat{x} = x \end{cases} \quad (4.13)$$

and the canonical commutation relation

$$\hat{p}_x\, \hat{x} = \hat{x}\, \hat{p}_x - i\tilde{\hbar} \quad (4.14)$$

According to this relation, in the expression (4.12) we can make the replacing

$$\hat{p}_x\, \hat{x} = -i\tilde{\hbar}\, \hat{x} \frac{\partial}{\partial x} - i\tilde{\hbar} \quad (4.15)$$

which leads us to the following expression of the operator D_t in the x representation

$$D_t = \frac{\partial}{\partial t} - \frac{\partial v_x}{\partial x} x \frac{\partial}{\partial x} - \frac{\partial v_x}{\partial x} \quad (4.16)$$

Therefore, we can write now the following Schrodinger equation

$$\hat{H} \Psi(x, t) = i\tilde{\hbar} D_t \Psi(x, t) \quad (4.17)$$

where the new Hamiltonian operator for the accelerated charge is given by

$$\hat{H} = i\tilde{\hbar} D_t = i\tilde{\hbar} \frac{\partial}{\partial t} - i\tilde{\hbar} \frac{\partial v_x}{\partial x} x \frac{\partial}{\partial x} - i\tilde{\hbar} \frac{\partial v_x}{\partial x} \quad (4.18)$$

The additional term is of complex nature and its real part signify the potential energy of the field of force in which the particle is moving.

5. The Damped Oscillator

Let us consider that the accelerated charge $(-q)$ is bound to the fixed charge (q) by an elastic force of the form

$$F_{el.} = -Kx \;(K = m_0\omega_0^2) \tag{5.1}$$

Taking the total time derivative of the equation (2.6), we can introduce, according to the special relativity theory, the following equation

$$\frac{dH}{dt} = mc\dot{x}\frac{d\alpha}{dt} = \vec{F}\vec{v} = \dot{x}F_x \tag{5.2}$$

where F_x is the resultant force which acts on the accelerating charge

$$F_x = \frac{dp_x}{dt} \tag{5.3}$$

This force must be the sum between the elastic force and the radiative reaction force

$$F_{rad.} = \frac{\mu_0 q^2}{6\pi c}\dddot{x} \tag{5.4}$$

Therefore, for non-relativistic velocities, we get the equation of motion

$$\ddot{x} - \frac{\mu_0 q^2}{6\pi c}\dddot{x} + \omega_0^2 x = 0 \tag{5.5}$$

We know that this equation is applicable only to the extent that the radiative reaction force is small compared with the elastic force

$$\frac{\mu_0 q^2}{6\pi c}\dddot{x} \ll \omega_0^2 x$$

This condition leads us to the following equation of motion

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{5.6}$$

where

$$\gamma = \frac{\mu_0 q^2 \omega_0^2}{12\pi m_0 c} \tag{5.7}$$

Thus, the approximate solution of the equation (5.6) can be written as

$$x(t) = Ae^{-\gamma t}cos(\omega t + \varphi_0) \tag{5.8}$$

where ω is the angular frequency of the damped oscillator

$$\omega = \sqrt{\omega_0^2 - \gamma^2} \tag{5.9}$$

Also, from the equation (5.2), for non-relativistic velocities, we can now deduce a new differential equation

$$\frac{d\alpha}{dt} = -\frac{\omega_0^2}{c}x - \frac{2\gamma}{c}\dot{x} \tag{5.10}$$

The integration of this equation leads us to the solution

$$\begin{cases} \alpha(x,t) = -2\mu\, x - \varphi(t) \\ \mu = \gamma/c \end{cases} \tag{5.11}$$

where $\varphi(t)$ is given by the expression

$$\varphi(t) = \frac{\omega_0^2}{c}\int x\, dt \tag{5.12}$$

But, according to the equation (1.13), we must have the condition

$$\frac{dx}{dt} = c\tanh\left(-2\mu x - \frac{\omega_0^2}{c}\int x\, dt\right) \tag{5.13}$$

Taking the total time derivative of this equation, we get

$$\frac{d^2x}{dt^2} = c\frac{1}{cosh^2\left(-2\mu x - \frac{\omega_0^2}{c}\int x\, dt\right)}\left(-2\mu\frac{dx}{dt} - \frac{\omega_0^2}{c}x\right)$$

Now, we can replace

$$\frac{1}{cosh^2\left(-2\mu x - \frac{\omega_0^2}{c}\int x\, dt\right)} = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \cong 1$$

Therefore, we get the equation

$$\frac{d^2x}{dt^2} = -2\mu c\frac{dx}{dt} - \omega_0^2 x$$

which is the same equation (5.6). Substituting this result into the equation (5.10), we can write the approximate equation

$$\frac{d\alpha}{dt} \cong \frac{1}{c}\ddot{x} \tag{5.14}$$

The solution of this equation is given by the expression

$$\alpha(x,t) = argtanh\left(\frac{\dot{x}}{c}\right) \tag{5.15}$$

which results from the equation (1.13). Taking now the total time derivative of the displacement (5.8), we get

$$\dot{x} = -\gamma Ae^{-\gamma t}cos(\omega t + \varphi_0) - \omega Ae^{-\gamma t}sin(\omega t + \varphi_0) \tag{5.16}$$

This expression can be written as a function of the displacement function $x(t)$, under the following form

$$\dot{x} = v_x(x,t) = -\gamma x - cu(t) \tag{5.17}$$

where the function $u(t)$ is given by the expression

$$u(t) = \frac{\omega}{c}Ae^{-\gamma t}sin(\omega t + \varphi_0) \tag{5.18}$$

In this way, the velocity becomes a function of the coordinate x, because here the displacement x is considered as a variable which no longer depends on time. Consequently, we may admit for $\alpha(x,t)$ a solution of the form

$$\alpha(x,t) = argtanh[-\mu x - u(t)] \tag{5.19}$$

6. The Damped Quantum Oscillator

Substituting now the solution (5.17) into the equation (4.18), we get, for non-relativistic velocities

$$\hat{H} = i\tilde{h}\frac{\partial}{\partial t} + i\gamma\tilde{h}x\frac{\partial}{\partial x} + i\gamma\tilde{h} \tag{6.1}$$

For the Hamiltonian operator of the oscillator, we take the expression

$$\hat{H} = -\frac{\tilde{h}^2}{2m_0}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m_0\omega_0^2 x^2 \tag{6.2}$$

Admitting for the wave function a solution of the form $\Psi(x,t) = R(x)T(t)$, after the separation of variables, we get the following two equations

$$-\frac{\tilde{h}^2}{2m_0}R''(x)+\frac{1}{2}m_0\omega_0^2x^2R(x)-i\gamma\tilde{h}[xR'(x)+Rx=ERx \quad (6.3)$$

$$i\tilde{h}\partial_t T(t)=E\,T(t) \quad (6.4)$$

where the constant E is the separation constant, and signifies the total energy of the quantum oscillator. Thus, after a rearrangement of the terms, the first equation can be rewritten as follows

$$R''(x)+2i\gamma\frac{m_0}{\tilde{h}}xR'(x)+\frac{2m_0}{\tilde{h}^2}\left(E+i\gamma\tilde{h}-\frac{1}{2}m_0\omega_0^2x^2\right)R(x)=0 \quad (6.5)$$

Further on, we can introduce a dimensionless variable

$$\rho=\sqrt{\frac{m_0\omega_0}{\tilde{h}}}\,x=\nu x$$

such that the equation becomes

$$R''(\rho)+2ib\rho R'(\rho)+(a-\rho^2)R=0 \quad (6.6)$$

in which we have introduced the notations

$$a=a_n+2ib \quad (6.7)$$

$$a_n=\frac{2E}{\tilde{h}\omega_0} \quad (6.8)$$

$$b=\frac{\gamma}{\omega_0} \quad (6.9)$$

Now, we can take for $R(\rho)$ a solution of the form

$$R(\rho)=w(\rho)e^{k\rho^2} \quad (6.10)$$

which generates the differential equation for the unknown function $w(\rho)$

$$w''+2\rho(2k+ib)w'+(2k+a)w+\left(4k^2+4ibk-1\right)\rho^2w=0 \quad (6.11)$$

The constant k can be determined from the equation

$$4k^2+4ibk-1=0 \quad (6.12)$$

In order to get a finite solution for the function $R(\rho)$, we choose the root

$$k=(-ib-d)/2 \quad (6.13)$$

where we have introduced the notation

$$d=\sqrt{1-b^2} \quad (6.14)$$

Therefore, the equation (6.11) becomes

$$w''(\rho)-2d\rho w'(\rho)+(a_n-d+ib)w(\rho)=0 \quad (6.15)$$

We can introduce now a power series expansion as solution

$$w(\rho)=\sum_{j=0}^{\infty}A_j\rho^j \quad (6.16)$$

Substituting this expansion into the differential equation (6.15), we get

$$\sum_j\left\{(j+1)(j+2)A_{j+2}-\left[2jd-\left(a_j-d+ib\right)\right]A_j\right\}y^j=0 \quad (6.17)$$

which leads us to the recursion relation

$$A_{j+2}=\frac{2jd-\left(a_j-d+ib\right)}{(j+1)(j+2)}A_j \quad (6.18)$$

Because the series must be finite, there exists some n such that when $j=n$, the numerator will be zero. Therefore, we get

$$a_n-d+ib=2nd \quad (6.19)$$

Substituting here the formula (6.8), we obtain the following complex expression for the total energy of the quantum oscillator

$$E=E_n-i\frac{\gamma}{2}\tilde{h} \quad (6.20)$$

where E_n are the quantized energies of the quantum oscillator

$$E_n=\tilde{h}\omega\left(n+\frac{1}{2}\right) \quad (6.21)$$

Substituting now the constant k into the expression (6.10), we get

$$R(\rho)=w(\rho)e^{-\frac{\sqrt{1-b^2}}{2}\rho^2}e^{-i\frac{b}{2}\rho^2} \quad (6.22)$$

Also, substituting the equation (6.19) into the equation (6.15), we get for the unknown function $w(\rho)$ the equation

$$w''(\rho)-2d\rho w'(\rho)+2ndw(\rho)=0 \quad (6.23)$$

If we introduce now a new dimensionless variable

$$r=\sqrt{d}\,\rho=\sqrt{\frac{m_0\omega}{\tilde{h}}}\,x=\tau x \quad (6.24)$$

the equation (6.23) becomes

$$H_n''(r)-2rH_n'(r)+2nH_n(r)=0 \quad (6.25)$$

where $H_n(r)$ are the Hermite polynomials. So, $w(r)=H_n(r)$, and the solution $R(x)$ can be written as

$$R(x)=N_nH_n(\tau x)e^{-\frac{\sqrt{1-b^2}}{2}\nu^2x^2}e^{-i\frac{b}{2}\nu^2x^2} \quad (6.26)$$

where N_n are the multiplication factors, whose expression can be determined by the normalization condition of the wave function.

For the second equation (6.4), we have the solution

$$T(t)=e^{-i(E/\tilde{h})\,t} \quad (6.27)$$

Substituting here the solution (6.20), we get the expression

$$T(t)=e^{-\frac{\gamma}{2}t}e^{-i(E_n/\tilde{h})t} \quad (6.28)$$

which describe the time evolution of the wave function for the accelerated charge.

7. The Metric of Space-Time

Let us write now the “distance” between two infinitesimally close events, into the inertial reference frame (O; x, y, z, t)

$$ds^2=c^2dt^2-dx^2 \quad (7.1)$$

Substituting here the equations (1.8) and (1.9), we get

$$ds^2=g_{00}c^2dt'^2+2g_{01}dx'cdt'+g_{11}dx'^2 \quad (7.2)$$

where the components of the metric tensor are

$$g_{00} = 1 + \frac{1}{c^2}\left(\frac{\partial\beta}{\partial t'}\right)^2 (x'^2 - c^2t'^2) + 2\frac{x'}{c}\frac{\partial\beta}{\partial t'} \quad (7.3)$$

$$g_{11} = -\left[1 - \left(\frac{\partial\beta}{\partial x'}\right)^2 (x'^2 - c^2t'^2) + 2ct'\frac{\partial\beta}{\partial x'}\right] \quad (7.4)$$

$$g_{01} = x'\frac{\partial\beta}{\partial x'} - t'\frac{\partial\beta}{\partial t'} + \frac{1}{c}\frac{\partial\beta}{\partial x'}\frac{\partial\beta}{\partial t'}(x'^2 - c^2t'^2) \quad (7.5)$$

But, according to the equations (1.17) and (5.11), we have

$$\beta(x', t') = \alpha(x,t) = -2\mu x - \varphi(t) \quad (7.6)$$

Then, according to the transformations (1.6), we can impose

$$x = \Lambda(v_x)(x' + v_x t') \quad (7.7)$$

$$t = \Lambda(v_x)\left(t' + \frac{v_x}{c^2}x'\right) \quad (7.8)$$

where

$$\Lambda(v_x) = \frac{1}{\sqrt{1 - v_x^2/c^2}} \quad (7.9)$$

Substituting now these expressions into the equation (7.6), we get

$$\beta(x', t') = -2\mu\ \Lambda(v_x)(x' + v_x t') - \varphi(x', t') \quad (7.10)$$

where

$$\varphi(x', t') = \varphi(t)\ \Big|_{t=\Lambda(v_x)\left(t' + \frac{v_x}{c^2}x'\right)} \quad (7.11)$$

Taking now the partial x' derivative of $\beta(x', t')$, we obtain

$$\frac{\partial\beta}{\partial x'} = -2\mu\Lambda(v_x) + \frac{d\varphi}{dt}\frac{\partial t}{\partial x'} \quad (7.12)$$

Also, taking the partial t' derivative of $\beta(x', t')$, we obtain

$$\frac{\partial\beta}{\partial t'} = -2\mu v_x\Lambda(v_x) + \frac{d\varphi}{dt}\frac{\partial t}{\partial t'} \quad (7.13)$$

But, according to the transformations (7.7) and (7.8), we can write

$$\frac{\partial t}{\partial x'} = \frac{v_x}{c^2}\Lambda(v_x) \quad (7.14)$$

$$\frac{\partial t}{\partial t'} = \Lambda(v_x) \quad (7.15)$$

$$\frac{d\varphi}{dt} = -\frac{\omega_0^2}{c}x = -\frac{\omega_0^2}{c}\Lambda(v_x)(x' + v_x t') \quad (7.16)$$

Substituting these expressions into the equations (7.12) and (7.13), we get

$$\frac{\partial\beta}{\partial x'} = -2\mu\Lambda(v_x) - \frac{\omega_0^2}{c^3}v_x\Lambda^2(v_x)(x' + v_x t') \quad (7.17)$$

$$\frac{\partial\beta}{\partial t'} = -2\mu v_x\Lambda(v_x) - \frac{\omega_0^2}{c}\Lambda^2(v_x)(x' + v_x t') \quad (7.18)$$

where the velocity v_x is given by the expression (5.17), which no longer can be written as a function of the coordinates (x', t'). So, in this particular case, the metric of space-time (7.2) becomes a Finsler metric, and the space becomes a Finsler space.

8. Conclusions

Intuitively, the non-uniformly accelerated linear motion of a oscillator must be equivalent to a uniform but variable gravitational field. However, the gravitational field described by the metric of space-time (7.2), is a non-uniform field. This is due to the particular mode of writing of the velocity of a particle as a function of the coordinates (x, t), and the function $\alpha(x, t)$, depending on the same coordinates, by means of the velocity. Also, from the equation (3.17), we can observe that the term $\frac{\partial v_x}{\partial x}p_x$ corresponds at the half of the radiative reaction force

$$\frac{\partial v_x}{\partial x}p_x = -\gamma m_0\dot{x} \cong \frac{1}{2}F_{rad.}$$

Therefore, the function $\alpha(x, t)$ which determines the Lorentz non-linear transformations, intervenes in the determination of the radiative reaction force of the EM field and, also, in the determination of the gravitational field which appears in the non-inertial reference frame. Indeed, if we impose now $\alpha = const.$, we get

$$\begin{cases} F_{rad.} = 0 \\ ds^2 = c^2dt'^2 - dx'^2 \end{cases}$$

so, both, the radiative reaction force and the gravitational field, vanish. On the other hand, the radiative reaction force vanishes again when the oscillator is a neutral particle, but the gravitational field does not vanish, because ω_0 is different from zero. This suggests that the EM energy carried by the EM radiation contributes to the energy of the gravitational field. Also we can conclude that the field which appears as an EM field into the inertial reference frame, appears as a gravitational field into the non-inertial reference frame.

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