

Analyzing Time near Horizon in Schwarzschild and De-Sitter Space-Time in Particle's Frame of Reference

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Abstract Here, we calculate the variation of distance due to gravitation in De-Sitter and Schwarzschild space. Then, we calculate motion of the particle in particle's frame of reference to get the result that the particle can't reach event horizon [$r=2GM$ for Schwarzschild space and $r=\sqrt{3/\lambda}$ for De-Sitter space] in any frame of reference.

Keywords Schwarzschild space, De-Sitter space, Length contraction

There have been works on past, from the evolution of general theory of relativity, about the singularity of the Schwarzschild metric. We clearly have, $r = 0$ and $r = 2GM$ as the singularities in Schwarzschild metric. Since Schwarzschild solution is the solution of Einstein's field equation in empty space, singularity at $r = 0$ is due to a finite mass concentrated at the point. However, the singularity at $r = 2GM$ is due to the use of general relativity to describe the physical phenomena and forms the centre of our discussion.

Lemaitre [1], Einstein and Rosen [2], Finkelstein [3], Fronsdal [4], Kruskal [5] etc. in past have concluded that the singularity at $r = 2GM$ is different from that at $r = 0$. With the motivation that curvature invariants are perfectly well behaved at $r = 2GM$, various efforts have been made by the likes of Lemaitre, Finkelstein, Kruskal etc. to remove the singularity there. However, we might have neglected in past that, for Schwarzschild metric, component of Ricci tensor R_{rr} is singular at $r = 2GM$, though it vanishes everywhere else [6]. The tool used to remove the singularity in each of the cases were coordinate transformation, and in every transformation from Schwarzschild metric r, t to r', t' there was the situation of either singular or infinite value of some variable at $r = 2GM$. Although the transformed metric becomes free from singularity, this shouldn't be thought of as the a great achievement. The two representations r, t and r', t' aren't equivalent representations as the change of parameter isn't proper i.e. the Jacobian of transformation is either zero or doesn't exists at $r = 2GM$ [7]. Hence, this doesn't inform us whether the singularity is essential singularity or coordinate singularity. In past, we might have focused too much to remove singularity, rather than analyzing the change in space by

gravitation in particle's frame of reference.

S. Chandrasekhar [8] had calculated the time taken by the particle to reach the event horizon for Schwarzschild metric, in particle's frame of reference. However, he didn't considered the variation of length by Gravitation, and used same elementary length dr given by Schwarzschild metric, in both non-inertial frame and particle's frame. The obvious result is the finite value of proper time to reach $r = 2GM$ in particle's frame of reference. Assuming the Principle of Equivalence between gravitation and inertia, variation of length by gravitation is considered in this article to get the result of infinite proper time to reach $r = 2GM$ in particle's frame of reference.

For spherically symmetric system with stationary gravitational field, we have the metric calculated by Schwarzschild [9] as;

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{1}{1 - \frac{2GM}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

Where, M is mass and G is Gravitational constant.

Now, from the Principle of Equivalence between gravitation and inertia, there exists corresponding locally inertial frame of the form,

$$d\tau^2 = dt'^2 - dr'^2 - r'^2(d\theta'^2 + d\phi'^2) \quad (2)$$

Considering only the radial component of motion and equating equations 1 and 2, we have,

$$\left(1 - \frac{2GM}{r}\right) dt^2 - \frac{1}{1 - \frac{2GM}{r}} dr^2 = dt'^2 - dr'^2 \quad (3)$$

Equation 3 gives the transformation relation from non-inertial frame of reference to the inertial frame of reference. One of the possible transformation relations compatible with Lorentz transformation in special case is,

$$dt' = \frac{1}{\sqrt{1-v^2}} \left[\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} dt - \frac{v}{\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}}} dr \right] \quad (4)$$

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$$dr' = \frac{1}{\sqrt{1-v^2}} \left[\frac{1}{\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}}} dr - v \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} dt \right] \quad (5)$$

So, equation 4 and 5 gives in general, the transformation relation from non-inertial frame to the inertial frame of reference moving with velocity v . In the case of gravitation, there exist inertial frame, which is locally inertial frame moving with the particle. Hence, v is not constant in the case of gravitation. We can resolve the problem by imagining an infinite no. of inertial systems moving with uniform velocity, one of which instantaneously matches the velocity of the particle [10] i.e. velocity of the particle with respect to the inertial frame is Δv . The particle is thus instantaneously at rest with respect to the inertial systems, which is moving with velocity v with respect to the given non-inertial frame. Since, velocity of the particle $\frac{dr}{dt} = \sqrt{\frac{2GM}{r}}$, velocity of our inertial systems varies as $\sqrt{\frac{2GM}{r}}$ and $v \rightarrow 1$ as $r \rightarrow 2GM$. If two frames have their relative velocities zero, i.e. $v=0$ then from above equations,

$$dt' = \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} dt \quad (6)$$

And

$$dr' = \frac{1}{\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}}} dr \quad (7)$$

These two relations give change in time and change in length (or distance) due to the transformation of coordinates from non-inertial to inertial frame of reference. Now, from 6 and 7 we have for photons,

$$\frac{dr'}{dt'} = \frac{1}{\left(1 - \frac{2GM}{r}\right)} \frac{dr}{dt} = 1 \quad \left(\text{for light, } \frac{dr}{dt} = \left(1 - \frac{2GM}{r}\right) \right)$$

Velocity of light is one in inertial frame of reference. In fact, the transformation rule of the form 4 and 5 is valid not only for Schwarzschild and De-Sitter space-time. But my fellow friend Narayan Khadka has shown, in his unpublished work that, it is in general valid for Kerr and KDS space-time also.

Equation 7 gives the variation in length due to gravitation. This equation shows that length gets smaller in non-inertial frame of reference [for $r \approx 2GM$, $dr' \gg dr$]. Similarly, equation 6 gives the relation for time dilation in non-inertial frame of reference.

We know that, for radial motion, equation of motion for the particle in Schwarzschild space is,

$$\frac{dr}{dt} = \sqrt{\left[1 - E \left(1 - \frac{2GM}{r}\right)\right]} \left(1 - \frac{2GM}{r}\right)$$

where $E = 0$ for photons

$E > 0$ for mass particles

Taking $E=1$, hereafter, we have,

$$\frac{dr}{dt} = \sqrt{\frac{2GM}{r}} \left(1 - \frac{2GM}{r}\right) \quad (8)$$

In the frame of reference moving with velocity v with respect to the frame given by Schwarzschild metric, we have, from equations 4 and 5,

$$\frac{dr'}{dt'} = \frac{\frac{dr}{dt} - v \left(1 - \frac{2GM}{r}\right)}{\left(1 - \frac{2GM}{r}\right) - v \frac{dr}{dt}}$$

Substituting the value of $\frac{dr}{dt}$ from equation 8, we have,

$$\frac{dr'}{dt'} = \frac{\sqrt{\frac{2GM}{r}} - v}{1 - v \sqrt{\frac{2GM}{r}}}, \quad v \neq 0$$

$$\frac{dr'}{dt'} = \sqrt{\frac{2GM}{r}}, \quad v = 0$$

As $\sqrt{\frac{2GM}{r}} \rightarrow v$ for the particle starting from infinity, we have, from above equation $\frac{dr'}{dt'} \rightarrow 0$. That is, particle comes to rest in the inertial frame of reference. Now, as $r = 2GM$, we have $\frac{dr'}{dt'} = 1$, which is impossible for material particle in inertial frame of reference. Hence, particle doesn't reach the event horizon in given inertial frame of reference.

In particle's frame of reference,

$$\frac{dr'}{d\tau} = \frac{1}{\sqrt{1-v^2}} \left[\frac{1}{\sqrt{1 - \frac{2GM}{r}}} \frac{dr}{dt} - v \sqrt{1 - \frac{2GM}{r}} \frac{dt}{dt} \right] \quad (\text{from equation 5})$$

Substituting the value of $\frac{dr}{dt} = \frac{2GM}{r}$ and $\frac{dt}{dt} = \frac{1}{1 - \frac{2GM}{r}}$ we have,

$$\frac{dr'}{d\tau} = \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \frac{1}{\sqrt{1-v^2}} \left[\sqrt{\frac{2GM}{r}} - v \right] \quad (9)$$

Analytic solution of equation 9 is very complicated. Integration near $r \approx 2GM$ and $v \approx 1$ gives,

$$\tau = 4(1-v) \left[v \ln \left(v - \sqrt{\frac{2GM}{r}} \right) - \sqrt{\frac{2GM}{r}} \right] \quad (10)$$

Here for $2GM = r$, we have, $d\tau = 0$, which is impossible for particles having mass. The result of integration 10, from our above argument, clearly diverges

as $r \approx 2GM$. And, as, $\sqrt{\frac{2GM}{r}} \rightarrow v$, velocity of the particle $\frac{dr'}{d\tau}$ goes to zero. Since, $v < 1$ for mass particle, we have, $r > 2GM$. Hence, the velocity of the particle, in particle's frame of reference, goes to zero before it reaches the event horizon. Due to the phenomena of length contraction, what we see as a small length in non-inertial frame of reference is very large length in inertial frame of reference. So, the particle takes infinite time ($\tau = \infty$) to reach $r \approx 2GM$. Thus, in any frame of reference, particle cannot cross $2GM = r$ in Schwarzschild space.

Similarly, in De-Sitter space, substituting the value $\frac{\lambda}{3}r^2$ instead of $\frac{2GM}{r}$ in all of the above equations, where λ is energy density, we have, for metric,

$$d\tau^2 = \left(1 - \frac{\lambda}{3}r^2\right) dt^2 - \frac{1}{1 - \frac{\lambda}{3}r^2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (11)$$

$$\frac{dr}{dt} = \sqrt{\frac{\lambda}{3}r^2 \left(1 - \frac{\lambda}{3}r^2\right)}$$

In the inertial frame of reference moving with velocity v with respect to the non-inertial frame of reference given by equation 10,

$$\frac{dr'}{dt'} = \frac{\sqrt{\frac{\lambda}{3}r^2 - v}}{1 - v\sqrt{\frac{\lambda}{3}r^2}}$$

Similarly, in particle's frame of reference,

$$\frac{dr'}{d\tau} = \frac{1}{\sqrt{1 - \frac{\lambda}{3}r^2}} \frac{1}{\sqrt{1 - v^2}} \left[\sqrt{\frac{\lambda}{3}r^2} - v \right] \quad (12)$$

By the same argument as above, equation 11 shows that particle cannot reach $r = \sqrt{\frac{3}{\lambda}}$ in De-Sitter space. As above, analytical solution of 12 is very complicated. Near $r \approx \sqrt{\frac{3}{\lambda}}$ and $v \approx 1$, we have,

$$\tau = \sqrt{(1 - v^2)} \ln\left(v - \sqrt{\frac{\lambda}{3}}r\right) \quad (13)$$

which clearly diverges for, $r \approx \sqrt{\frac{3}{\lambda}}$.

Conclusions

Hence, we have seen that the event horizon is the region of isolation for the particle in Schwarzschild space and De-Sitter space. Particle can't cross event horizon in any frame of reference. So, event horizon is the singularity in the particle's frame of reference also.

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