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Hamiltonian Control Systems

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Abstract In this paper it is intended to elaborate a framework in which we can incorporate external forces in the systems prescription with emphasis on Hamiltonian systems with external forces and on the consequences of external forces. An appealing tool for this case is the language of symplectic geometry. Definitions of Hamiltonian systems with external forces are given and it is shown how they fit very naturally into the framework. It is also shown that forces are basic variables and that they have to be included in the definitions of mechanical systems.

Keywords Hamiltonian, Control systems, Controllability, Observability

1. Introduction

The dynamics of a system can be formulated using either the Newtonian, the Lagrangian or the Hamiltonian approaches. In the Newtonian and the Lagrangian formulations it is assumed that if all the coordinates and velocities are simultaneously specified, the state of the system can be completely determined and its consequent motion calculated. This gives rise to solving a system of second order ordinary differential equations. On the other hand, the Hamiltonian formulation, we seek to describe the motion in terms of first order equations of motion parametized by generalized coordinates and momenta. The transition from the Newtonian and Lagrangian formulations to the Hamiltonian formulation corresponds to changing the variables in the system from $(q,\ \dot{q},\ t)$ to $(q,\ p,\ t)$ where $q=(q_1,\ ...,\ q_n)$ are the generalized coordinates and $p=(p_1\ ...,\ p_n)$ are the generalized momenta [1] . However the Hamiltonian methods are not superior to the other methods for direct solutions of mechanical problems. Rather we gain another more powerful method of working with physical systems. The usefulness of the Hamiltonian viewpoint lies in providing a framework for theoretical extension in many areas of physics such as statistical mechanics and quantum mechanics.

A Hamiltonian system is characterized by an existence of a symplectic structure on a smooth even-dimensional manifold [2]. The symplectic approach allows one to extend the local description of a dynamical system to a global description. [3] has shown how network modelling of lumped-parameter physical systems naturally leads to a geometrically defined class of systems, called port-controlled Hamiltonian systems with dissipation. The structural properties of these systems were discussed, in particular the existence of Casimir functions and their implications for stability. [4] has shown the geometric property and structure of the Hamilton--Jacobi equation arising from nonlinear control theory are investigated using symplectic geometry. [5] made an analysis on Hamiltonian systems and his results revealed a systematic geometric frame for generalized controlled Hamiltonian systems. The pseudo-Poisson manifold and the ω-manifold are proposed as the state space of the generalized controlled Hamiltonian systems were established. However the above results did not consider the external forces as basic variables.

In this paper it is therefore intended to show that external forces should. It will be shown that it is necessary to maintain forces as basic variables and those have to be included in the definition of mechanical systems.

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2. Hamilton's Equation

Let $L(q,\ \dot{q},\ t)$ be the Lagrangian function for a given system. Then the Hamiltonian function $H(q,\ p,\ t)$ for the system is defined by ([1], [6]).

$$H(q,\ p,\ t)=\sum_{i=1}^{n} p_i\dot{q}_i - L(q,\ \dot{q},\ t) \tag{1}$$

The differential of H is given by

$$dH=\sum_{i=1}^{n}\frac{\partial H}{\partial q_i}dq_i+\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}dp_i+\frac{\partial H}{\partial t}dt \cdot \tag{2}$$

Also

$$dH=\sum_{i=1}^{n}\dot{q}_i dp_i+\sum_{i=1}^{n}p_i d\dot{q}_i-\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}$$

$$d\dot{q}_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \tag{3}$$

Using $p_i = \frac{\partial L}{\partial \dot{q}_i}$ from the generalized momenta, equation (3) reduces to

$$dH = \sum_{i=1}^{n} \dot{q}_i dp_i - \sum_{i=1}^{n} \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt \tag{4}$$

Comparing (4) and (2) we get

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ -\dot{p}_i = \dfrac{\partial H}{\partial q_i} \end{cases} \quad i = 1, \ldots, n \tag{5}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad i = 1, \ldots, n \tag{6}$$

Equations (5) consist of a set of $2n$ first order ordinary differential equations called the Hamilton's equations of motion and (q, p) are called canonical coordinates.

2.1. Hamiltonian Control Systems

According to [2] we let

- X be a manifold M of the state space in a manifold with symplectic form ω,
- W be a manifold of the space of external variables with symplectic form ω^e ('e' for external). In local coordinates, $\omega^e = \sum_{i=1}^{n} du_i \wedge dy_i$ where $(u_1, \ldots, u_n)$ are the external forces (inputs) and $(y_1, \ldots, y_n)$ are the observations (outputs),
- A fiber bunder B be over M,
- A smooth function (smooth meaning C^∞) such that $f : B \to TM \times W$ (TM is a tangent bundle over M.

Then

(i) $\sum(M, W, B, f)$ with M and W symplectic manifolds is called a full Hamiltonian system if $f(B)$ is a Lagrangian submanifold of $(TM \times W, \Omega)$ where

$$\Omega = \sum_i \left(d\dot{p}_i \wedge dq_i - dq_i \wedge d\dot{p}_i \right) - \sum_j du_j \wedge dy_j$$

is derived from the local coordinates.

(ii) $\sum(M, W, B, f)$ is called degenerate Hamiltonian system if there exists a full Hamiltonian system $\sum(M, W, B', f')$ such that $f(B)$ is a submanifold of $f'(B')$.

The definition of Hamiltonian control system depends on the submanifold $f(B)$ and not on f and B separately. It can be observed that the set of external variables W can be split into inputs $\{u_1, \ldots, u_m\}$ and outputs $\{y_1, \ldots, y_m\}$. The inputs are the external forces (controls) and the outputs are the observations. If the external forces are constant, then the dynamics of the system are described by a Hamiltonian vector field on M.

2.2. Affine Hamiltonian Control Systems

Let (M, ω) be a symplectic manifold. Let Y be an observation manifold. Define the symplectic form Ω on $TM \times T^*Y$ by $\Omega = \pi_1^* \dot{\omega} - \pi_2^* \omega^e$ (T^*Y is the cotangent bundle of Y, π_1^* and π_2^* are pullbacks of $\dot{\omega}$ and ω^e by π_1 and π_2 respectively [2]. Then according to [7] an affine Hamiltonian system $\sum(M, T^*Y, D)$ is given by a submanifold $D \subset TM \times T^*Y$ such that

(i) D can be parametized by the coordinates of M and the coordinates of the fibres of T^*Y,

(ii) D is a Lagrangian submanifold of $(TM \times T^*Y, \Omega)$,

(iii) The value of the Y-coordinates of a point on D is a function only of the M-coordinates of this point.

This system is thus given by ([8])

$$\begin{cases} \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H}{\partial p_i} \\ -\dfrac{\partial H}{\partial q_i} \end{pmatrix} + \sum_{j=1}^{m} u_j \begin{pmatrix} \dfrac{\partial C_j}{\partial p_i} \\ -\dfrac{\partial C_j}{\partial q_i} \end{pmatrix}; \\ y_j = C_j(q, p); j = 1, \ldots, m \end{cases}$$

$$i = 1, \ldots, n \tag{7}$$

In vector form we denote the system (7) by

$$\begin{cases} \dot{x} = X_H(x) + \sum_{i=1}^{m} u_i X_{C_i}(x) \\ y_i = C_i(x) \end{cases} \quad i = 1, \ldots, m \tag{8}$$

Because of (ii) above, D has a generating function. Because of (i) and (ii), this generating function has then form $H(q, p) - \sum_{j=1}^{m} u_j C_j(q, p)$ with (q, p) canonical coordinates for M and (y, u) natural coordinates for

T^*Y. Therefore $(\dot{q},\ \dot{p})$ coordinates of D are given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{pmatrix} + \sum_{j=1}^{m} u_j \begin{pmatrix} \frac{\partial C_j}{\partial p_i} \\ -\frac{\partial C_j}{\partial q_i} \end{pmatrix},\ i = 1,\ \ldots,\ n. \quad (9)$$

and the y coordinates by

$$y_j = C_j(q,\ p).$$

This is equivalent to linearizing $H(q,\ p,\ u)$ with respect to u. We note that without condition (iii), the external forces enter the system in a nonlinear way and the generating function of D is $H(q,\ p,\ u)$ which locally gives

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i}(q,\ p,\ u) \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}(q,\ p,\ u) \qquad i = 1,\ \ldots,\ n,\ j = 1,\ \ldots,\ m \\ y_j = -\frac{\partial H}{\partial u_j}(q,\ p,\ u) \end{cases} \quad (10)$$

This is just the usual general input-output Hamiltonian system. We note that if there are no dynamics i.e. no state space M, then D is just a Lagrangian manifold of T^*Y and this describes statistic mechanical systems. If there are no inputs and outputs i.e. no T^*Y, then D is a Lagrangian submanifold of $(TM,\ \dot{\omega})$. This describes a locally Hamiltonian vector field.

2.3. Linear Hamiltonian Control Systems

The Linear system in state form $\sum(A,\ B,\ C,\ D)$ given by

$$\begin{cases} \dot{x} = Ax = Bu & x \in \mathbf{X},\ u \in \mathbf{U} \\ w = Cx + Du & w \in \mathbf{W} \end{cases} \quad (11)$$

is a linear Hamiltonian system if $\mathbf{X}$ and $\mathbf{W}$ are symplectic linear spaces. It is a result in symplectic geometry that there exists a nondegenerate skew-symmetric bilinear form $\mathbf{J}$ and $\mathbf{J}^e$ on the state space $\mathbf{X}$ and on the set of external variables $\mathbf{W}$ respectively ([2]). In the same language o symplectic geometry if $\dim \mathbf{X} = 2n$ and $\dim \mathbf{W} = 2m$ then there exists bases of $\mathbf{X}$ and $\mathbf{W}$ such that in these bases

$$\mathbf{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \text{ and } \mathbf{J}^e = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \text{ respectively.}$$

[9] has established and proved by the following theorem the conditions required for the system $\sum(A,\ B,\ C,\ D)$ above to be a full linear Hamiltonian system.

Theorem 1:

Let $\sum(A,\ B,\ C,\ D)$ be a linear system given by (11) above. If $\begin{pmatrix} B \\ D \end{pmatrix}$ is injective and $(X,\ J)$, $(W,\ J^e)$ are linear symplectic spaces, then $\sum$ is a full Hamiltonian system if $A,\ B,\ C$ and D satisfy the following:

$$\begin{aligned} A^T\mathbf{J} + \mathbf{J}A - C^T\mathbf{J}^e C &= \mathbf{0} \\ B^T\mathbf{J} \quad - D^T\mathbf{J}^e C &= \mathbf{0} \\ D^T\mathbf{J}^e D &= \mathbf{0} \\ \operatorname{rank} D &= m \end{aligned} \quad (11)$$

If the feedback given by $A \to A + BF$, $C \to C + DF$ is applied to $\sum(A,\ B,\ C,\ D)$, then necessarily $\sum(A',\ B',\ C',\ D')$ has to satisfy

$$\begin{aligned} A'^T T + \mathbf{J}A' &= \mathbf{0} \\ B'^T\mathbf{J} &= \overline{\mathbf{C}} \\ \mathbf{C}' &= \begin{pmatrix} \overline{C}' \\ 0_m \end{pmatrix} \\ \mathbf{D}' &= \begin{pmatrix} 0_m \\ I_m \end{pmatrix} \end{aligned} \quad (12)$$

where

$$\overline{\mathbf{C}} = \begin{pmatrix} C \\ 0_m \end{pmatrix}.$$

The condition that $\begin{pmatrix} B \\ D \end{pmatrix}$ is injective is similar to the condition that $f : B \to TM \times W$ is an embedding for the case of nonlinear systems.

3. Controllability and Observability for Hamiltonian Systems

We shall consider the ideas of controllability and observability only for affine nonlinear Hamiltonian systems. Consider the system given by

$$\begin{cases} \dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) \\ y_i = C_i(x) \qquad i = 1,\ \ldots,\ m \end{cases} \quad (13)$$

Defined on a symplectic manifold $(M,\ \omega)$ where A is a locally Hamiltonian vector field i.e. the Lie derivative $\pounds_A \omega = 0$. And B_i is the Hamiltonian vector fields such that $\omega(B_i,\ -) = dC_i$ ([10]).

Definition 1

Consider the affine system (13) above. We define $F_{\circ} = (C_1, ..., C_p)$ and $F_k = £_{\Gamma} F_{k-1} + F_{k-1}$. This system is locally weakly observable if $G = \bigcup_{k \geq 0} F_k$ satisfies $dG(x) = T_x^* M \quad \forall x \in M$. Here $dG(x)$ is the linear subspace of $T_x^* M$ spanned by $dh(x)$ with $h \in G$ [11].

For Hamiltonian systems, since $£_A \omega = 0$ and there exists $H : M \to \mathbb{R}$ such that $A = X_H$ then the F_k's defined above satisfy $F_k = \{F, F_{k-1}\} + F_{k-1}$ with $F = H + (C_1, ..., C_m)$ the affine subspace of vector space of functions on M. ([6])

We define Controllability and Observability as follows:

Controllability: Controllability is the set of points reachable from x_0 in time $T > 0$ by applying input functions $\bar{u} : [0, T] \to \mathbb{R}$ to $\dot{x} = X_H(x) + \sum_{j=1}^{m} u_j X_{c_j}(x)$ with initial condition $x(0) = x_0$ contains an open subset of M for every $T > 0$ and for every ([9]).

Observability:

For every $x_0 \in M$ and every $T > 0$ there exists a neighbourhood V of x_0 such that for x_1 and x_2 in V and $x_1 \neq x_2$, there exists input functions $u : [0, T] \to \mathbb{R}$ such that, if we denote the solutions of $\dot{x} = X_H(x) + \sum_{j=1}^{m} u_j X_{C_j}(x)$ on $[0, T]$ corresponds to initial conditions $x_1(0)$ and $x_2(0)$ by $\bar{x}_1$ and $\bar{x}_2$ respectively, then the output functions $C_1(\bar{x}_1), ..., C_m(\bar{x}_1)$ and $C_1(\bar{x}_2), ..., C_m(\bar{x}_2)$ are different while the trajectories $\bar{x}_1$ and $\bar{x}_2$ remain in V ([9]).

4. Feedback for Hamiltonian Systems

Define a function $P : Y \to \mathbb{R}$ such that $\beta = dP$, where β is a closed one-form on the output Y, and consider the output feedback given by

$$v \equiv \bar{\alpha}(y, u) = \frac{\partial P}{\partial y}(y) + u \qquad (14)$$

Let F be the graph of β and F be a Lagrangian submanifold i.e. $F \subset T^* Y$. Accordingly we say that the output feedback given by equation (14) is Hamiltonian ([5]).

Proposition 1

Let $\sum(M, T^* Y, L)$ be an affine Hamiltonian system given by

$$\begin{cases} \dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) \\ y_i = C_i(x) \end{cases} \quad i = 1, ..., m$$

with $£_A \omega = 0$ and so $A = X_H$ and $\omega(B_i, -) = dC_i$ i.e. $B_i - X_{C_i}$ ([5]). Let $u \mapsto v = \alpha(x, u)$ be a feedback for this system. After feedback, this system will again be an affine Hamiltonian system.

$$\begin{cases} \dot{x} = \tilde{A}(x) + \sum_{i=1}^{m} v_i \tilde{B}_i(x) \\ y_i = C_i(x) \end{cases} \quad i = 1, ..., m$$

iff α is a Hamiltonian feedback i.e. there exists a function $P : Y \to \mathbb{R}$ such that $\tilde{A}$ and $\tilde{B}_i$ satisfy

(i) $\tilde{B}_i = B_i$, $i = 1, ..., m$,

(ii) $\tilde{A} = X_{\tilde{H}}$, with $\tilde{H} = H + P \circ C$.

Hence Hamiltonian feedback adds a potential function which is only a function of the output.

Let us now consider a solution of Disturbance decoupling by observation feedback (DDOF). The formulation of DDOF is as follows: Let $\sum(A, B, C)$ be a Hamiltonian system on a symplectic space (X, J) where $\mathbf{J} = \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix}$. It is assumed that there are disturbances in this system and it is intended to control the state space. The system can be described by

$$\begin{cases} \dot{x} = Ax + Bu + Ed, & d \in \mathbb{R}^r \\ y = Cx \\ z = Hx, & z \in \mathbb{R}^2 \end{cases}$$

with d the disturbances and z the variables which are to be regulated. We shall call $\sum(A, B, C, E, H)$ with $\sum(A, B, C)$ Hamiltonian and $E^T J = H$ a Hamiltonian system with disturbances. Then the DDOF problem is to find a compensator

$$\begin{cases} \dot{w} = Nw + My \\ u = Lw + Ky \end{cases}, \quad w \in W$$

Such that the closed-loop system

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} E \\ 0 \end{pmatrix} d \\ z = \begin{pmatrix} H & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \end{cases}$$

decouples the disturbances d from z.

We shall require W to posses a symplectic form J_w and

$$\begin{aligned} N^T J_W + J_W N &= 0 \\ M^T J_W &= L \\ K &= K^T \end{aligned}$$

Proposition 2

Let $\sum(A, B, C, E, H)$ be a Hamiltonian system with disturbances. Then

(i) DDOF is solvable iff there exists an (A, B)-invariant subspace V contained in $Ker\ H$ and which is coisotropic ([10]).

(ii) DDOF is solvable if the pullback $V^*(Ker\ H)$ is coisotropic ([7]).

Let $\sum(A, B, C, E, H)$ be a Hamiltonian system with disturbances. Let $V \subset Ker\ H$ be (A, B)-invariant and Lagrangian, so DDOF is solvable by static output feedback $u = Ky$. Then also the Hamiltonian output feedback $u = \frac{1}{2}(K + K^T) y$ solves DDOF. ([7]).

5. Discussions and Conclusions

In the past, most treatments in classical mechanics have dealt only with analytical mechanics. This part of mechanics confines itself to the study of mechanical systems without external influences. When forces are present they are assumed as coming from a potential field. In this context one observes the motions, makes classifications etc. i.e. one does only descriptive work but cannot influence the behaviour of the system. This restriction entails a heavy loss of generality because external forces do come up at various places for instance experimental devices and technical applications and mostly cannot be derived from a potential function. Control theory on the other hand does prescriptive work. One attempts to express all models in input/output form so that the variables which may be manipulated and observed are clearly distinguished. Also one tries to find methods for regulating the response of systems by altering the equations of motion. Problems of this type are very natural i.e. in engineering, operational research, and in economics where the problems are laid out in terms of the decision variable (inputs) which have to be chosen on the basis of certain observations (outputs) and the mathematical model involves the interrelations between these variables. The Newtonian and Lagrangian approaches are as good as any other but it is more advantageous to develop the Hamiltonian point of view because it reveals certain structural features e.g. we can anlyze the non-linear systems globally; conservation laws follow easily. In this paper, a framework in which we can incorporate external forces in the systems prescription with emphasis on Hamiltonian systems with external forces and on the consequences of external forces has been elaborated.

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