International Journal of Theoretical and Mathematical Physics 2016, 6(1): 1-25
DOI: 10.5923/j.ijtmp.20160601.01

Two-Body Problem of Classical Electrodynamics with Radiation Terms - Periodic Solution (II)

Vasil G. Angelov

Department of Mathematics, University of Mining and Geology "St. I. Rilski", Sofia, Bulgaria

Abstract This paper is the second in a series of papers dedicated to the two-body problem of classical electrodynamics. In the first part we have derived equations of motion describing two-body problem of classical electrodynamics with radiation terms based on W. Pauli, J. L. Synge and P. A. M. Dirac results. The system obtained is a neutral one with respect to the unknown velocities with both retarded and advanced arguments depending on the unknown trajectories. We introduce a suitable operator whose fixed point is a periodic solution of the problem in question. Using fixed point theorem we prove the main result. Since we consider two charged particles in an internal frame of reference, we prove an existence-unique- ness of a periodic solution that implies an existence of closed orbits. In other words N. Bohr stationary states are a consequence of classical electrodynamics. We generalize A. Sommerfeld result where he has proved an existence of elliptic orbits of the classical Kepler problem. Our existence result gives also a method of overcoming the singularities.

Keywords Two-Body problem, Dirac radiation term, Neutral equations, Periodic solution, Fixed point theorem

1. Introduction

The primary goal of the present paper is to prove an existence-uniqueness of a periodic solution of the two-body system with radiation terms, derived in a recent paper [1]. We have already mentioned in [1] that using the relativistic form of Lienard-Wiechert retarded potentials (cf. [2]) J. L. Synge [3] has formulated the two-body problem of classical electrodynamics and has suggested the idea to generalize the model including Dirac radiation terms [4]. Developing Synge idea we have derived a new form of Dirac radiation terms [1]. So instead of Synge two-body system [2]

$$m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2} F_{rs}^{(2)} \lambda_s^{(1)} - \frac{2}{3}\frac{e_1^2}{c^2}\left(\ddot{\lambda}_r^{(1)} - \lambda_r^{(2)} \dot{\lambda}_s^{(1)} \dot{\lambda}_s^{(1)}\right);$$

$$m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2} F_{rs}^{(1)} \lambda_s^{(2)} - \frac{2}{3}\frac{e_2^2}{c^2}\left(\ddot{\lambda}_r^{(2)} - \lambda_r^{(1)} \dot{\lambda}_s^{(2)} \dot{\lambda}_s^{(2)}\right)$$

we have introduced the system

$$m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2}\left(F_{rs}^{(2)} \lambda_1^{(1)} + F_{rs}^{(1)rad} \lambda_s^{(1)}\right);$$

$$m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2}\left(F_{rs}^{(1)} \lambda_s^{(2)} + F_{rs}^{(2)rad} \lambda_s^{(2)}\right).$$

where the radiation terms $F_{rs}^{(1)rad}, F_{rs}^{(2)rad}$ have a new form, although $F_{rs}^{(1)rad}, F_{rs}^{(2)rad}$ are derived on the basis of the original Dirac physical assumptions [4]. In [1] we have proved that the 4th and the 8ht equations are consequences from the rest ones and after some transformations we have reached a system of 6 equations as there are unknown functions.

We emphasize that classical Lorentz-Dirac radiation term and two- body equations of motion do not satisfy the basic relativistic equation

$$\lambda_n^{(p)} \lambda_n^{(p)} = \frac{\gamma_p u_\alpha^{(p)}(t)}{c} \frac{\gamma_p u_\alpha^{(p)}(t)}{c} + \left(i\gamma_p\right)^2 = -1$$

obviously fulfilled in our case [1].

We also note that on the base of Wheeler-Feynman formalism [8], [9] the authors in [10]-[14] have been obtained interesting results.

Our purpose, however, is to show the existence and uniqueness of a periodic solution of the two-body equations of motion obtained in [1]. We present the system in question in a suitable operator form and the fixed point of this operator appears a periodic solution of the system.

Section 1 is an introduction and we rearrange equations of motion in more convenient form. Main results are given in Section 2. In Subsection 2.1 we derive radiation terms in an explicit form and show that they are bounded, that is, free of singularities. In Subsection 2.2 we formulate the main periodic problem and prove some preliminary assertions. In order to apply fixed point theorems from [5] we introduce suitable function spaces and operator whose fixed points are solutions of the periodic problem mentioned and give some lemmas. Supplement 1 contains

* Corresponding author:
angelov@mgu.bg (Vasil G. Angelov)
Published online at http://journal.sapub.org/ijtmp

preliminary estimates that imply the operator maps the solution set into itself. In Supplement 2 Lipschitz estimates of the operator and its derivatives are obtained. Subsection 2.3 contains the main result: two-body system has a unique periodic solution. Subsection 2.4 includes numerical test results. Section 3 is a conclusion and shows the adequacy of the main theorem.

Recall denotations from [1]:

$\left(x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)} = ict\right)$ $(p=1,2)$ are the space- time coordinates of the moving particles;

$u^{(p)} = \left\{u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t)\right\} = \left\{\dot{x}_1^{(p)}(t), \dot{x}_2^{(p)}(t), \dot{x}_3^{(p)}(t)\right\}$ $(p=1,2)$ – velocities of the moving particles;

$\xi^{(pq)} = \left(\xi_1^{(pq)}, \xi_2^{(pq)}, \xi_3^{(pq)}, \xi_4^{(pq)}\right) = \left(x_1^{(p)}(t) - x_1^{(q)}(t-\tau_{pq}),\right.$ $\left. x_2^{(p)}(t) - x_2^{(q)}(t-\tau_{pq}), x_3^{(p)}(t) - x_3^{(q)}(t-\tau_{pq}),\ ic\tau_{pq}\right)$

$(pq) = (12),(21)$;

$\langle .,. \rangle$ – dot product in 3-dimensional Euclidian space.

In [1] we have derived the following equations of motion $\alpha = 1,2,3; (pq) = (12),(21)$:

$$\dot{u}_\alpha^{(p)} + \frac{u_\alpha^{(p)}}{\Delta_p^2}\langle u^{(p)}, \dot{u}^{(p)}\rangle = \frac{e_p e_q \Delta_p}{m_p c^2}\left(A_{pq}\xi_\alpha^{(pq)} + B_{pq}u_\alpha^{(q)} + C_{pq}\dot{u}_\alpha^{(q)}\right) + \frac{e_p^2 \Delta_p}{2m_p c^2}\left(A_{(p)r}\xi_\alpha^{(p)r} + B_{(p)r}u_\alpha^{(p)r} + C_{(p)r}\dot{u}_\alpha^{(p)r} - \right.$$
$$\left. - A_{(p)a}\xi_\alpha^{(p)a} - B_{(p)a}u_\alpha^{(p)a} - C_{(p)a}\dot{u}_\alpha^{(p)a}\right).$$

where

$$\Delta_p = \sqrt{c^2 - \left\langle u^{(p)}(t), u^{(p)}(t)\right\rangle};$$

$$\Delta_{pq} = \sqrt{c^2 - \left\langle u^{(q)}(t-\tau_{pq}), u^{(q)}(t-\tau_{pq})\right\rangle};$$

$$\Delta_{(p)r} = \sqrt{c^2 - \left\langle u^{(p)}(t-\tau), u^{(p)}(t-\tau)\right\rangle};$$

$$\Delta_{(p)a} = \sqrt{c^2 - \left\langle u^{(p)}(t+\tau), u^{(p)}(t+\tau)\right\rangle};$$

$$\left|Q_p\right| = \frac{\left|e_p e_q\right|}{m_p};\ r(t) = \sqrt{\sum_{\gamma=1}^{3}\left(x_\gamma^{(1)}(t) - x_\gamma^{(2)}(t)\right)^2};$$

$$\tau_{pq}(t) = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t) - x_\gamma^{(q)}(t-\tau_{pq}(t)\right]^2} = \frac{1}{c}\sqrt{\left\langle \xi^{(pq)}, \xi^{(pq)}\right\rangle};$$

$$D_{pq} = \frac{c^2\tau_{pq} - \langle \xi^{(pq)}, u^{(q)}\rangle}{c^2\tau_{pq} - \langle \xi^{(pq)}, u^{(p)}\rangle};$$

$$H_{pq} = \Delta_{pq}^2 + D_{pq}\frac{\Delta_{pq}^2\langle \xi^{(pq)}, \dot{u}^{(q)}\rangle + \left(\langle \xi^{(pq)}, u^{(q)}\rangle - c^2\tau_{pq}\right)\langle u^{(q)}, \dot{u}^{(q)}\rangle}{\Delta_{pq}^2};$$

$$D_{(p)r} = \frac{c^2\tau^{(p)r} - \langle \xi^{(p)r}, u^{(p)r}\rangle}{c^2\tau^{(p)r} - \langle \xi^{(p)r}, u^{(p)}\rangle};$$

$$H_{(p)r} = \Delta_{(p)r}^2 + D_{(p)r}\times\frac{\Delta_{(p)r}^2\langle \xi^{(p)r}, \dot{u}^{(p)r}\rangle + \left(\langle \xi^{(p)r}, u^{(p)\mathrm{r}}\rangle - c^2\tau^{(p)r}\right)\langle u^{(p)r}, \dot{u}^{(p)r}\rangle}{\Delta_{(p)r}^2};$$

$$D_{(p)a} = \frac{c^2\tau^{(p)a} - \langle \xi^{(p)a}, u^{(p)a}\rangle}{c^2\tau^{(p)a} - \langle \xi^{(p)a}, u^{(p)}\rangle};$$

$$H_{(p)a} = \Delta_{(p)a}^2 + D_{(p)a}\times\frac{\Delta_{(p)a}^2\langle \xi^{(p)a}, \dot{u}^{(p)a}\rangle + \left(\langle \xi^{(p)a}, u^{(p)a}\rangle - c^2\tau^{(p)a}\right)\left\langle u^{(p)a}, \dot{u}^{(p)a}\right\rangle}{\Delta_{(p)a}^2};$$

$$A_{pq} = \frac{H_{pq}\left(c^2 - \langle u^{(p)}, u^{(q)}\rangle\right)}{\left(c^2\tau_{pq} - \langle \xi^{(pq)}, u^{(q)}\rangle\right)^3} - D_{pq}\frac{\Delta_{pq}^2\langle u^{(p)}, \dot{u}^{(q)}\rangle + \left(\langle u^{(p)}, u^{(q)}\rangle - c^2\right)\langle u^{(q)}, \dot{u}^{(q)}\rangle}{\Delta_{pq}^2\left(\langle \xi^{(pq)}, u^{(q)}\rangle - c^2\tau_{pq}\right)^2};$$

$$B_{pq}=-\frac{\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta_{pq}^{2}\left(c^{2}\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)}-\frac{H_{pq}\left(c^{2}\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle\right)}{\left(c^{2}\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^{3}};$$

$$C_{pq}=-\frac{1}{c^{2}\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle};$$

$$A_{(p)r}=\frac{H_{(p)r}\left(c^{2}-\langle u^{(p)},u^{(p)r}\rangle\right)}{\left(c^{2}\tau^{(p)r}-\langle u^{(p)},u^{(p)r}\rangle\right)^{3}}-D_{(p)r}\frac{\Delta_{(p)r}^{2}\langle u^{(p)},\dot{u}^{(p)r}\rangle+\left(\langle u^{(p)},u^{(p)r}\rangle-c^{2}\right)\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta_{(p)r}^{2}\left(\langle\xi^{(p)r},u^{(p)r}\rangle-c^{2}\tau^{(p)r}\right)^{2}};$$

$$B_{(p)r}=-\frac{\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta_{(p)r}^{2}\left(c^{2}\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)}-\frac{H_{(p)r}\left(c^{2}\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)}\rangle\right)}{\left(c^{2}\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^{3}};$$

$$C_{(p)r}=-\frac{1}{c^{2}\tau^{(p)r}-\left\langle\xi^{(p)r},u^{(p)r}\right\rangle};$$

$$A_{(p)a}=\frac{H_{(p)a}\left(c^{2}-\langle u^{(p)},u^{(p)a}\rangle\right)}{\left(c^{2}\tau^{(p)a}-\langle u^{(p)},u^{(p)a}\rangle\right)^{3}}--D_{(p)a}\frac{\Delta_{(p)a}^{2}\langle u^{(p)},\dot{u}^{(p)a}\rangle+\left(\langle u^{(p)},u^{(p)a}\rangle-c^{2}\right)\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta_{(p)a}^{2}\left(\langle\xi^{(p)a},u^{(p)a}\rangle-c^{2}\tau^{(p)a}\right)^{2}};$$

$$B_{(p)a}=-\frac{\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta_{(p)a}^{2}\left(c^{2}\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)}a\rangle\right)}-\frac{H_{(p)a}\left(c^{2}\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)}\rangle\right)}{\left(c^{2}\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^{3}};$$

$$C_{(p)a}=-\frac{1}{c^{2}\tau^{(p)a}-\left\langle\xi^{(p)a},u^{(p)a}\right\rangle}.$$

Substituting D_{pq} (resp. $D_{(p)r},D_{(p)a}$) and H_{pq} (resp. $H_{(p)r},H_{(p)a}$) into A_{pq}, B_{pq} (resp. $A_{(p)r},B_{(p)r},A_{(p)a},B_{(p)a}$), transforming the expressions obtained under Dirac assumption $\tau^{(p)r}=\tau^{(p)a}=\tau=const.>0$ (cf. [4]) we have:

$$A_{pq}=\frac{\Delta_{pq}\left(c^{2}-\left\langle u^{(p)},u^{(q)}\right\rangle\right)}{\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^{3}}+\frac{\left(c^{2}-\left\langle u^{(p)},u^{(q)}\right\rangle\right)\langle\xi^{(pq)},\dot{u}^{(q)}\rangle}{\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^{2}\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(p)}\right\rangle\right)}-$$
$$-\frac{\left\langle u^{(p)},\dot{u}^{(q)}\right\rangle}{\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(p)}\right\rangle\right)};$$

$$B_{pq}=-\frac{\Delta_{pq}\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(p)}\right\rangle\right)+\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)\langle\xi^{(pq)},\dot{u}^{(q)}\rangle}{\left(c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^{3}};$$

$$C_{pq}=-\frac{1}{c^{2}\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle};$$

$$A_{(p)r}=\frac{\Delta_{(p)r}^{2}\left(c^{2}-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)}{\left(c^{2}\tau-\left\langle\xi^{(p)r},u^{(p)r}\right\rangle\right)^{3}}+\frac{\left(c^{2}-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)\langle\xi^{(p)r},\dot{u}^{(p)r}\rangle}{\left(c^{2}\tau-\left\langle\xi^{(p)r},u^{(p)r}\right\rangle\right)^{2}\left(c^{2}\tau-\left\langle\xi^{(p)r},u^{(p)}\right\rangle\right)}-$$
$$-\frac{\left(c^{2}\tau-\left\langle\xi^{(p)r},u^{(p)r}\right\rangle\right)\left\langle u^{(p)},\dot{u}^{(p)r}\right\rangle}{\left(c^{2}\tau-\left\langle\xi^{(p)r},u^{(p)r}\right\rangle\right)^{2}\left(c^{2}\tau-\left\langle\xi^{(p)r},u^{(p)}\right\rangle\right)};$$

$$B_{(p)r}=-\frac{\Delta_{(p)r}^{2}\left(c^{2}\tau-\left\langle \xi^{(p)r},u^{(p)}\right\rangle\right)-\left(c^{2}\tau-\left\langle \xi^{(p)r},u^{(p)r}\right\rangle\right)\left\langle \xi^{(p)r},\dot{u}^{(p)r}\right\rangle}{\left(c^{2}\tau-\left\langle \xi^{(p)r},u^{(p)r}\right\rangle\right)^{3}};$$

$$C_{(p)r}=-\frac{1}{c^{2}\tau-\left\langle \xi^{(p)r},u^{(p)r}\right\rangle};$$

$$A_{(p)a}=\frac{\Delta_{(p)a}^{2}\left(c^{2}-\langle u^{(p)},u^{(p)a}\rangle\right)}{\left(c^{2}\tau-\langle \xi^{(p)a},u^{(p)a},\rangle\right)^{3}}+\frac{\left(c^{2}-\left\langle u^{(p)},u^{(p)a}\right\rangle\right)\langle \xi^{(p)a},\dot{u}^{(p)a}\rangle}{\left(c^{2}\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle\right)^{2}\left(c^{2}\tau-\left\langle \xi^{(p)a)},u^{(p)}\right\rangle\right)}-$$

$$-\frac{\left(c^{2}\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle\right)\left\langle u^{(p)},\dot{u}^{(p)a}\right\rangle}{\left(c^{2}\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle\right)^{2}\left(c^{2}\tau-\left\langle \xi^{(p)a)},u^{(p)}\right\rangle\right)};$$

$$B_{(p)a}=-\frac{\Delta_{(p)a}^{2}\left(c^{2}\tau-\langle u^{(p)},\xi^{(p)a}\rangle\right)-\left(c^{2}\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle\right)\left\langle \xi^{(p)a},\dot{u}^{(p)a}\right\rangle}{\left(c^{2}\tau-\langle u^{(p)a},\xi^{(p)a}\rangle\right)^{3}};$$

$$C_{(p)a}=-\frac{1}{c^{2}\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle}.$$

Denoting by $G_{\alpha}^{(p)}$ the right-hand sides of the above system we obtain

$$\dot{u}_{\alpha}^{(p)}+\frac{u_{\alpha}^{(p)}}{\Delta_{p}^{2}}\langle u^{(p)},\dot{u}^{(p)}\rangle=G_{\alpha}^{(p)}.$$

Solving with respect to $\dot{u}_{1}^{(p)}(t),\dot{u}_{2}^{(p)}(t),\dot{u}_{3}^{(p)}(t)$ under assumption

$$\textbf{(C)}:\ \sqrt{\left\langle u^{(p)}(t),u^{(p)}(t)\right\rangle}\le\overline{c}<c$$

we reach the neutral system of six equations
$(\alpha=1,2,3;(pq)=(12),(21))$:

$$\dot{u}_{1}^{(p)}(t)=U_{1}^{(p)}\equiv\frac{c^{2}-\left(u_{1}^{(p)}(t)\right)^{2}}{c^{2}}G_{1}^{(p)}-\frac{u_{1}^{(p)}(t)u_{2}^{(p)}(t)}{c^{2}}G_{2}^{(p)}-\frac{u_{1}^{(p)}(t)u_{3}^{(p)}(t)}{c^{2}}G_{3}^{(p)}$$

$$\dot{u}_{2}^{(p)}(t)=U_{2}^{(p)}\equiv-\frac{u_{1}^{(p)}(t)u_{2}^{(p)}(t)}{c^{2}}G_{1}^{(p)}+\frac{c^{2}-\left(u_{2}^{(p)}(t)\right)^{2}}{c^{2}}G_{2}^{(p)}-\frac{u_{2}^{(p)}(t)u_{3}^{(p)}(t)}{c^{2}}G_{3}^{(p)}$$

$$\dot{u}_{3}^{(p)}(t)=U_{3}^{(p)}\equiv-\frac{u_{1}^{(p)}(t)u_{3}^{(p)}(t)}{c^{2}}G_{1}^{(p)}-\frac{u_{2}^{(p)}(t)u_{3}^{(p)}(t)}{c^{2}}G_{2}^{(p)}+\frac{c^{2}-\left(u_{3}^{(p)}(t)\right)^{2}}{c^{2}}G_{3}^{(p)}$$

$$(1.p)$$

for six unknown functions $u_{1}^{(1)}(t),u_{2}^{(1)}(t),u_{3}^{(1)}(t),u_{1}^{(2)}(t),$ $u_{2}^{(2)}(t),u_{3}^{(2)}(t)$, where $G_{\alpha}^{(p)}=G_{\alpha}^{(pq)}+G_{\alpha}^{(p)rad}$. The summands $G_{\alpha}^{(pq)}=\frac{e_{1}e_{2}\Delta_{p}}{m_{p}c^{2}}\left(A_{pq}\xi_{\alpha}^{(pq)}+B_{pq}u_{\alpha}^{(q)}+C_{pq}\dot{u}_{\alpha}^{(q)}\right)$ are called Lorentz terms, while the summand

$$G_{\alpha}^{(p)rad}=\frac{e_{p}^{2}\Delta_{p}}{2m_{p}c^{2}}\left(A_{(p)r}\xi_{\alpha}^{(p)r}+B_{(p)r}u_{\alpha}^{(p)r}+C_{(p)r}\dot{u}_{\alpha}^{(p)r}-A_{(p)a}\xi_{\alpha}^{(p)a}-B_{(p)a}u_{\alpha}^{(p)a}-C_{(p)a}\dot{u}_{\alpha}^{(p)a}\right)$$

– radiation terms.

2. Main Results

2.1. Explicit Form of the Radiation Term

In what follows we derive the explicit form of the radiation terms. By $C^{\infty}_{T_0}[0,\infty)$ we denote the set of all infinitely differentiable T_0 -periodic functions. We introduce the space of functions:

$$M = \Big\{u \in C^{\infty}_{T_0}[0,\infty) : \left|(u(t))^{(n)}\right| \le U_0 \omega^n e^{\mu(t-kT_0)}, \\ t \in [kT_0,(k+1)T_0]\Big\} \\ (k=0,1,2,...);(n=0,1,2,...),(pq)=(12),(21) \tag{2.1}$$

where U_0, ω, T_0 are positive constants and following A. Sommerfeld [7] $0 < \beta = \bar{c}/c < 1$.

Recall that $\tau^{(p)r} = \tau^{(p)a} = \tau$ is assumed to be infinitely small parameter because $\tau = \tau_0\sqrt{1-\beta^2}$, $\left(\tau_0 = r_e/c \approx 10^{-24}\sec\right)$ (cf. [2]). In fact $\tau_0 = r_e/c \approx 9{,}4.10^{-24}$ sec .

Since we work in spaces of infinitely smooth functions, using the Taylor expansions we obtain

$$\xi^{(p)a}_{\alpha} = x^{(p)}_{\alpha}(t+\tau) - x^{(p)}_{\alpha}(t) = \tau u^{(p)}_{\alpha}(t) + \frac{\tau^2}{2!}\dot{u}^{(p)}_{\alpha}(t) + ... \Rightarrow \xi^{(p)a}_{\alpha} = \tau u^{(p)}_{\alpha}(t) + O(\tau^2) \Rightarrow \xi^{(p)a}_{\alpha} \approx \tau u^{(p)}_{\alpha}(t);$$

$$\xi^{(p)r}_{\alpha} = x^{(p)}_{\alpha}(t) - x^{(p)}_{\alpha}(t-\tau) \approx u^{(p)}_{\alpha}(t)\tau;$$

$$u^{(p)}_{\alpha}(t+\tau) = u^{(p)}_{\alpha}(t) + \frac{\tau}{1!}\dot{u}^{(p)}_{\alpha}(t) + \frac{\tau^2}{2!}\ddot{u}^{(p)}_{\alpha}(t) + \frac{\tau^3}{3!}\dddot{u}^{(p)}_{\alpha}(t) + ... \Rightarrow u^{(p)}_{\alpha}(t+\tau) = u^{(p)}_{\alpha}(t) + O(\tau);$$

$$u^{(p)}_{\alpha}(t-\tau) = u^{(p)}_{\alpha}(t) - \frac{\tau}{1!}\dot{u}^{(p)}_{\alpha}(t) + \frac{\tau^2}{2!}\ddot{u}^{(p)}_{\alpha}(t) - \frac{\tau^3}{3!}\dddot{u}^{(p)}_{\alpha}(t) + ... \Rightarrow u^{(p)}_{\alpha}(t-\tau) = u^{(p)}_{\alpha}(t) - O(\tau);$$

$$u^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t+\tau) = \left(u^{(p)}_{\alpha}(t)\right)^2 + \frac{\tau^2}{1!}\dot{u}^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t) + \frac{\tau^2}{2!}\ddot{u}^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t) + ... \Rightarrow u^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t+\tau) = \left(u^{(p)}_{\alpha}(t)\right)^2 + O(\tau);$$

$$u^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t-\tau) = \left(u^{(p)}_{\alpha}(t)\right)^2 - \frac{\tau}{1!}\dot{u}^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t) + \frac{\tau^2}{2!}\ddot{u}^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t) - ... \Rightarrow u^{(p)}_{\alpha}(t)u^{(p)}_{\alpha}(t-\tau) = \left(u^{(p)}_{\alpha}(t)\right)^2 - O(\tau);$$

$$\left\langle u^{(p)}, u^{(p)a}\right\rangle = \left\langle u^{(p)}, u^{(p)}(t+\tau)\right\rangle = \sum_{\gamma=1}^{3} u^{(p)}_{\gamma}(t)u^{(p)}_{\gamma}(t+\tau) \approx \sum_{\gamma=1}^{3} u^{(p)}_{\gamma}(t)u^{(p)}_{\gamma}(t) = \left\langle u^{(p)}, u^{(p)}\right\rangle;$$

$$\left\langle u^{(p)}, u^{(p)r}\right\rangle \approx \left\langle u^{(p)}, u^{(p)}\right\rangle; \ \left\langle u^{(p)a}, u^{(p)a}\right\rangle \approx \left\langle u^{(p)}, u^{(p)a}\right\rangle; \ \left\langle u^{(p)r}, u^{(p)r}\right\rangle \approx \left\langle u^{(p)}, u^{(p)r}\right\rangle;$$

$$c^2\tau^{(p)r} - \left\langle \xi^{(p)r}, u^{(p)r}\right\rangle = c^2\tau - \tau\left\langle u^{(p)}(t), u^{(p)}(t-\tau)\right\rangle \approx \tau\left(c^2 - \left\langle u^{(p)}(t), u^{(p)}(t)\right\rangle\right);$$

$$c^2\tau^{(p)a} - \left\langle \xi^{(p)a}, u^{(p)a}\right\rangle = c^2\tau - \tau\left\langle u^{(p)}(t), u^{(p)}(t+\tau)\right\rangle \approx \tau\left(c^2 - \left\langle u^{(p)}(t), u^{(p)}(t)\right\rangle\right).$$

Then

$$G^{(p)rad}_{\alpha} = \frac{e_p^2\Delta_p}{2m_pc^2}\left(A_{(p)r}\xi^{(p)r}_{\alpha} - A_{(p)a}\xi^{(p)a}_{\alpha} + B_{(p)r}u^{(p)r}_{\alpha} - B_{(p)a}u^{(p)a}_{\alpha} + C_{(p)r}\dot{u}^{(p)r}_{\alpha} - C_{(p)a}\dot{u}^{(p)a}_{\alpha}\right)$$

$$= \frac{e_p^2\Delta_p}{2m_pc^2}\left[\xi^{(p)r}_{\alpha}\left(\frac{\Delta^2_{(p)r}\left(c^2 - \left\langle u^{(p)}, u^{(p)r}\right\rangle\right)}{\left(c^2\tau - \left\langle \xi^{(p)r}, u^{(p)r}\right\rangle\right)^3} + \frac{\left(c^2 - \left\langle u^{(p)}, u^{(p)r}\right\rangle\right)\langle \xi^{(p)r}, \dot{u}^{(p)r}\rangle - \left(c^2\tau - \left\langle \xi^{(p)r}, u^{(p)r}\right\rangle\right)\left\langle u^{(p)}, \dot{u}^{(p)r}\right\rangle}{\left(c^2\tau - \left\langle \xi^{(p)r}, u^{(p)}\right\rangle\right)\left(c^2\tau - \left\langle \xi^{(p)r}, u^{(p)r}\right\rangle\right)^2}\right)\right.$$

$$\left. - \xi^{(p)a}_{\alpha}\left(\frac{\Delta^2_{(p)a}\left(c^2 - \langle u^{(p)}, u^{(p)a}\rangle\right)}{\left(c^2\tau - \langle \xi^{(p)a}, u^{(p)a},\rangle\right)^3} + \frac{\left(c^2 - \left\langle u^{(p)}, u^{(p)a}\right\rangle\right)\langle \xi^{(p)a}, \dot{u}^{(p)a}\rangle - \left(c^2\tau - \left\langle \xi^{(p)a}, u^{(p)a}\right\rangle\right)\left\langle u^{(p)}, \dot{u}^{(p)a}\right\rangle}{\left(c^2\tau - \left\langle \xi^{(p)a)}, u^{(p)}\right\rangle\right)\left(c^2\tau - \left\langle \xi^{(p)a}, u^{(p)a}\right\rangle\right)^2}\right) + \right.$$

$$+\left(\frac{\Delta^2_{(p)a}\left(c^2\tau-\langle u^{(p)},\xi^{(p)a}\rangle\right)-\left(c^2\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle\right)\left\langle \xi^{(p)a},\dot u^{(p)a}\right\rangle}{\left(c^2\tau-\langle u^{(p)a},\xi^{(p)a}\rangle\right)^3}\right)u^{(p)a}_\alpha-$$

$$-\left(\frac{\Delta^2_{(p)r}\left(c^2\tau-\left\langle \xi^{(p)r},u^{(p)}\right\rangle\right)-\left(c^2\tau-\left\langle \xi^{(p)r},u^{(p)r}\right\rangle\right)\left\langle \xi^{(p)r},\dot u^{(p)r}\right\rangle}{\left(c^2\tau-\left\langle \xi^{(p)r},u^{(p)r}\right\rangle\right)^3}\right)u^{(p)r}_\alpha-\frac{\dot u^{(p)a}_\alpha}{c^2\tau-\left\langle \xi^{(p)a},u^{(p)a}\right\rangle}+\frac{\dot u^{(p)r}_\alpha}{c^2\tau-\left\langle \xi^{(p)r},u^{(p)r}\right\rangle}\Bigg]$$

$$=\frac{e_p^2\Delta_p}{2m_pc^2}\Bigg[\tau u^{(p)}_\alpha(t)\left(\frac{\Delta^2_{(p)r}\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)}{\tau^3\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)^3}+\frac{\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)\tau\langle u^{(p)},\dot u^{(p)r}\rangle-\tau\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)\left\langle u^{(p)},\dot u^{(p)r}\right\rangle}{\tau^3\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)^2}\right)$$

$$-\tau u^{(p)}_\alpha(t)\left(\frac{\Delta^2_{(p)a}\left(c^2-\langle u^{(p)},u^{(p)a}\rangle\right)}{\tau^3\left(c^2-\langle u^{(p)},u^{(p)a},\rangle\right)^3}+\frac{\left(c^2-\left\langle u^{(p)},u^{(p)a}\right\rangle\right)\tau\langle u^{(p)},\dot u^{(p)a}\rangle-\tau\left(c^2-\left\langle u^{(p)},u^{(p)a}\right\rangle\right)\left\langle u^{(p)},\dot u^{(p)a}\right\rangle}{\tau^3\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)\left(c^2-\left\langle u^{(p)},u^{(p)a}\right\rangle\right)^2}\right)-$$

$$+\left(\Delta^2_{(p)a}\left(c^2\tau-\tau\langle u^{(p)},u^{(p)}\rangle\right)-\left(c^2\tau-\tau\left\langle u^{(p)},u^{(p)a}\right\rangle\right)\tau\left\langle u^{(p)},\dot u^{(p)a}\right\rangle\right)\times\frac{u^{(p)a}_\alpha}{\tau^3\left(c^2-\langle u^{(p)a},u^{(p)}\rangle\right)^3}-$$

$$-\Delta^2_{(p)r}\left(c^2\tau-\tau\left\langle u^{(p)},u^{(p)}\right\rangle\right)-\left(c^2\tau-\tau\left\langle u^{(p)},u^{(p)r}\right\rangle\right)\tau\left\langle u^{(p)},\dot u^{(p)r}\right\rangle\times\frac{u^{(p)r}_\alpha}{\tau^3\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)^3}-\frac{\dot u^{(p)a}_\alpha}{c^2\tau-\tau\left\langle u^{(p)},u^{(p)a}\right\rangle}+\frac{\dot u^{(p)r}_\alpha}{c^2\tau-\tau\left\langle u^{(p)},u^{(p)r}\right\rangle}\Bigg]$$

$$=\frac{e_p^2\Delta_p}{2m_pc^2}\Bigg[\left(\frac{\Delta^2_{(p)r}}{\left(c^2-\left\langle u^{(p)},u^{(p)r}\right\rangle\right)^2}-\frac{\Delta^2_{(p)a}}{\left(c^2-\langle u^{(p)},u^{(p)a}\rangle\right)^2}\right)\frac{u^{(p)}_\alpha}{\tau^2}+\frac{\Delta^2_{(p)a}-\tau\left\langle u^{(p)},\dot u^{(p)a}\right\rangle}{\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^2}\frac{u^{(p)a}_\alpha}{\tau^2}-\frac{\Delta^2_{(p)r}-\tau\left\langle u^{(p)},\dot u^{(p)r}\right\rangle}{\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)^2}\frac{u^{(p)r}_\alpha}{\tau^2}-$$

$$-\frac{\left\langle u^{(p)},\dot u^{(p)a}\right\rangle u^{(p)a}_\alpha-\left\langle u^{(p)},\dot u^{(p)r}\right\rangle u^{(p)r}_\alpha}{\tau\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)^2}-\frac{\dot u^{(p)a}_\alpha-\dot u^{(p)r}_\alpha}{\tau\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)}\Bigg]\approx\frac{e_p^2\Delta_p}{2m_pc^2}\Bigg[\frac{\Delta^2_{(p)r}-\Delta^2_{(p)a}}{\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)^2}\frac{u^{(p)}_\alpha}{\tau^2}+\frac{\Delta^2_{(p)a}-\Delta^2_{(p)r}}{\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^2}\frac{u^{(p)}_\alpha}{\tau^2}-$$

$$-\frac{\left\langle u^{(p)},\dot u^{(p)a}\right\rangle-\left\langle u^{(p)},\dot u^{(p)r}\right\rangle}{\tau\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^2}u^{(p)}_\alpha-\frac{\dot u^{(p)a}_\alpha-\dot u^{(p)r}_\alpha}{\tau\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)}\Bigg]$$

$$=\frac{e_p^2\Delta_p}{m_pc^2}\left(-\frac{u^{(p)}_\alpha}{\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^2}\left\langle u^{(p)},\frac{\dot u^{(p)a}-\dot u^{(p)r}}{2\tau}\right\rangle-\frac{1}{c^2-\left\langle u^{(p)},u^{(p)}\right\rangle}\frac{\dot u^{(p)a}_\alpha-\dot u^{(p)r}_\alpha}{2\tau}\right).$$

In explicit form the radiation term is:

$$G^{(p)rad}_\alpha=-\frac{e_p^2\Delta_p}{m_pc^2}\left(\frac{u^{(p)}_\alpha(t)}{\left(c^2-\langle u^{(p)}(t),u^{(p)}(t)\rangle\right)^2}\left\langle u^{(p)}(t),\frac{\dot u^{(p)}(t+\tau)-\dot u^{(p)}(t-\tau)}{2\tau}\right\rangle+\frac{\dot u^{(p)}_\alpha(t+\tau)-\dot u^{(p)}_\alpha(t-\tau)}{2\tau}\frac{1}{\left(c^2-\left\langle u^{(p)}(t),u^{(p)}(t)\right\rangle\right)}\right)$$

or

$$G^{(p)rad}_\alpha=-\frac{e_p^2\Delta_p}{m_pc^2}\left[\frac{u^{(p)}_\alpha(t)}{\left(c^2-\langle u^{(p)}(t),u^{(p)}(t)\rangle\right)^2}\sum_{\gamma=1}^{3}u_\gamma{}^{(p)}(t)\frac{\dot u_\gamma{}^{(p)}(t+\tau)-\dot u_\gamma{}^{(p)}(t-\tau)}{2\tau}+\frac{1}{\left(c^2-\left\langle u^{(p)}(t),u^{(p)}(t)\right\rangle\right)}\frac{\dot u^{(p)}_\alpha(t+\tau)-\dot u^{(p)}_\alpha(t-\tau)}{2\tau}\right].$$

Lemma 2.1 The radiation term obtained is bounded provided $\tau\omega<2$.

Proof: We need an estimate of H. A. Schwartz difference quotient. In view of $u_\alpha^{(p)}(.) \in M$ we obtain

$$\left|\frac{\dot{u}_\alpha^{(p)}(t+\tau)-\dot{u}_\alpha^{(p)}(t-\tau)}{2\tau}\right| = \left|\left(u_\alpha^{(p)}(t)\right)^{(2)} + \frac{\tau^2}{3!}\left(u_\alpha^{(p)}(t)\right)^{(4)} + \frac{\tau^4}{5!}\left(u_\alpha^{(p)}(t)\right)^{(6)} + \ldots\right|$$

$$\le U_0\omega^2 + \frac{\tau^2}{3!}U_0\omega^4 + \frac{\tau^4}{5!}U_0\omega^6 + \ldots \le U_0\omega^2\left(1+\frac{\tau^2\omega^2}{3!}+\frac{\tau^4\omega^4}{5!}+\ldots\right)$$

$$\le U_0\omega^2\left(1+\frac{\tau^2\omega^2}{2^2}+\frac{\tau^4\omega^4}{2^4}+\ldots\right) \le \frac{4U_0\omega^2}{4-\tau^2\omega^2}$$

and then

$$\left|G_\alpha^{(p)rad}\right| \le \frac{e_p^2}{m_p c^2}\left|\frac{u_\alpha^{(p)}c\sqrt{3}}{\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^{3/2}}\frac{4U_0\omega^2}{4-\tau^2\omega^2} + \frac{1}{\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)^{1/2}}\frac{4U_0\omega^2}{4-\tau^2\omega^2}\right| \le$$

$$\le \frac{e_p^2}{m_p c^2}\left(\frac{c^2\sqrt{3}}{c^3\left(1-\beta^2\right)^{3/2}} + \frac{1}{c\left(1-\beta^2\right)^{1/2}}\right)\frac{4U_0\omega^2}{4-\tau^2\omega^2} \le \frac{e_p^2}{m_p}\frac{\sqrt{3}+1}{c^3\left(1-\beta^2\right)^{3/2}}.$$

Lemma 2.1 is thus proved.

2.2. Preliminary Assertions, Formulation of Main Periodic Problem and Lemmas

Our main purpose is to obtain an existence-uniqueness of T_0-periodic solution of the equations of motion (1.*p*).

We consider (1.*p*) jointly with functional equations

$$\tau_{pq}(t) = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(q)}(t-\tau_{pq}(t))\right]^2}\ , t\in(-\infty,\infty), (pq)=(12),(21) \tag{2.2}$$

and the initial value problem

$$\dot{u}_\alpha^{(p)}(t) = U_\alpha^{(p)}\ (\alpha=1,2,3),\ t\in[0,\infty) \tag{2.3}$$

$$u_\alpha^{(p)}(t) = u_{\alpha 0}^{(p)}(t),\ \ \dot{u}_\alpha^{(p)}(t) = \dot{u}_{\alpha 0}^{(p)}(t),\ t\in[-T_0,0],$$

if $T_0 \ge \tau^0 = \max\left\{\tau_{12}^0, \tau_{21}^0\right\}$; $\tau_{pq}^0 = \max\left\{\tau_{pq}(t): t\in[0,T_0]\right\}$.

Otherwise we take the initial interval $[-\tau^0, 0]$.

We have however already proved in [15], [16] that (2.2) has a unique continuous solution for every Lipschitz continuous trajectories. That is why we can consider only (2.3).

Remark 2.1 Here instead of inequality $\tau_{pq} \ge \frac{r(t)}{2c}$ from [15], [16] we can use $\tau_{pq}(t) = \frac{r(t)}{c}$. It follows immediately from $\tau_{pq}(t) = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(q)}(t-\tau_{pq}(t)\right]^2}$.

Indeed, considering J. Kepler problem we put $x_\gamma^{(q)} = 0$ which implies the relation required. We prove some additional properties of $\tau_{pq}(t)$.

Lemma 2.2 If $\left(x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t)\right)$ are smooth T_0- periodic trajectories with velocities $\left(u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t)\right)$ satisfying $\sqrt{\sum_{\gamma=1}^{3}\left(u_\gamma^{(p)}(t)\right)^2} \le \overline{c} < c$, then:

1) (3.1) has a unique smooth T_0-periodic solution.

2) The derivative $\dot{\tau}_{pq}(t) = \frac{1}{c}\frac{\left\langle \xi^{(pq)}, u^{(p)}\right\rangle - \left\langle \xi^{(pq)}, u^{(q)}\right\rangle}{\sqrt{\left\langle \xi^{(pq)}, \xi^{(pq)}\right\rangle} - \left\langle \xi^{(pq)}, u^{(q)}\right\rangle}$ satisfies inequalities $\dot{\tau}_{pq}(t) < 1$; $\frac{1}{1-\dot{\tau}_{pq}(t)} \le \frac{1+\beta}{1-\beta}$.

Proof:

1. The proof can be accomplished as in [15], [16].

2. Differentiating

$$\tau_{pq}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_{\gamma}^{(p)}(t)-x_{\gamma}^{(q)}(t-\tau_{pq}(t)\right]^{2}}$$

and solving with respect to $\dot{\tau}_{pq}(t)$ we obtain

$$\dot{\tau}_{pq}(t)=\frac{\left\langle \xi^{(pq)},u^{(p)}\right\rangle-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}{c\sqrt{\left\langle \xi^{(pq)},\xi^{(pq)}\right\rangle}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}.$$

Using that (2.2) has a unique solution we have

$$1-\dot{\tau}_{pq}(t)=1-\frac{\left\langle \xi^{(pq)},u^{(p)}\right\rangle-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}{c^{2}\tau_{pq}(t)-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}=\frac{c^{2}\tau_{pq}(t)-\left\langle \xi^{(pq)},u^{(q)}\right\rangle-\left\langle \xi^{(pq)},u^{(p)}\right\rangle+\left\langle \xi^{(pq)},u^{(q)}\right\rangle}{c^{2}\tau_{pq}(t)-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}$$

$$=\frac{c^{2}\tau_{pq}(t)-\left\langle \xi^{(pq)},u^{(p)}\right\rangle}{c^{2}\tau_{pq}(t)-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}\geq\frac{c^{2}\tau_{pq}(t)-c\tau_{pq}(t)\overline{c}}{c^{2}\tau_{pq}(t)+c\tau_{pq}(t)\overline{c}}=\frac{1-\beta}{1+\beta}>0.$$

Obviously $1-\dot{\tau}_{pq}(t)\neq 0$ and besides

$$1-\dot{\tau}_{pq}(t)\geq\frac{1-\beta}{1+\beta}\Rightarrow\frac{1}{1-\dot{\tau}_{pq}(t)}\leq\frac{1+\beta}{1-\beta}.$$

Lemma 2.2 is thus proved.

The main difficulty is to define a suitable operator whose fixed points are solutions sought.

Assuming that the initial point is $t_0=0$ we introduce the function set:

$$M_0=\left\{u(.)\in M:\int_{kT_0}^{(k+1)T_0}u(t)dt=0\ (k=0,1,2,3,...)\right\}, \tag{2.4}$$

$\mu,\ \omega, U_0.T_0=const.>0,(k=0,1,2,...),(m=0,1,2,...);\ U_0e^{\mu T_0}\leq\overline{c}\leq c.$

Introduce a family of pseudo-metrics

$$\begin{aligned}&\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)})=\\&=\max\left\{e^{-\mu(t-kT_0)}\omega^{-n}\left|u^{(n)}(t)-\overline{u}^{(n)}(t)\right|:t\in[kT_0,(k+1)T_0]\right\},\\&(n=0,1,2,...\,;\ k=0,1,2,...\ ).\end{aligned} \tag{2.5}$$

It follows that the terms are uniformly bounded

$$\begin{aligned}&e^{-\mu(t-kT_0)}\omega^{-n}\left|u^{(n)}(t)-\overline{u}^{(n)}(t)\right|\leq\\&\leq e^{-\mu(t-kT_0)}\omega^{-n}\,2\,\omega^{n}e^{\mu(t-kT_0)}U_0=2U_0<\infty.\end{aligned}$$

Therefore $\sup\left\{\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)}):n=0,1,2,...\right\}<\infty$ and we put $\rho_{(k,\infty)}(u,\overline{u})=\sup\left\{\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)}):n=0,1,2,...\right\}$.

Further on we put

$$\begin{aligned}\rho_{(k,n)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right)=\max\Big\{&\rho_{(k,n)}(u_1,\overline{u}_1),\rho_{(k,n)}(u_2,\overline{u}_2),\rho_{(k,n)}(u_3,\overline{u}_3),\\&\rho_{(k,n)}(u_4,\overline{u}_4),\rho_{(k,n)}(u_5,\overline{u}_5),\rho_{(k,n)}(u_6,\overline{u}_6)\Big\}\end{aligned}$$

and

$$\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right)=\max\Big\{\rho_{(k,\infty)}(u_1,\overline{u}_1),\rho_{(k,\infty)}(u_2,\overline{u}_2),\rho_{(k,\infty)}(u_3,\overline{u}_3),$$

$\rho_{(k,\infty)}(u_4,\overline{u}_4),\rho_{(k,\infty)}(u_5,\overline{u}_5),\rho_{(k,\infty)}(u_6,\overline{u}_6)\}$.

In fact

$$(u_1,u_2,u_3,u_4,u_5,u_6)=\left(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)}\right).$$

Assuming $u_\alpha^{(p)}(0)=0$ $\left(\Rightarrow u_\alpha^{(p)}(kT_0)=0\right)$ introduce the operator B as a 6-tuple

$$B(t)=\left(B_1^{(1)}(k)(t),\, B_2^{(1)}(k)(t),B_3^{(1)}(k)(t),B_1^{(2)}(k)(t),B_2^{(2)}(k)(t),B_3^{(2)}(k)(t)\right),$$

where

$$B_\alpha^{(1)}(k)(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)})(t):=\int_{kT_0}^{t}U_\alpha^{(1)}(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(1)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U_\alpha^{(1)}(s)dsdt$$

$$B_\alpha^{(2)}(k)(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)})(t):=\int_{kT_0}^{t}U_\alpha^{(2)}(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(2)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U_\alpha^{(2)}(s)dsdt \quad (2.6)$$

$\left(\alpha=1,2,3;\ k=0,1,2,...\right)$ $t\in[kT_0,(k+1)T_0]$.

In the right-hand-sides

$$U_1^{(p)}=\frac{c^2-\left(u_1^{(p)}(t)\right)^2}{c^2}G_1^{(p)}-\frac{u_1^{(p)}(t)u_2^{(p)}(t)}{c^2}G_2^{(p)}-\frac{u_1^{(p)}(t)u_3^{(p)}(t)}{c^2}G_3^{(p)}$$

$$U_2^{(p)}=-\frac{u_1^{(p)}(t)u_2^{(p)}(t)}{c^2}G_1^{(p)}+\frac{c^2-\left(u_2^{(p)}(t)\right)^2}{c^2}G_2^{(p)}-\frac{u_2^{(p)}(t)u_3^{(p)}(t)}{c^2}G_3^{(p)}$$

$$U_3^{(p)}=-\frac{u_1^{(p)}(t)u_3^{(p)}(t)}{c^2}G_1^{(p)}-\frac{u_2^{(p)}(t)u_3^{(p)}(t)}{c^2}G_2^{(p)}+\frac{c^2-\left(u_3^{(p)}(t)\right)^2}{c^2}G_3^{(p)}$$

(p = 1, 2) we substitute the functions with retarded arguments

$$u_\alpha^{(p)}(t-\tau_{pq}(t)),\ u_\alpha^{(p)}(t-\tau),\dot{u}_\alpha^{(p)}(t-\tau_{pq}(t)),\ \dot{u}_\alpha^{(p)}(t-\tau)$$

by the initial functions $u_{\alpha 0}^{(p)}(t)$ translated to the right on the interval $[0,\infty)$. By necessity we assume that $u_{\alpha 0}^{(p)}(t)$ are such that their translated image on $[0,\infty)$ belong to M_0.

We recall some assertions from [17]:

Lemma 2.3 [17] If $u_\alpha^{(p)}(.)\in M_0$ then

$x_\alpha^{(p)}(t)=x_{\alpha 0}^{(p)}+\int_{kT_0}^{t}u_\alpha^{(p)}(\tau)d\tau$ is a T_0-periodic function.

Lemma 2.4 [17] If the translated function $u_{\alpha 0}^{(p)}(.)\in C_{T_0}[-T_0,\infty)$, on $[0,\infty)$ satisfies

$\int_{kT_0}^{(k+1)T_0}u_\alpha^{(p)}(t)dt=0$ $(k=0,1,2,...)$ and $u_\alpha^{(p)}\in M_0$ then $U_\alpha^{(p)}(t)$ are T_0-periodic functions.

Lemma 2.5 [17] For every $u_\alpha^{(p)}\in M_0$ it follows

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}U_\alpha^{(p)}(\theta)d\theta ds=\int_{(k+1)T_0}^{(k+2)T_0}\int_{(k+1)T_0}^{s}U_\alpha^{(p)}(\theta)d\theta ds\ \ (k=0,1,2,...).$$

Lemma 2.6 [17] The function $B_{\alpha,k}^{(p)}(.)(t)$ belongs to M_0.

Lemma 2.7 The following inequalities are fulfilled:

1) $c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle)>cr_0\left(1-\beta\right)>0$;

2) $\left|\xi_\alpha^{(pq)}\right|=\left|x_\alpha^{(p)}(t)-x_\alpha^{(q)}(t-\tau_{pq})\right|\le\frac{2e^{\mu_0}}{\mu}\left(\omega/\mu\right)^{n-1}U_0$ for arbitrary n;

3) $\left|u_\alpha^{(q)}(t-\tau_{pq})\right| \le \left(e^{\mu_0}+1\right)\left(\dfrac{\omega}{\mu}\right)^n U_0$; $\left|\dot u_\alpha^{(q)}(t-\tau_{pq})\right| \le \left(e^{\mu_0}+1\right)(\omega/\mu)^n U_0$ for arbitrary n.

Proof:

1) $$c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle \ge c^2\tau_{pq}-\sqrt{\sum_{\alpha=1}^{3}\left(\xi_\alpha^{(pq)}\right)^2}\sqrt{\sum_{\alpha=1}^{3}\left(u_\alpha^{(q)}\right)^2} \ge c^2\tau_{pq}-c\tau_{pq}\overline{c}=cr(t)(1-\beta)>cr_0(1-\beta)>0;$$

2) $$\left|\xi_\alpha^{(pq)}\right|=\left|x_\alpha^{(p)}(t)-x_\alpha^{(q)}(t-\tau_{pq})\right|=\left|\int_{kT_0}^{t}u_\alpha^{(p)}(s)ds-\int_{kT_0}^{t-\tau_{pq}}u_\alpha^{(q)}(s)ds\right| \le \left|\int_{kT_0}^{t}u_\alpha^{(p)}(s)ds\right|+\left|\int_{kT_0}^{t-\tau_{pq}}u_\alpha^{(q)}(s)ds\right| \le$$

$$\le\left|\int_{kT_0}^{t}\int_{kT_0}^{s_1}\dots\int_{kT_0}^{s_{n-1}}\left(u_\alpha^{(p)}(s_n)\right)^{(n-1)}ds_n\right|+\left|\int_{kT_0}^{t-\tau_{pq}}\int_{kT_0}^{s_1}\dots\int_{kT_0}^{s_{n-1}}\left(u_\alpha^{(p)}(s_n)\right)^{(n-1)}ds_n\right| \le \frac{e^{\mu(t-kT_0)}-1}{\mu^n}U_0\omega^{n-1}+\left|\frac{e^{\mu(t-\tau_{pq}-kT_0)}-1}{\mu^n}\right|U_0\omega^{n-1}$$

$$\le\frac{e^{\mu(t-kT_0)}-1}{\mu^n}U_0\omega^{n-1}+\left|\frac{e^{\mu(t-\tau_{pq}-kT_0)}+1}{\mu^n}\right|U_0\omega^{n-1}\le\frac{e^{\mu(t-kT_0)}-1+e^{\mu(t-kT_0)}+1}{\mu^n}U_0\omega^{n-1}=\frac{2e^{\mu_0}}{\mu}(\omega/\mu)^{n-1}U_0;$$

3) $$\left|u(t-\tau_{pq})\right|=\left|\int_{kT_0}^{t-\tau_{pq}}\dot u(s_1)ds_1\right|=\left|\int_{kT_0}^{t-\tau_{pq}}\int_{kT_0}^{s_1}\ddot u(s_2)ds_2ds_1\right|=\dots=\left|\int_{kT_0}^{t-\tau_{pq}}\int_{kT_0}^{s_1}\dots\int_{kT_0}^{s_{n-1}}u^{(n)}(s_n)ds_n\dots ds_1\right|\le\frac{U_0\omega^n}{\mu^{n-1}}.\left|\int_{kT_0}^{t-\tau_{pq}}e^{\mu(s_1-kT_0)}ds_1\right|$$

$$=\frac{U_0\omega^n}{\mu^{n-1}}.\left|\frac{e^{\mu(t-\tau_{pq}-kT_0)}-1}{\mu}\right|\le\frac{U_0\omega^n}{\mu^n}\left|e^{\mu(t-\tau_{pq}-kT_0)}+1\right|\le\left(e^{\mu_0}+1\right)(\omega/\mu)^nU_0.$$

Lemma 2.7 is thus proved.

Lemma 2.8 The following inequalities are fulfilled:

$$\left|\int_{kT_0}^{(k+1)T_0}U_\gamma^{(p)}(s)ds\right|\le\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{(k+1)T_0}G_\alpha^{(pq)}ds\right|+\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{(k+1)T_0}G_\alpha^{(p)rad}ds\right|\le3\left|Q_p\right|\left(\frac{4e^{\mu_0}}{\mu^2r_0^3}+\frac{2e^{\mu_0}+2}{\mu cr_0^2}\right)\frac{e^{\mu_0}-1}{(1-\beta)^3}(\omega/\mu)^{n-1}U_0.$$

Proof: We use the inequality

$$\left|\int_a^b f(s)\dot g(s)ds\right|\le\left|\int_a^b\left|f(s)\right|dg(s)\right|\le\max\left|f(s)\right|\left|g(b)-g(a)\right|.$$

In the first summand of $\left|\int_{kT_0}^{(k+1)T_0}A_{pq}\xi_\alpha^{(pq)}ds\right|$ we use the estimate $\left|\xi_\alpha^{(pq)}\right|\le\dfrac{2e^{\mu_0}}{\mu}(\omega/\mu)^{n-1}U_0$, while in the second and third ones – $\left|\xi_\alpha^{(pq)}\right|\le\sqrt{\left\langle\xi^{(pq)},\xi^{(pq)}\right\rangle}=c\tau_{pq}$:

$$\left|\int_{kT_0}^{(k+1)T_0}A_{pq}\xi_\alpha^{(pq)}ds\right|\le\left|\int_{kT_0}^{(k+1)T_0}\frac{2c^4}{c^3(1-\beta)^3r^3(t)}\frac{2e^{\mu_0}}{\mu}\left(\frac{\omega}{\mu}\right)^{n-1}U_0ds\right|+\left|\int_{kT_0}^{(k+1)T_0}\frac{2c^2\langle\xi^{(pq)},\dot u^{(q)}\rangle}{c^6(1-\beta)^3\tau^3_{pq}}c\tau_{pq}ds\right|+\left|\int_{kT_0}^{(k+1)T_0}\frac{\left\langle u^{(p)},\dot u^{(q)}\right\rangle}{c^4(1-\beta)^2\tau^2_{pq}}c\tau_{pq}ds\right|$$

$$\le\frac{2e^{\mu_0}}{\mu}\left(\frac{\omega}{\mu}\right)^{n-1}U_0\frac{2c}{(1-\beta)^3r_0^3}\int_{kT_0}^{(k+1)T_0}e^{\mu(s-kT_0)}ds+$$

$$+\frac{2\sqrt3}{c(1-\beta)^3r_0}\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{(k+1)T_0}\frac{du_\gamma{}^{(q)}(s-\tau_{pq})}{1-\dot\tau_{pq}}\right|+\frac{2\sqrt3}{c(1-\beta)^2r_0}\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{(k+1)T_0}u_\gamma{}^{(p)}\frac{du_\gamma{}^{(q)}(s-\tau_{pq})}{1-\dot\tau_{pq}}\right|\le$$

$$\le\frac{2e^{\mu_0}}{\mu}\left(\frac{\omega}{\mu}\right)^{n-1}U_0\frac{2c}{(1-\beta)^3r_0^3}\frac{e^{\mu_0}-1}{\mu}+\frac{2\sqrt3}{c(1-\beta)^3r_0}\frac{1+\beta}{1-\beta}\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{(k+1)T_0}du_\gamma{}^{(q)}(s-\tau_{pq})\right|+$$

$$+\frac{2\sqrt{3}}{c(1-\beta)^2 r_0}\frac{1+\beta}{1-\beta}\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{(k+1)T_0} du_\gamma{}^{(q)}(s-\tau_{pq})\right| = \frac{4ce^{\mu_0}\left(e^{\mu_0}-1\right)}{\mu^2(1-\beta)^3 r_0^3}(\omega/\mu)^{n-1}U_0 +$$

$$+\frac{4\sqrt{3}}{c(1-\beta)^3 r_0}\frac{1+\beta}{1-\beta}\left|\sum_{\gamma=1}^{3} u_\gamma{}^{(q)}((k+1)T_0-\tau_{pq})-u_\gamma{}^{(q)}(kT_0-\tau_{pq})\right| = \frac{4ce^{\mu_0}\left(e^{\mu_0}-1\right)}{\mu^2(1-\beta)^3 r_0^3}(\omega/\mu)^{n-1}U_0.$$

In $\left|\int_{kT_0}^{(k+1)T_0} B_{pq}u_\alpha^{(q)}ds\right|$ we estimate the first summand via $\left|u_\alpha^{(q)}\right| \le \left(e^{\mu_0}+1\right)U_0(\omega/\mu)^n$, while in the second one – $\left|u_\alpha^{(q)}\right| \le c$:

$$\left|\int_{kT_0}^{(k+1)T_0} B_{pq}u_\alpha^{(q)}ds\right| \le \left|\int_{kT_0}^{(k+1)T_0}\left(\frac{\Delta_{pq}^2\left(c^2\tau_{pq}-\left\langle\xi^{(pq)},u^{(p)}\right\rangle\right)}{\left(c^2\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^3}u_\alpha^{(q)}+\frac{\langle\xi^{(pq)},\dot u^{(q)}\rangle}{\left(c^2\tau_{pq}-\left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^2}u_\alpha^{(q)}\right)ds\right|$$

$$\le\left|\int_{kT_0}^{(k+1)T_0}\left(\frac{2}{(1-\beta)^3 r_0^2}\left(e^{\mu_0}+1\right)U_0(\omega/\mu)^n+\frac{c\langle\xi^{(pq)},\dot u^{(q)}\rangle}{c^4(1-\beta)^2\tau^2{}_{pq}}\right)ds\right| \le \frac{2\left(e^{\mu_0}+1\right)}{(1-\beta)^3 r_0^2}\left(\frac{\omega}{\mu}\right)^n U_0\left|\int_{kT_0}^{(k+1)T_0} e^{\mu(s-kT_0)}ds\right| +$$

$$+\frac{4}{c(1-\beta)^3 r_0}\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{(k+1)T_0} du_\gamma{}^{(q)}(s-\tau_{pq})\right| = \frac{e^{\mu_0}-1}{\mu}\frac{2\left(e^{\mu_0}+1\right)}{(1-\beta)^3 r_0^2}(\omega/\mu)^n U_0;$$

$$\left|\int_{kT_0}^{(k+1)T_0} C_{pq}\dot u_\alpha^{(q)}ds\right| = \frac{1}{(c-\bar c)r_0}\frac{1+\beta}{1-\beta}\left|\int_{kT_0}^{(k+1)T_0} du_\alpha^{(q)}\right| = 0.$$

Therefore

$$\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{(k+1)T_0} G_\alpha^{(pq)}ds\right| \le \frac{\left|Q_p\right|}{c}\sum_{\alpha=1}^{3}\left(\left|\int_{kT_0}^{(k+1)T_0} A_{pq}\xi_\alpha^{(pq)}ds\right|+\left|\int_{kT_0}^{(k+1)T_0} B_{pq}u_\alpha^{(q)}ds\right|+\left|\int_{kT_0}^{(k+1)T_0} C_{pq}\dot u_\alpha^{(q)}ds\right|\right)\le$$

$$\le 3\left|Q_p\right|\left[\frac{4e^{\mu_0}\left(e^{\mu_0}-1\right)}{\mu^2(1-\beta)^3 r_0^3}\left(\frac{\omega}{\mu}\right)^{n-1}+\frac{e^{\mu_0}-1}{\mu}\frac{2\left(e^{\mu_0}+1\right)}{c(1-\beta)^3 r_0^2}\left(\frac{\omega}{\mu}\right)^n\right]U_0 \le 3\left|Q_p\right|\left(\frac{4e^{\mu_0}}{\mu^2 r_0^3}+\frac{2e^{\mu_0}+2}{\mu c r_0^2}\right)\frac{e^{\mu_0}-1}{(1-\beta)^3}(\omega/\mu)^{n-1}U_0.$$

For the radiation part we obtain

$$\left|\int_{kT_0}^{(k+1)T_0} G_\alpha^{(p)rad}dt\right| \le \frac{e_p^2}{m_p c^2}\frac{c}{\left(c^2-\bar c^2\right)^{3/2}}\left(\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{(k+1)T_0} u_\gamma{}^{(p)}(t)\frac{\dot u_\gamma{}^{(p)}(t+\tau)-\dot u_\gamma{}^{(p)}(t-\tau)}{2\tau}dt\right|+\right.$$

$$\left.+\left|\int_{kT_0}^{(k+1)T_0} u_\alpha{}^{(p)}(t)\frac{\dot u_\alpha{}^{(p)}(t+\tau)-\dot u_\alpha{}^{(p)}(t-\tau)}{2\tau}dt\right|\right)\le$$

$$\le\frac{e_p^2}{m_p c^2}\frac{c^2}{\left(c^2-\bar c^2\right)^{3/2}}\left(\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{(k+1)T_0}\frac{\dot u_\gamma{}^{(p)}(t+\tau)-\dot u_\gamma{}^{(p)}(t-\tau)}{2\tau}dt\right|+\left|\int_{kT_0}^{(k+1)T_0}\frac{\dot u_\alpha{}^{(p)}(t+\tau)-\dot u_\alpha{}^{(p)}(t-\tau)}{2\tau}dt\right|\right)=0.$$

Lemma 2.8 is thus proved.

Lemma 2.9 (Main Lemma) **The periodic problem (2.3) has a unique solution** $(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)})$ iff the operator B has a fixed point, belonging to $(M_0)^6$.

Proof: Let $(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)})$ be a T_0 - periodic solution of (2.3). Then after integration in view of $u_\alpha^{(p)}(kT_0)=0$ $(k=0,1,2,..)$ we obtain

$$u_\alpha^{(p)}(t)=\int_{kT_0}^{t} U_\alpha^{(p)}(t)(t)dt \;\Rightarrow\; u_\alpha^{(p)}((k+1)T_0)=\int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(t)dt \Rightarrow \int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(t)dt=0.$$

Therefore operator $B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)$ becomes

$$B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)=\int_{kT_0}^{t}U_\alpha^{(p)}(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds\ .$$

We have supposed that the system has a periodic solution. Then changing the order of integration we obtain

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U_\alpha^{(p)}(s)dsdt=\int_{kT_0}^{(k+1)T_0}\left[(k+1)T_0-s\right]U_\alpha^{(p)}(s)ds$$

$$=(k+1)T_0\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds-\int_{kT_0}^{(k+1)T_0}sU_\alpha^{(p)}(s)ds=-\int_{kT_0}^{(k+1)T_0}sU_\alpha^{(p)}(s)ds.$$

But

$$\int_{kT_0}^{(k+1)T_0}s\frac{du_\alpha^{(p)}(s)}{ds}ds=\int_{kT_0}^{(k+1)T_0}sdu_\alpha^{(p)}(s)=su_\alpha^{(p)}(s)-\int_{kT_0}^{(k+1)T_0}u_\alpha^{(p)}(s)ds=\left[(k+1)T_0u_\alpha^{(p)}((k+1)T_0)-kT_0u_\alpha^{(p)}(kT_0)\right]=0$$

Therefore the following equality is satisfied

$$B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)=\int_{kT_0}^{t}U_\alpha^{(p)}(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U_\alpha^{(p)}(s)dsdt.$$

Conversely, let B has a fixed point

$$(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)})\in(M_0)^6,$$

that is, $u_\alpha^{(p)}=B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)}),\ (p=1,2;\ \alpha=1,2,3)$.

Therefore $u_\alpha^{(p)}(kT_0)=B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(kT_0)$ or

$$0=u_\alpha^{(p)}(kT_0)=B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(kT_0)=\int_{kT_0}^{kT_0}U_\alpha^{(p)}(s)ds-$$

$$-\left(\frac{kT_0-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}U_\alpha^{(p)}(\theta)d\theta ds=$$

$$=\frac{1}{2}\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}U_\alpha^{(p)}(\theta)d\theta ds.$$

It follows $\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}U_\alpha^{(p)}(\theta)d\theta ds=\frac{1}{2}\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds$ .We show that $\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds=0$. Indeed, if $\left|\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds\right|\neq 0$ for sufficiently large μ the inequality of Lemma 2.8 might be violated.

Therefore the operator

$$B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t):=\int_{kT_0}^{t}U_\alpha^{(p)}(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U_\alpha^{(p)}(s)dsdt$$

becomes $B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)=\int_{kT_0}^{t}U_\alpha^{(p)}(s)ds$. Differentiating the last equalities we obtain that the fixed point of the operator is a periodic solution of (2.3).

Lemma 2.9 is thus proved.

Remark 2.2 We use the equality

$$\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}U_\alpha^{(p)}(\theta)d\theta ds=\frac{1}{2}\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds$$

for further estimations.

2.3. Existence-Uniqueness of a Periodic Solution of the Two-Body System

Here we prove the main result:

Theorem 2.1 (Main result) Let the following conditions be fulfilled:

IN-1) the initial velocities $u_{\alpha 0}^{(p)}(t), t\in[-T_0;0]$ are T_0-periodic infinitely differentiable functions and initial trajectories are such that

$$0 < r_0 \le r(t) = \sqrt{\sum_{\gamma=1}^{3}\left(x_{\gamma 0}^{(p)}(t) - x_{\gamma 0}^{(q)}(t)\right)} \quad t\in[-T_0;0].$$

IN-2) the translations (to the right) of $u_{\alpha 0}^{(p)}(t)$ on $[0;T_0]$ are restrictions of some functions from M_0.

Besides the following inequalities are satisfied:

1) $3\left|Q_p\right|\left(e^{\mu_0}+1\right)\left(\frac{8}{\mu^2 r_0^3}+\frac{4}{\mu c r_0^2}+\frac{104}{\mu c^2 r_0}+\frac{1}{\mu c^3}\frac{4\omega^2}{4-\tau^2\omega^2}\right)\times\left(\omega/\mu\right)^{n-1}\frac{U_0}{\left(1-\beta\right)^4}\le U_0;$

2) $3\left|Q_p\right|\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\omega/\mu\right)^{n-2}\omega U_0\times\left[\left(\frac{4}{\mu^2 r_0^3}+\frac{2}{\mu c r_0^2}+\frac{3\sqrt{3}+2}{\mu c^2 r_0}\right)\frac{e^{\mu_0}+1}{\left(1-\beta\right)^3}+\frac{1}{\mu c^3\left(1-\beta\right)^{3/2}}\frac{4\omega^2}{4-\tau^2\omega^2}\right]\le\omega U_0;$

3) $K=\left|Q_p\right|\left(\frac{3}{c^2 r_0}+\frac{1}{c^3}\frac{4}{4-\tau^2}\right)\frac{18\left(e^{\mu_0}+1\right)}{\left(1-\beta\right)^3}\left(\frac{\omega}{\mu}\right)^n<1;$

4) $\dot{K}=\frac{9\left|Q_p\right|\left(e^{\mu_0}+1\right)}{\left(1-\beta\right)^3}\frac{12}{c^2 r_0}\left(\frac{\omega}{\mu}\right)^n<1.$

Then there exists a unique T_0-periodic solution

$$\left(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)}\right) \text{ of (2.3) for } t\ge 0.$$

Proof: With accordance of the Main lemma 2.9 we have to prove that operator defined by (2.6) possesses a unique fixed point which means that (2.3) has a unique periodic solution.

We use function spaces M and M_0 defined by (2.1) and (2.4), respectively, and family of pseudo-metrics $\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)})$ defined by (2.5).

The set M_0 is endowed with a countable family of pseudo-metrics

$$\rho_{(k,n)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right)=\max\left\{\rho_{(k,n)}(u_1,\overline{u}_1),\rho_{(k,n)}(u_2,\overline{u}_2),\rho_{(k,n)}(u_3,\overline{u}_3),\right.$$
$$\left.\rho_{(k,n)}(u_4,\overline{u}_4),\rho_{(k,n)}(u_5,\overline{u}_5),\rho_{(k,n)}(u_6,\overline{u}_6)\right\}$$

whose index set is $A=\left\{\bigcup_{k,n=0}^{\infty}(k,n)\right\}\cup\left\{\bigcup_{k=0}^{\infty}(k,\infty)\right\}$.

The lemmas from Subsection 2.2 imply that the function

$$B_\alpha^{(p)}(k)(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)})(t):=\int_{kT_0}^{t}U_\alpha^{(p)}(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U_\alpha^{(p)}(s)dsdt$$

$(\alpha=1,2,3;\ p=1,2)$, $t\in[kT_0,(k+1)T_0]$ $(k=0,1,2,...)$

is T_0-periodic one.

Estimates from Supplement 1.1 and condition 1) of the Main theorem imply

$$\left|B_\alpha^{(p)}(k)(u_1^{(1)},...,u_3^{(2)})(t)\right|\le e^{\mu(t-kT_0)}3\left|Q_p\right|\left(e^{\mu_0}+1\right)\left(\frac{\omega}{\mu}\right)^{n-1}\frac{U_0}{\left(1-\beta\right)^4}\times\left(\frac{8}{\mu^2 r_0^3}+\frac{4}{\mu c r_0^2}+\frac{104}{\mu c^2 r_0}+\frac{1}{\mu c^3}\frac{4\omega^2}{4-\tau^2\omega^2}\right)\le e^{\mu(t-kT_0)}U_0.$$

Estimates from Supplement 1.2 and condition 2) of the Main theorem imply

$$\left|\dot{B}_{\alpha}^{(p)}(k)(u_1^{(1)},...,u_3^{(2)})(t)\right| \le e^{\mu(t-kT_0)}3\left|Q_p\right|\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\frac{\omega}{\mu}\right)^{n-2}\times$$
$$\times\left[\left(\frac{4}{\mu^2 r_0^3}+\frac{2}{\mu c r_0^2}+\frac{3\sqrt{3}+2}{\mu c^2 r_0}\right)\left(e^{\mu_0}+1\right)+\frac{1}{\mu c^3}\frac{4\omega^2}{4-\tau^2\omega^2}\right]\frac{\omega U_0}{(1-\beta)^3} \le \omega U_0 e^{\mu(t-kT_0)}.$$

In a similar way we obtain for derivatives

$$\left(B_{\alpha}^{(p)}(k)(t)\right)^{(n)} \le \omega^n U_0 e^{\mu(t-kT_0)}, t\in[kT_0,(k+1)T_0], \ (n=0,1,2,...)$$

that is, the operator B maps the set M_0 into itself.

It remains to show that operator B is a contractive one. Indeed, in view of the inequalities obtained in Supplement 2 we have

$$\left|B_{\alpha}^{(p)}(k)(u_1^{(1)},...,u_3^{(2)})(t)-B_{\alpha}^{(p)}(k)(\overline{u}_1^{(1)},...,\overline{u}_3^{(2)})(t)\right| \le$$
$$\le e^{\mu(t-kT_0)}\left|Q_p\right|\left(\frac{3}{c^2 r_0}+\frac{1}{c^3}\frac{4}{4-\tau^2}\right)\frac{18\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\left(\frac{\omega}{\mu}\right)^n\times$$
$$\times\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right).$$

Multiplying by $e^{-\mu(t-kT_0)}$ and taking the supremum in t we obtain

$$\rho_{(k,0)}\left(B_{\alpha}^{(p)}(k),\overline{B}_{\alpha}^{(p)}(k)\right) \le K\,\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right)$$

where $K=\left|Q_p\right|\left(\frac{3}{c^2 r_0}+\frac{1}{c^3}\frac{4}{4-\tau^2}\right)\frac{18\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\left(\frac{\omega}{\mu}\right)^n<1$.

For the derivatives we obtain

$$\left|\dot{B}_{\alpha}^{(p)}(k)(u_1^{(1)},...,u_3^{(2)})(t)-\dot{B}_{\alpha}^{(p)}(k)(\overline{u}_1^{(1)},...,\overline{u}_3^{(2)})(t)\right| \le \frac{9\left|Q_p\right|\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\frac{12}{c^2 r_0}\left(\frac{\omega}{\mu}\right)^n\times$$
$$\times\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right)$$

and therefore

$$\rho_{(k,1)}\left(\dot{B}_{\alpha}^{(p)}(k),\dot{\overline{B}}_{\alpha}^{(p)}(k)\right) \le K_1\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right)$$

where $K_1=\frac{9\left|Q_p\right|\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\frac{12}{c^2 r_0}\left(\frac{\omega}{\mu}\right)^n<1$.

Analogously we have

$$\rho_{(k,n)}\left(\left(B_{\alpha}^{(p)}(k)\right)^{(n)},\left(\overline{B}_{\alpha}^{(p)}(k)\right)^{(n)}\right) \le K_n\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right).$$

Taking the supremum in n we obtain

$$\rho_{(k,\infty)}\left(\left(B_{\alpha}^{(p)}(k)\right),\left(\overline{B}_{\alpha}^{(p)}(k)\right)\right) \le K_\infty\rho_{(k,\infty)}\left((u_1,u_2,u_3,u_4,u_5,u_6),(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{u}_5,\overline{u}_6)\right),$$

where $\max\{K,K_1,K_2,...,K_n,....\}\le K_\infty<1$. This means B is a contractive operator in the sense of definition given in [5]. Its unique fixed point is a periodic solution of two-body system of equations.

Theorem 2.1 is thus proved.

2.4. Numerical Test Results

Let us show that the inequalities of Theorem 2.1 are satisfied:

1) $\dfrac{3|Q_p|\left(e^{\mu_0}+1\right)}{\mu(1-\beta)^4}\left(\dfrac{8}{\mu r_0^3}+\dfrac{4}{cr_0^2}+\dfrac{104}{c^2r_0}+\dfrac{1}{c^3}\dfrac{4\omega^2}{4-\tau^2\omega^2}\right)\left(\dfrac{\omega}{\mu}\right)^{n-1}\le 1;$

2) $3|Q_p|\left(1+\dfrac{e^{\mu_0}-1}{\mu_0}\right)\left[\dfrac{4}{\mu r_0^3}+\dfrac{2}{cr_0^2}+\dfrac{3\sqrt{3}+2}{c^2r_0}+\dfrac{\omega^2}{c^3}\right]\dfrac{e^{\mu_0}+1}{\mu(1-\beta)^3}\left(\dfrac{\omega}{\mu}\right)^{n-1}\le 1$;

3) $K=|Q_p|\left(3\dfrac{e^{\mu_0}+1}{c^2 r_0}+\dfrac{1}{c^3}\right)\left(\dfrac{\omega}{\mu}\right)^n\dfrac{18}{(1-\beta)^3}<1;$

4) $\dot{K}=|Q_p|\dfrac{9\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\dfrac{12}{c^2 r_0}\left(\dfrac{\omega}{\mu}\right)^n<1.$

We recall that

$$|Q_1|=\frac{|e_1e_2|}{m_1}\approx 2{,}8.10^{-8};\ |Q_2|=\frac{|e_1e_2|}{m_2}\approx\frac{2{,}8.10^{-8}}{1836};\ c=3.10^8\,m/\sec;\ \tau_0=\frac{r_e}{c}=\frac{2{,}82.10^{-15}}{3.10^8}\approx 0{,}94.10^{-23}=9{,}4.10^{-24}\sec .$$

Since the radius of first Bohr orbit is $r_0=0{,}53.10^{-10}\,m$

and its velocity is $v_0=\dfrac{c}{137}=r_0\omega_0$ (1/137 is Sommerfeld fine structure constant), then

$$\frac{c}{137}=r_0\omega_0\Rightarrow\omega_0=\frac{c}{137r_0}=\frac{3.10^8}{137.0{,}53.10^{-10}}\approx 4.10^{16};$$

$$T_0=\frac{2\pi}{\omega_0}=\frac{2\pi.137r_0}{c}\approx 1{,}52.10^{-16};\ f_0=\frac{1}{T_0}\approx\frac{1}{1{,}52.10^{-16}}Hz;$$

$$\lambda_0=\frac{c}{f_0}=\frac{3.10^8}{1/T_0}=3.10^8.1{,}52.10^{-16}\approx 4{,}56.10^{-8}m=456.10^{-10}\overset{\circ}{A}.$$

In the above inequalities n could be chosen arbitrarily large because the solution belongs to the space of infinitely differentiable functions. Our estimates require $\mu T_0=\mu_0$ to be a constant. Here $\omega=\omega_0$ and we have to take $\mu>\omega_0$, for instance $\mu=6.10^{16}$. Then

$$\mu T_0=6.10^{16}.1{,}52.10^{-16}=9{,}12=\mu_0 .$$

Therefore $\dfrac{\omega_0}{\mu}=\dfrac{2}{3}$ and $\tau_0\omega_0=9{,}4.10^{-24}.4.10^{16}=3{,}76.10^{-7}$, that is condition $\tau_0\omega_0=9{,}4.10^{-24}.4.10^{16}<2$ is satisfied.

We choose $\beta=\dfrac{1}{137}\Rightarrow 1-\beta\approx 1$. For the inequalities we get

1) $\left(1{,}2.10^{-5}+6{,}1.10^{-8}+2{,}9.10^{-25}+7{,}7.10^{-13}\right)(2/3)^{n-1}\le 1;$

2) $\left(6.10^{-3}+3.10^{-5}+2.10^{-23}+10^{-9}\right)(2/3)^{n-1}\le 1;$

3) $K\approx 2{,}96142.10^{-9}(2/3)^n<1;$

4) $\dot{K}=5{,}92230.10^{-9}(2/3)^n<1.$

We notice that the initial conditions of Theorem 2.1 exclude the condition from [16] for escape trajectories.

3. Conclusions

It is easy to see that the inequalities are satisfied for every radius of "larger" orbit $r_n>r_0$.

Following [18] and [19] we would like to recall some difficulties of planetary model of the atom.

Difficulty (1): It is known that the properties of an atom are determined by its electric field. The identity of their properties means that all of the hydrogen atoms possess identical electrical fields. In other words the electrons of all these atoms move in strictly identical orbits (for instance, circles or ellipses and so on). The shape of electron orbits must depend on the initial conditions of formation of the atom. It is clear that these initial conditions can be very different. Completely incomprehensible is why at various initial conditions the electron attaches the same orbit.

Difficulty (2): In the planetary model of the hydrogen atom the electron moves in a stable orbit. This contradicts electrodynamics because the radiation of the electron leads to decreasing of its radius as a result it should be collided to the nucleus. This time is 10^{-8} sec [18], [19].

Our conclusion: It is not natural to expect that the applying of the methods of classical mechanics to relativistic objects will give adequate results. Namely:

1) The orbits in question [18], [19] are obtained in the frame of classical mechanics, while we consider two-body problem in the relativistic case;
2) We have found initial conditions (these are the conditions of our main Theorem 2.1) which guarantee an existence-uniqueness of a closed orbit;
3) We have considered equations with radiation terms and nevertheless there is unique periodic orbit for every stationary state without radiation. In other words the radiation is so small that it does not affect the stability of the atom. This fact is confirmed experimentally. In our model the hydrogen atom exists infinitely long.

Finally, we would like to say that basic difficulties of the planetary model are overcome provided the considerations to be relativistic ones.

Supplement 1. The Operator Maps Solution Set into Itself

1.1. Estimates of the Right-Hand Sides

We have to show that the operator

$$\left|B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)\right| = \left|\int_{kT_0}^{t} U_\alpha^{(p)}(s)ds - \left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(s)ds - \frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} U_\alpha^{(p)}(s)dsdt\right|$$
$$\le U_0 e^{\mu(t-kT_0)} \quad (\alpha=1,2,3;\ p=1,2)$$

maps $(M_0)^6$ into itself.

In view the previous Remark 5.2 we have:

$$\left|B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)\right| \le \left|\int_{kT_0}^{t} U_\alpha^{(p)}(s)ds\right| + \left|\frac{t-kT_0}{T_0}-\frac{1}{2}\right|\left|\int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(s)ds\right| + \left|\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} U_\alpha^{(p)}(s)dsdt\right|$$
$$= \left|\int_{kT_0}^{t} U_\alpha^{(p)}(s)ds\right| + \left|\frac{t-kT_0}{T_0}-\frac{1}{2}\right|\left|\int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(s)ds\right| + \left|\frac{1}{2}\int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(s)dst\right| \le \left|\int_{kT_0}^{t} U_\alpha^{(p)}(s)ds\right| + \left|\int_{kT_0}^{(k+1)T_0} U_\alpha^{(p)}(s)ds\right|$$
$$\le \sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t}\left(G_\alpha^{(pq)}(s)+G_\alpha^{(p)rad}(s)\right)ds\right| + \sum_{\alpha=1}^{3}\left|\int_{kT_0}^{(k+1)T_0}\left(G_\alpha^{(pq)}(s)+G_\alpha^{(p)rad}(s)\right)ds\right|.$$

From the proof of the Main Lemma and the assertion of Lemma 2.8 we obtain

$$\left|B_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)\right| \le \sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t} G_\gamma^{(pq)}(s)ds\right| + \sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t} G_\lambda^{(p)rad}(s)ds\right| + \sum_{\lambda=1}^{3}\left|\int_{kT_0}^{(k+1)T_0} G_\gamma^{(pq)}(s)ds\right|.$$

We have

$$\left|\int_{kT_0}^{t} G_\alpha^{(pq)}ds\right| \le \frac{|e_p e_q|}{m_p c}\left(\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t} A_{pq}\xi_\alpha^{(pq)}ds\right| + \sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t} B_{pq}u_\alpha^{(q)}ds\right| + \sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t} C_{pq}\dot{u}_\alpha^{(q)}ds\right|\right).$$

Having in mind Lemma 2.7 we use the estimate $\left|\xi_\alpha^{(pq)}\right| \le \frac{2e^{\mu_0}}{\mu}\left(\frac{\omega}{\mu}\right)^{n-1}U_0$ in the first summand of $\left|\int_{kT_0}^{t} A_{pq}\xi_\alpha^{(pq)}ds\right|$, while in the second and third ones – $\left|\xi_\alpha^{(pq)}\right| \le c\tau_{pq}$. Then in view of $\frac{1}{c\tau_{pq}} = \frac{1}{r(t)}$ we have

$$\left|\int_{kT_0}^{t} A_{pq}\xi_\alpha^{(pq)}ds\right| \le \left|\int_{kT_0}^{t} \frac{2c^4}{c^6\left(1-\beta\right)^3\tau^3_{pq}}\frac{2e^{\mu_0}}{\mu^n}U_0\omega^{n-1}ds\right| + \left|\int_{kT_0}^{t} \frac{2c^2\langle\xi^{(pq)},\dot{u}^{(q)}\rangle}{c^6\left(1-\beta\right)^3\tau^3_{pq}}c\tau_{pq}ds\right| + \left|\int_{kT_0}^{t} \frac{\left\langle u^{(p)},\dot{u}^{(q)}\right\rangle}{c^4\left(1-\beta\right)^2\tau^2_{pq}}c\tau_{pq}ds\right|$$

$$\le \frac{2e^{\mu_0}}{\mu^n}U_0\omega^{n-1}\frac{2c}{\left(1-\beta\right)^3 r_0^3}\int_{kT_0}^{t} e^{\mu(s-kT_0)}ds +$$

$$+\frac{2}{c\left(1-\beta\right)^3 r_0}\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{t}\frac{du_\gamma{}^{(q)}(s-\tau_{pq})}{1-\dot{\tau}_{pq}}\right| + \frac{1}{c\left(1-\beta\right)^2 r_0}\left|\sum_{\gamma=1}^{3}\int_{kT_0}^{t} u_\gamma{}^{(p)}\frac{du_\gamma{}^{(q)}(s-\tau_{pq})}{1-\dot{\tau}_{pq}}\right| \le$$

$$\le \frac{2e^{\mu_0}}{\mu^n}U_0\omega^{n-1}\frac{2c}{\left(1-\beta\right)^3 r_0^3}\frac{e^{\mu(t-kT_0)}-1}{\mu} + \frac{2}{c\left(1-\beta\right)^3 r_0}\frac{1+\beta}{1-\beta}\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t} du_\gamma{}^{(q)}(s-\tau_{pq})\right| +$$

$$+\frac{1}{c\left(1-\beta\right)^2 r_0}\frac{1+\beta}{1-\beta}\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t} du_\gamma{}^{(q)}(s-\tau_{pq})\right| \le e^{\mu(t-kT_0)}\frac{4e^{\mu_0}c}{\mu^2\left(1-\beta\right)^3 r_0^3}\left(\frac{\omega}{\mu}\right)^{n-1}U_0 +$$

$$+\left(\frac{4}{c\left(1-\beta\right)^4 r_0} + \frac{2}{c\left(1-\beta\right)^3 r_0}\right)\sum_{\gamma=1}^{3}\left|u_\gamma{}^{(q)}(t-\tau_{pq}) - u_\gamma{}^{(q)}(kT_0-\tau_{pq})\right| \le$$

$$\le e^{\mu(t-kT_0)}\frac{4e^{\mu_0}c}{\mu^2(1-\beta)^3 r_0^3}\left(\frac{\omega}{\mu}\right)^{n-1}U_0 + \frac{36}{c(1-\beta)^4 r_0}\left(e^{\mu_0}+1\right)\left(\frac{\omega}{\mu}\right)^n U_0 \le$$

$$\le e^{\mu(t-kT_0)}\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^4}\left(\frac{4c}{\mu^2 r_0^3} + \frac{36}{cr_0}\right)\left(\frac{\omega}{\mu}\right)^{n-1}U_0.$$

In a similar way in the first summand of $\left|\int_{kT_0}^{t} B_{pq}u_\alpha^{(q)}ds\right|$ we use $\left|u_\alpha^{(q)}(t-\tau_{pq})\right| \le \left(e^{\mu_0}+1\right)(\omega/\mu)^n U_0$, while in the second one – $\left|u_\alpha^{(q)}(t-\tau_{pq})\right| \le c$ and get

$$\left|\int_{kT_0}^{t} B_{pq}u_\alpha^{(q)}ds\right| \le \left|\int_{kT_0}^{t}\left(\frac{\Delta_{pq}^2\left(c^2\tau_{pq} - \left\langle\xi^{(pq)},u^{(p)}\right\rangle\right)}{\left(c^2\tau_{pq} - \left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^3}u_\alpha^{(q)} + \frac{\langle\xi^{(pq)},\dot{u}^{(q)}\rangle u_\alpha^{(q)}}{\left(c^2\tau_{pq} - \left\langle\xi^{(pq)},u^{(q)}\right\rangle\right)^2}\right)ds\right| \le$$

$$\le \left|\int_{kT_0}^{t}\left(\frac{2}{\left(1-\beta\right)^3 r_0^2}\frac{U_0\omega^n\left(e^{\mu_0}+1\right)}{\mu^n} + \frac{c\langle\xi^{(pq)},\dot{u}^{(q)}\rangle}{c^4\left(1-\beta\right)^2\tau^2_{pq}}\right)ds\right| \le \frac{2U_0\left(e^{\mu_0}+1\right)}{\mu^{n-1}\left(1-\beta\right)^3 r_0^2}\left|\int_{kT_0}^{t} e^{\mu(s-kT_0)}ds\right| + \frac{2}{c\left(1-\beta\right)^3 r_0}\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t} du_\gamma{}^{(q)}(s-\tau_{pq})\right| \le$$

$$\le e^{\mu(t-kT_0)}\frac{2\left(e^{\mu_0}+1\right)}{\mu\left(1-\beta\right)^3 r_0^2}\left(\frac{\omega}{\mu}\right)^n U_0 + \frac{2}{c\left(1-\beta\right)^3 r_0}\sum_{\gamma=1}^{3}\left|u_\gamma^{(q)}(t-\tau_{pq}) - u_\gamma^{(q)}(kT_0-\tau_{pq})\right| \le$$

$$\le e^{\mu(t-kT_0)}\frac{2\left(e^{\mu_0}+1\right)}{\mu\left(1-\beta\right)^3 r_0^2}\left(\frac{\omega}{\mu}\right)^n U_0 + \frac{12\left(e^{\mu_0}+1\right)}{c\left(1-\beta\right)^3 r_0}\left(\frac{\omega}{\mu}\right)^n U_0 \le e^{\mu(t-kT_0)}\frac{\left(e^{\mu_0}+1\right)}{\left(1-\beta\right)^3}\left(\frac{2}{\mu r_0^2} + \frac{12}{cr_0}\right)\left(\frac{\omega}{\mu}\right)^n U_0.$$

For $\left|\int_{kT_0}^{t} C_{pq}\dot{u}_\alpha^{(q)}ds\right|$ we have

$$\left|\int_{kT_0}^{t} C_{pq}\dot{u}_\alpha^{(q)}ds\right| \le \frac{1}{c(1-\beta)^2 r_0}\frac{1+\beta}{1-\beta}\left|\int_{kT_0}^{t} du_\alpha^{(q)}\right| \le \frac{2\left|u_\alpha^{(q)}(t-\tau_{pq}) - u_\alpha^{(q)}(kT_0-\tau_{pq})\right|}{c(1-\beta)^3 r_0} \le \frac{4\left(e^{\mu_0}+1\right)}{c(1-\beta)^3 r_0}\left(\frac{\omega}{\mu}\right)^n U_0.$$

Then

$$\left|\int_{kT_0}^{t} G_{\alpha}^{(pq)}ds\right| \le \frac{|e_1e_2|}{m_pc}\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t} A_{pq}\xi_{\alpha}^{(pq)}ds\right| + \frac{|e_1e_2|}{m_pc}\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t} B_{pq}u_{\alpha}^{(q)}ds\right| + \frac{|e_1e_2|}{m_pc}\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t} C_{pq}\dot{u}_{\alpha}^{(q)}ds\right| \le$$

$$\le e^{\mu(t-kT_0)}\frac{|Q_p|}{c}\left(e^{\mu_0}+1\right)\left(\frac{1}{(1-\beta)^4}\left(\frac{4c}{\mu^2 r_0^3}+\frac{36}{cr_0}\right)+\right.$$

$$\left.+\frac{1}{(1-\beta)^3}\left(\frac{2}{\mu r_0^2}+\frac{12}{cr_0}\right)+\frac{4}{c(1-\beta)^3 r_0}\right)\left(\frac{\omega}{\mu}\right)^{n-1}U_0 \le e^{\mu(t-kT_0)}|Q_p|\frac{e^{\mu_0}+1}{(1-\beta)^4}\left(\frac{4}{\mu^2 r_0^3}+\frac{2}{\mu c r_0^2}+\frac{52}{c^2 r_0}\right)\left(\frac{\omega}{\mu}\right)^{n-1}U_0.$$

In a similar way we obtain

$$\left|\int_{kT_0}^{(k+1)T_0} G_{\alpha}^{(pq)}ds\right| \le e^{\mu(t-kT_0)}\frac{|Q_p|}{c}\left(\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^4}\frac{4c}{\mu^2 r_0^3}\left(\frac{\omega}{\mu}\right)^{n-1}+\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\frac{2}{\mu r_0^2}\left(\frac{\omega}{\mu}\right)^{n}\right)U_0$$

$$\le e^{\mu(t-kT_0)}|Q_p|\frac{e^{\mu_0}+1}{\mu(1-\beta)^4}\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}\right)\left(\frac{\omega}{\mu}\right)^{n-1}U_0.$$

For the radiation term in view of

$$u_{\alpha}^{(q)}(t+\tau)=u_{\alpha}^{(q)}(t)+\tau\dot{u}_{\alpha}^{(q)}(t)+\frac{\tau^2}{2!}\ddot{u}_{\alpha}^{(q)}(t)+\frac{\tau^3}{3!}\dddot{u}_{\alpha}^{(q)}(t)+...;$$

$$u_{\alpha}^{(q)}(t-\tau)=u_{\alpha}^{(q)}(t)-\tau\dot{u}_{\alpha}^{(q)}(t)+\frac{\tau^2}{2!}\ddot{u}_{\alpha}^{(q)}(t)-\frac{\tau^3}{3!}\dddot{u}_{\alpha}^{(q)}(t)+...;$$

$$\frac{u_{\alpha}^{(q)}(t+\tau)-u_{\alpha}^{(q)}(t-\tau)}{2\tau}=\dot{u}_{\alpha}^{(q)}(t)+\frac{\tau^2}{3!}\dddot{u}_{\alpha}^{(q)}(t)+\frac{\tau^4}{5!}\left(u_{\alpha}^{(q)}\right)^{V}(t)+...;$$

$$\frac{u_{\alpha}^{(q)}(t+\tau)-u_{\alpha}^{(q)}(t-\tau)}{2\tau}=\dot{u}_{\alpha}^{(q)}(t)+\frac{\tau^2}{3!}\dddot{u}_{\alpha}^{(q)}(t)+\frac{\tau^4}{5!}\left(u_{\alpha}^{(q)}\right)^{V}(t)+...;$$

$$\left|\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{u}_{\gamma}^{(p)}(t-\tau)}{2\tau}\right| \le \left|\ddot{u}_{\gamma}^{(p)}(t)\right|+\frac{\tau^2}{3!}\left|\left(u_{\gamma}^{(p)}(t)\right)^{(4)}\right|+...$$

$$\le\left|\int_{kT_0}^{t}...\int_{kT_0}^{s_{n-1}}\left(u_{\gamma}^{(p)}(t)\right)^{(n+2)}ds_n...ds_1\right|+\left|\int_{kT_0}^{t}...\int_{kT_0}^{s_{n-1}}\left(u_{\gamma}^{(p)}(t)\right)^{(n+4)}ds_n...ds_1\right|+....$$

$$\le U_0\omega^2\frac{\omega^n}{\mu^n}+\frac{\tau^2}{3!}U_0\omega^4\frac{\omega^n}{\mu^n}+...\le U_0\omega^2\frac{\omega^n}{\mu^n}\left(1+\frac{\tau^2\omega^2}{2^2}+...\right)=\left(\frac{\omega}{\mu}\right)^n\frac{4}{4-\tau^2\omega^2}\omega^2U_0$$

we obtain

$$\left|\int_{kT_0}^{t} G_{\alpha}^{(p)rad}(s)ds\right| = \left|\frac{e_p^2}{m_pc^2}\left(\frac{u_{\alpha}^{(p)}}{\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^{3/2}}\left\langle u^{(p)},\frac{\dot{u}^{(p)a}-\dot{u}^{(p)r}}{2\tau}\right\rangle++\frac{\dot{u}_{\alpha}^{(p)a}-\dot{u}_{\alpha}^{(p)r}}{2\tau}\frac{1}{\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)^{1/2}}\right)\right|$$

$$\le\frac{e_p^2}{m_pc^2}\left(\frac{c}{\left(c^2-\overline{c}^2\right)^{3/2}}\left|\int_{kT_0}^{t}\left\langle u^{(p)},\frac{\dot{u}^{(p)a}-\dot{u}^{(p)r}}{2\tau}\right\rangle ds\right|+\frac{1}{\left(c^2-\overline{c}^2\right)^{1/2}}\left|\int_{kT_0}^{t}\frac{\dot{u}_{\alpha}^{(p)a}-\dot{u}_{\alpha}^{(p)r}}{2\tau}ds\right|\right)\le$$

$$\leq \frac{e_p^2}{m_p c^2}\left[\frac{c}{c^3\left(1-\beta^2\right)^{3/2}}\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t} u_\gamma^{(p)}\frac{\dot{u}_\gamma^{(p)}(s+\tau)-\dot{u}_\gamma^{(p)}(s-\tau)}{2\tau}ds\right|+\frac{1}{c\left(1-\beta^2\right)^{1/2}}\left|\int_{kT_0}^{t}\frac{\dot{u}_\alpha^{(p)}(s+\tau)-\dot{u}_\alpha^{(p)}(s-\tau)}{2\tau}ds\right|\right]$$

$$\leq \frac{e_p^2}{m_p c^2}\frac{1}{c\left(1-\beta^2\right)^{3/2}}\left[\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t}\frac{\dot{u}_\gamma^{(p)}(s+\tau)-\dot{u}_\gamma^{(p)}(s-\tau)}{2\tau}ds\right|+\right.$$

$$\left.+\left|\int_{kT_0}^{t}\frac{\dot{u}_\alpha^{(p)}(s+\tau)-\dot{u}_\alpha^{(p)}(s-\tau)}{2\tau}ds\right|\right]\leq \frac{e_p^2}{m_p c^2}\frac{1}{c(1-\beta)^{3/2}}\frac{\omega^n}{\mu^n}\frac{4}{4-\tau^2\omega^2}\omega^2 U_0\times$$

$$\times\left[\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t}e^{\mu(s-kT_0)}ds\right|+\left|\int_{kT_0}^{t}e^{\mu(s-kT_0)}ds\right|\right]\leq e^{\mu(t-kT_0)}\frac{\left|Q_p\right|}{\mu c^3(1-\beta)^{3/2}}\left(\frac{\omega}{\mu}\right)^n\frac{4}{4-\tau^2\omega^2}\omega^2 U_0.$$

Therefore the integral of the radiation term is bounded.

We have already obtained that $\left|\int_{kT_0}^{(k+1)T_0}G_\alpha^{(p)rad}(s)ds\right|=0$.

Finally we obtain

$$\left|B_{\gamma,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)\right|\leq \sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t}G_\alpha^{(pq)}(s)ds\right|+\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{(k+1)T_0}G_\alpha^{(pq)}(s)ds\right|+\sum_{\alpha=1}^{3}\left|\int_{kT_0}^{t}G_\alpha^{(p)rad}(s)ds\right|$$

$$\leq e^{\mu(t-kT_0)}3\left|Q_p\right|\left(\frac{2\left(e^{\mu_0}+1\right)}{\mu(1-\beta)^4}\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}+\frac{52}{c^2 r_0}\right)\left(\frac{\omega}{\mu}\right)^{n-1}+\frac{1}{\mu c^3(1-\beta)^{3/2}}\left(\frac{\omega}{\mu}\right)^n\frac{4}{4-\tau^2\omega^2}\omega^2\right)U_0\leq$$

$$\leq\frac{e^{\mu(t-kT_0)}3\left|Q_p\right|\left(e^{\mu_0}+1\right)}{(1-\beta)^4}U_0\left(\frac{\omega}{\mu}\right)^{n-1}\times\left(\frac{8}{\mu^2 r_0^3}+\frac{4}{\mu c r_0^2}+\frac{104}{\mu c^2 r_0}+\frac{1}{\mu c^3}\frac{4\omega^2}{4-\tau^2\omega^2}\right)\leq U_0 e^{\mu(t-kT_0)}.$$

1.2. Estimates of the Derivatives

For the derivative

$$\dot{B}_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t):=U_\alpha^{(p)}(t)-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds\quad\left(\alpha=1,2,3;p=1,2\right)$$

We use the above estimates and for $\left|A_{pq}\xi_\alpha^{(pq)}\right|$ we obtain

$$\left|A_{pq}\xi_\alpha^{(pq)}\right|\leq\left|\frac{2c^4}{c^6(1-\beta)^3\tau^3{}_{pq}}\frac{2e^{\mu_0}}{\mu}\left(\frac{\omega}{\mu}\right)^{n-1}U_0\right|+\left|\frac{2c^2\langle\xi^{(pq)},\dot{u}^{(q)}\rangle}{c^6(1-\beta)^3\tau^3{}_{pq}}c\tau_{pq}\right|+\left|\frac{\left\langle u^{(p)},\dot{u}^{(q)}\right\rangle}{c^4(1-\beta)^2\tau^2{}_{pq}}c\tau_{pq}\right|\leq$$

$$\leq\left|\frac{2cU_0}{(1-\beta)^3 r_0^3}\frac{2e^{\mu_0}}{\mu}\left(\frac{\omega}{\mu}\right)^{n-1}\right|+\left|\frac{2c^2c\tau_{pq}\sqrt{3}}{c^6(1-\beta)^3\tau^3{}_{pq}}c\tau_{pq}\left(e^{\mu_0}+1\right)\left(\frac{\omega}{\mu}\right)^n U_0\right|+\left|\frac{c\sqrt{3}}{c^4(1-\beta)^2\tau^2{}_{pq}}c\tau_{pq}\left(e^{\mu_0}+1\right)\left(\frac{\omega}{\mu}\right)^n U_0\right|$$

$$\leq\left(\frac{4c}{\mu r_0^3}+\frac{3\sqrt{3}}{cr_0}\right)\left(\frac{\omega}{\mu}\right)^{n-1}\frac{\left(e^{\mu_0}+1\right)U_0}{(1-\beta)^3}.$$

In a similar way in the first summand of $\left|B_{pq}u_\alpha^{(q)}\right|$ we use

$\left|u_\alpha^{(q)}(t-\tau_{pq})\right|\leq...\leq\left(e^{\mu_0}+1\right)(\omega/\mu)^n U_0$**, while in the second one–** $\left|u_\alpha^{(q)}(t-\tau_{pq})\right|\leq c$ **and have**

$$\left|B_{pq}u_\alpha^{(q)}\right| \le \left|\frac{\Delta_{pq}^2\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^3}u_\alpha^{(q)}+\frac{\langle \xi^{(pq)},\dot{u}^{(q)}\rangle}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^2}u_\alpha^{(q)}\right|$$

$$\le \frac{2U_0}{(1-\beta)^3 r_0^2}\left(e^{\mu_0}+1\right)(\omega/\mu)^n+\frac{\sqrt{3}U_0}{c(1-\beta)^2 r_0}\left(e^{\mu_0}+1\right)(\omega/\mu)^n \le \left(\frac{2}{r_0^2}+\frac{1}{cr_0}\right)(\omega/\mu)^n\frac{\left(e^{\mu_0}+1\right)U_0}{(1-\beta)^3}.$$

For $\left|C_{pq}\dot{u}_\alpha^{(q)}\right|$ we have $\left|C_{pq}\dot{u}_\alpha^{(q)}\right| \le \frac{e^{\mu_0}+1}{c(1-\beta)^2 r_0}(\omega/\mu)^n U_0$.

Therefore

$$\left|G_\alpha^{(pq)}\right| \le \frac{|e_1e_2|}{m_pc}\sum_{\alpha=1}^{3}\left|A_{pq}\xi_\alpha^{(pq)}\right|+\frac{|e_1e_2|}{m_pc}\sum_{\alpha=1}^{3}\left|B_{pq}u_\alpha^{(q)}\right|+\frac{|e_1e_2|}{m_pc}\sum_{\alpha=1}^{3}\left|C_{pq}\dot{u}_\alpha^{(q)}\right| \le$$

$$\le |Q_p|\left(\left(\frac{4}{\mu r_0^3}+\frac{3\sqrt{3}}{c^2r_0}\right)\left(\frac{\omega}{\mu}\right)^{n-1}\frac{\left(e^{\mu_0}+1\right)U_0}{(1-\beta)^3}+\left(\frac{2}{cr_0^2}+\frac{1}{c^2r_0}\right)\left(\frac{\omega}{\mu}\right)^n\frac{\left(e^{\mu_0}+1\right)U_0}{(1-\beta)^3}+\frac{\left(e^{\mu_0}+1\right)U_0}{c^2(1-\beta)^2r_0}\left(\frac{\omega}{\mu}\right)^n\right) \le$$

$$\le |Q_p|\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}+\frac{3\sqrt{3}+2}{c^2r_0}\right)\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\left(\frac{\omega}{\mu}\right)^{n-1}U_0$$

and

$$\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}G_\alpha^{(pq)}ds\right| \le |Q_p|\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}+\frac{3\sqrt{3}+2}{c^2r_0}\right)\times\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\left(\frac{\omega}{\mu}\right)^{n-1}U_0\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}e^{\mu(t-kT_0)}dt$$

$$\le \frac{e^{\mu_0}-1}{\mu_0}|Q_p|\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}+\frac{3\sqrt{3}+2}{c^2r_0}\right)\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\left(\frac{\omega}{\mu}\right)^{n-1}U_0.$$

For the radiation term we can use the estimate from Lemma 2.1, namely

$$\left|G_\alpha^{(p)rad}\right| \le \frac{e_p^2}{m_pc^2}\frac{1}{c(1-\beta)^{3/2}}\frac{\omega^n}{\mu^n}\frac{4}{4-\tau^2\omega^2}\omega^2U_0 \le \frac{|Q_p|}{c^3(1-\beta)^{3/2}}\left(\frac{\omega}{\mu}\right)^n\frac{4\omega^2}{4-\tau^2\omega^2}U_0$$

and then

$$\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}G_\alpha^{(p)rad}ds\right| \le \frac{|Q_p|}{c^3(1-\beta)^{3/2}}(\omega/\mu)^n\frac{4\omega^2}{4-\tau^2\omega^2}U_0\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}e^{\mu(t-kT_0)}dt$$

$$\le \frac{|Q_p|}{c^3(1-\beta)^{3/2}}(\omega/\mu)^n\frac{4\omega^2}{4-\tau^2\omega^2}U_0\frac{e^{\mu_0}-1}{\mu_0}.$$

Therefore

$$\left|\dot{B}_{\alpha,k}^{(p)}(u_1^{(1)},\ldots,u_3^{(2)})(t)\right| \le \left|U_\alpha^{(p)}(t)\right|+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}U_\alpha^{(p)}(s)ds\right| \le$$

$$\le 3|Q_p|\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\left[\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}+\frac{3\sqrt{3}+2}{c^2r_0}\right)\frac{\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\left(\frac{\omega}{\mu}\right)^{n-1}+\frac{1}{c^3(1-\beta)^{3/2}}\left(\frac{\omega}{\mu}\right)^n\frac{4\omega^2}{4-\tau^2\omega^2}\right]U_0$$

$$\le 3|Q_p|\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\left[\left(\frac{4}{\mu r_0^3}+\frac{2}{cr_0^2}+\frac{3\sqrt{3}+2}{c^2r_0}\right)\frac{e^{\mu_0}+1}{(1-\beta)^3}+\frac{1}{c^3(1-\beta)^{3/2}}\frac{4\omega^2}{4-\tau^2\omega^2}\right](\omega/\mu)^{n-1}U_0$$

$$\le 3\left|Q_p\right|\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu}\left[\left(\frac{4}{\mu r_0^3}+\frac{2}{c r_0^2}+\frac{3\sqrt{3}+2}{c^2 r_0}\right)\frac{e^{\mu_0}+1}{\left(1-\beta\right)^3}+\frac{1}{c^3\left(1-\beta\right)^{3/2}}\frac{4\omega^2}{4-\tau^2\omega^2}\right]\left(\frac{\omega}{\mu}\right)^{n-2}\omega U_0 \le \omega U_0 .$$

Supplement 2. Lipschitz Estimate

2.1. Lipschitz Estimates of the Right-Hand Sides

Prior to obtain Lipschitz estimates we notice

$$\left|u(t-\tau_{pq})-\overline{u}(t-\tau_{pq})\right|=\left|\int_{kT_0}^{t-\tau_{pq}}\dot{u}(s_1)ds_1-\int_{kT_0}^{t-\tau_{pq}}\dot{\overline{u}}(s_1)ds_1\right|=\left|\int_{kT_0}^{t-\tau_{pq}}\int_{kT_0}^{s_1}...\int_{kT_0}^{s_{n-1}}\left(u(s_n)\right)^{(n)}ds_n...ds_1-\int_{kT_0}^{t-\tau_{pq}}\int_{kT_0}^{s_1}...\int_{kT_0}^{s_{n-1}}\left(\overline{u}(s_n)\right)^{(n)}ds_n...ds_1\right|$$

$$\le\left|\int_{kT_0}^{t-\tau_{pq}}\int_{kT_0}^{s_1}...\int_{kT_0}^{s_{n-1}}e^{\mu(s_n-kT_0)}ds_n...ds_1\right|\omega^n\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)})\le$$

$$\le\frac{\omega^n}{\mu^{n-1}}\left|\int_{kT_0}^{t-\tau_{pq}}e^{\mu(s_1-kT_0)}ds_1\right|\rho_k(u^{(n)},\overline{u}^{(n)})=\frac{\omega^n}{\mu^{n-1}}\left|\frac{e^{\mu(t-\tau_{pq}-kT_0)}-1}{\mu}\right|\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)})$$

$$\le\frac{\omega^n}{\mu^n}\left|e^{\mu(t-\tau_{pq}-kT_0)}+1\right|\rho_{(k,n)}(u^{(n)},\overline{u}^{(n)})\le\left(\omega/\mu\right)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}\left((u_1,...,u_6),(\overline{u}_1,...,\overline{u}_6)\right)$$

and for the derivatives we have the similar inequalities.

Therefore we can obtain Lipschitz estimates only for derivatives $\left|\dot{u}(t-\tau_{pq})-\dot{\overline{u}}(t-\tau_{pq})\right|$. They are

$$\frac{\partial U_1^{(p)}}{\partial\dot{u}_\beta^{(q)}}=\frac{c^2-(u_1^{(p)})^2}{c^2}\frac{\partial G_1^{(pq)}}{\partial\dot{u}_\beta^{(q)}}-\frac{u_1^{(p)}u_2^{(p)}}{c^2}\frac{\partial G_2^{(pq)}}{\partial\dot{u}_\beta^{(q)}}-\frac{u_1^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_3^{(pq)}}{\partial\dot{u}_\beta^{(q)}};$$

$$\frac{\partial U_2^{(p)}}{\partial\dot{u}_\beta^{(q)}}=-\frac{u_1^{(p)}u_2^{(p)}}{c^2}\frac{\partial G_1^{(pq)}}{\partial\dot{u}_\beta^{(q)}}+\frac{c^2-(u_2^{(p)})^2}{c^2}\frac{\partial G_2^{(pq)}}{\partial\dot{u}_\beta^{(q)}}-\frac{u_2^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_3^{(pq)}}{\partial\dot{u}_\beta^{(q)}};$$

$$\frac{\partial U_3^{(p)}}{\partial\dot{u}_\beta^{(q)}}=-\frac{u_1^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_1^{(pq)}}{\partial\dot{u}_\beta^{(q)}}-\frac{u_2^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_2^{(pq)}}{\partial\dot{u}_\beta^{(q)}}+\frac{c^2-(u_3^{(p)})^2}{c^2}\frac{\partial G_3^{(pq)}}{\partial\dot{u}_\beta^{(q)}}.$$

In order to simplify the next expressions we introduce the following denotation for H. A. Schwartz difference quotient:

$$H(t,\tau,u)=\frac{u(t+\tau)-u(t-\tau)}{2\tau},\ H(t,\tau,\dot{u})=\frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau}.$$

Then

$$\frac{\partial U_1^{(p)}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}=\frac{c^2-\left(u_1^{(p)}\right)^2}{c^2}\frac{\partial G_1^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}-\frac{u_1^{(p)}u_2^{(p)}}{c^2}\frac{\partial G_2^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}-\frac{u_1^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_3^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)};$$

$$\frac{\partial U_2^{(p)}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}=-\frac{u_1^{(p)}u_2^{(p)}}{c^2}\frac{\partial G_1^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}+\frac{c^2-\left(u_2^{(p)}\right)^2}{c^2}\frac{\partial G_2^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}-\frac{u_2^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_3^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)};$$

$$\frac{\partial U_3^{(p)}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}=-\frac{u_1^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_1^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}-\frac{u_2^{(p)}u_3^{(p)}}{c^2}\frac{\partial G_2^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}+\frac{c^2-(u_3^{(p)})^2}{c^2}\frac{\partial G_3^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}$$

and in view of

$$\frac{\partial G_\alpha^{(pq)}}{\partial\dot{u}_\beta{}^{(q)}}=\frac{\Delta_p Q_p}{c^2}\left(\frac{\partial A_{pq}}{\partial\dot{u}_\beta{}^{(q)}}\xi_\alpha^{(pq)}+\frac{\partial B_{pq}}{\partial\dot{u}_\beta{}^{(q)}}u_\alpha^{(q)}+\frac{\partial C_{pq}}{\partial\dot{u}_\beta{}^{(q)}}\dot{u}_\alpha^{(q)}+C_{pq}\right)$$

$(\alpha=1,2,3)$ we have

$$\frac{\partial A_{pq}}{\partial \dot{u}_1^{(q)}}=\frac{\partial}{\partial \dot{u}_1^{(q)}}\frac{\left(c^2-\left\langle u^{(q)},u^{(q)}\right\rangle\right)\left(c^2-\left\langle u^{(p)},u^{(q)}\right\rangle\right)}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^3}+\frac{\partial}{\partial \dot{u}_1^{(q)}}\frac{\left(c^2-\left\langle u^{(p)},u^{(q)}\right\rangle\right)\langle \xi^{(pq)},\dot{u}^{(q)}\rangle}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^2\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}$$

$$-\frac{\partial}{\partial \dot{u}_1^{(q)}}\frac{\left\langle u^{(p)},\dot{u}^{(q)}\right\rangle}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}$$

$$=\frac{\left(c^2-\left\langle u^{(p)},u^{(q)}\right\rangle\right)\xi_1^{(pq)}}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^2\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}-\frac{u_1^{(p)}}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)};$$

$$\frac{\partial B_{pq}}{\partial \dot{u}_1^{(q)}}=-\frac{\partial}{\partial \dot{u}_1^{(q)}}\frac{\left(c^2-\left\langle u^{(q)},u^{(q)}\right\rangle\right)\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)+\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)\langle \xi^{(pq)},\dot{u}^{(q)}\rangle}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^3}=\frac{-\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)\xi_1^{(pq)}}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^3};\quad \left|\frac{\partial C_{pq}}{\partial \dot{u}_1^{(q)}}\right|=0\,.$$

Then

$$\sum_{\beta=1}^{3}\left|\int_{kT_0}^{t}\frac{\partial G_\alpha^{(pq)}}{\partial \dot{u}_\beta^{(q)}}\left(\dot{u}_\beta^{(q)}-\dot{\bar{u}}_\beta^{(q)}\right)ds\right|\le\frac{\Delta_p\left|Q_p\right|}{c^2}\times\sum_{\beta=1}^{3}\left|\int_{kT_0}^{t}\left|\frac{\partial A_{pq}}{\partial \dot{u}_\beta^{(q)}}\xi_\alpha^{(pq)}+\frac{\partial B_{pq}}{\partial \dot{u}_\beta^{(q)}}u_\alpha^{(q)}+C_{pq}\right|\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds\right|\le$$

$$\le\frac{\left|Q_p\right|}{c}\left|\int_{kT_0}^{t}\left(\frac{\left(c^2-\left\langle u^{(p)},u^{(q)}\right\rangle\right)c\tau_{pq}.c\tau_{pq}}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^2\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}-\frac{c.c\tau_{pq}}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}\right)\times\right.$$

$$\left.\times\sum_{\beta=1}^{3}\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds\right|+\frac{\left|Q_p\right|}{c}\left(\left|\int_{kT_0}^{t}\frac{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)c\tau_{pq}c}{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle\right)^3}\times\right.\right.$$

$$\left.\left.\times\sum_{\beta=1}^{3}\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds\right|++\left|\int_{kT_0}^{t}\frac{\sum_{\beta=1}^{3}\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds}{c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}\right|\right)\le$$

$$\le\left|Q_p\right|\left(\frac{2}{(1-\beta)^3c^2r_0}+\frac{1}{c^2r_0(1-\beta)^2}+\frac{2}{(1-\beta)^2c^2r_0}+\frac{1}{c^2r_0(1-\beta)}\right)\times\left|\int_{kT_0}^{t}\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds\right|\le$$

$$\le\frac{3\left|Q_p\right|}{(1-\beta)^3}\frac{6}{c^2r_0}(\omega/\mu)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}((u_1,...,u_6),(\bar{u}_1,...,\bar{u}_6));$$

$$\left|\int_{kT_0}^{(k+1)T_0}\frac{\partial G_\alpha^{(pq)}}{\partial \dot{u}_\beta^{(q)}}\left(\dot{u}_\beta^{(q)}-\dot{\bar{u}}_\beta^{(q)}\right)ds\right|\le\frac{\Delta_p\left|Q_p\right|}{c^2}\times\left|\int_{kT_0}^{t}\left|\frac{\partial A_{pq}}{\partial \dot{u}_\beta^{(q)}}\xi_\alpha^{(pq)}+\frac{\partial B_{pq}}{\partial \dot{u}_\beta^{(q)}}u_\alpha^{(q)}+C_{pq}\right|\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds\right|\le$$

$$\le\left|Q_p\right|\left(\frac{2}{(1-\beta)^3c^2r_0}+\frac{1}{c^2r_0(1-\beta)^2}+\frac{2}{(1-\beta)^3c^2r_0}+\frac{1}{c^2r_0(1-\beta)}\right)\times\left|\int_{kT_0}^{(k+1)T_0}\left(\dot{u}_\beta^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_\beta^{(q)}(s-\tau_{pq})\right)ds\right|=0.$$

For the radiation term we obtain

$$\left|\frac{\partial G_\alpha^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_\beta^{(p)}\right)}\right|=\frac{e_p^2}{m_pc^2}\left|\frac{u_\alpha^{(p)}u_\beta^{(p)}}{\left(c^2-\langle u^{(p)},u^{(p)}\rangle\right)^{3/2}}+\frac{1}{\left(c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\right)^{1/2}}\right|\le\frac{\left|Q_p\right|}{c^2}\left(\frac{c^2}{c^3\left(1-\beta^2\right)^{3/2}}+\frac{1}{c\left(1-\beta^2\right)^{1/2}}\right)\le\frac{2\left|Q_p\right|}{c^3\left(1-\beta^2\right)^{3/2}}.$$

Since

$$\left|\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t+\tau)-\left(\dot{u}_{\gamma}^{(p)}(t-\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t-\tau)\right)}{2\tau}\right| \le \left|\ddot{u}_{\gamma}^{(p)}(t)-\ddot{\bar{u}}_{\gamma}^{(p)}(t)\right| + \frac{\tau^2}{3!}\left|\left(u_{\gamma}^{(p)}(t)\right)^{IV} - \left(\bar{u}_{\gamma}^{(p)}(t)\right)^{IV}\right| +$$

$$+\frac{\tau^4}{5!}\left|\left(u_{\gamma}^{(p)}(t)\right)^{VI} - \left(\bar{u}_{\gamma}^{(p)}(t)\right)^{VI}\right| + \ldots \le e^{\mu(t-kT_0)}\left[\frac{\omega^n}{\mu^n}\rho_{(k,n+2)}\left(\left(u_{\gamma}^{(p)}\right)^{(n+2)},\left(\bar{u}_{\gamma}^{(p)}\right)^{(n+2)}\right)+\right.$$

$$\left.+\frac{\tau^2}{3!}\frac{\omega^n}{\mu^n}\rho_{(k,n+4)}\left(\left(u_{\gamma}^{(p)}\right)^{(n+4)},\left(\bar{u}_{\gamma}^{(p)}\right)^{(n+4)}\right)+\frac{\tau^4}{5!}\frac{\omega^n}{\mu^n}\rho_{(k,n+4)}\left(\left(u_{\gamma}^{(p)}\right)^{(n+6)},\left(\bar{u}_{\gamma}^{(p)}\right)^{(n+6)}\right)+\ldots\right] \le$$

$$\le e^{\mu(t-kT_0)}\left(1+\frac{\tau^2}{2^2}+\frac{\tau^4}{2^4}+\ldots\right)\frac{\omega^n}{\mu^n}\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right) = e^{\mu(t-kT_0)}\frac{4}{4-\tau^2}\left(\omega/\mu\right)^n\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right)$$

we have

$$\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{t}\frac{\partial G_{\alpha}^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_{\beta}^{(p)}\right)}\left(\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{u}_{\gamma}^{(p)}(t-\tau)}{2\tau}-\frac{\dot{\bar{u}}_{\gamma}^{(p)}(t+\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t-\tau)}{2\tau}\right)ds\right| \le$$

$$\le \frac{6\left|Q_p\right|}{c^3\left(1-\beta^2\right)^{3/2}}e^{\mu(t-kT_0)}\frac{4}{4-\tau^2}\left(\omega/\mu\right)^n\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right) \le$$

$$\le e^{\mu(t-kT_0)}\frac{6\left|Q_p\right|}{c^3\left(1-\beta\right)^{3/2}}\frac{4}{4-\tau^2}\left(\omega/\mu\right)^n\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right)$$

and

$$\sum_{\gamma=1}^{3}\left|\int_{kT_0}^{(k+1)T_0}\frac{\partial G_{\alpha}^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_{\beta}^{(p)}\right)}\left(\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{u}_{\gamma}^{(p)}(t-\tau)}{2\tau}-\frac{\dot{\bar{u}}_{\gamma}^{(p)}(t+\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t-\tau)}{2\tau}\right)ds\right| = 0..$$

Therefore the Lipschitz estimate for the operator is

$$\left|B_{\beta,k}^{(p)}(u_1^{(1)},\ldots,u_3^{(2)})(t)-B_{\beta,k}^{(p)}(\bar{u}_1^{(1)},\ldots,\bar{u}_3^{(2)})(t)\right| \le \left|\int_{kT_0}^{t}U_{\beta}^{(p)}(u_1^{(1)},\ldots,u_3^{(2)})(s)-U_{\beta}^{(p)}(\bar{u}_1^{(1)},\ldots,\bar{u}_3^{(2)})(t)(s)ds\right| +$$

$$\le \frac{9\left|Q_p\right|}{(1-\beta)^3}\frac{6}{c^2 r_0}(\omega/\mu)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right)+$$

$$\le \frac{9\left|Q_p\right|}{\left(1-\beta\right)^3}\frac{6}{c^2 r_0}\left(\omega/\mu\right)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right)+$$

$$+e^{\mu(t-kT_0)}\frac{18\left|Q_p\right|}{c^3\left(1-\beta\right)^{3/2}}\frac{4}{4-\tau^2}\left(\frac{\omega}{\mu}\right)^n\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right) \le$$

$$\le e^{\mu(t-kT_0)}\left|Q_p\right|\left(3\frac{e^{\mu_0}+1}{c^2 r_0}+\frac{1}{c^3}\frac{4}{4-\tau^2}\right)\left(\frac{\omega}{\mu}\right)^n\frac{18}{\left(1-\beta\right)^3}\times\rho_{(k,\infty)}\left((u_1,\ldots,u_6),(\bar{u}_1,\ldots,\bar{u}_6)\right).$$

2.2. Lipschitz Estimates of the Derivatives

In view of $\tau_{pq} < \frac{T_0}{2}$ we obtain the following inequality:

$$\sum_{\gamma=1}^{3}\left|\frac{\partial G_{\gamma}^{(pq)}}{\partial \dot{u}_{\alpha}^{(q)}}\left(\dot{u}_{\alpha}^{(q)}(t-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(t-\tau_{pq})\right)\right|\le 3\left|Q_p\right|e^{\mu_0}\left(\frac{4}{r_0^3}+\frac{2\sqrt{3}}{cr_0^2}+\frac{4\sqrt{3}+2}{c^2 r_0}\right)U_0\left|\dot{u}_{\alpha}^{(q)}(t-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(t-\tau_{pq})\right|\le$$

$$\le\left(\frac{2}{(1-\beta)^3 c^2 r_0}+\frac{1}{c^2 r_0(1-\beta)^2}+\frac{2}{(1-\beta)^2 c^2 r_0}+\frac{1}{c^2 r_0(1-\beta)}\right)\times\left|Q_p\right|\left|\dot{u}_{\alpha}^{(q)}(t-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(t-\tau_{pq})\right|\le$$

$$\le\frac{3\left|Q_p\right|}{(1-\beta)^3}\frac{6}{c^2 r_0}(\omega/\mu)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}\left((u_1,...,u_6),(\bar{u}_1,...,\bar{u}_6)\right);$$

$$\left|\sum_{\alpha=1}^{3}\sum_{\gamma=1}^{3}\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\frac{\partial G_{\gamma}^{(pq)}}{\partial \dot{u}_{\alpha}^{(q)}}\left(\dot{u}_{\alpha}^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(s-\tau_{pq})\right)ds\right|=0.$$

For the radiation term we have

$$\sum_{\gamma=1}^{3}\left|\frac{\partial G_{\alpha}^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_{\gamma}^{(p)}\right)}\left(\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{u}_{\gamma}^{(p)}(t-\tau)}{2\tau}-\frac{\dot{\bar{u}}_{\gamma}^{(p)}(t+\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t-\tau)}{2\tau}\right)\right|\le\frac{3\left|Q_p\right|}{(1-\beta)^3}\frac{6\left(e^{\mu_0}+1\right)}{c^2 r_0}(\omega/\mu)^n\rho_{(k,\infty)}\left((u_1,...,u_6),(\bar{u}_1,...,\bar{u}_6)\right)$$

and

$$\sum_{\gamma=1}^{3}\left|\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\frac{\partial G_{\alpha}^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_{\gamma}^{(p)}\right)}\times\left(\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{u}_{\gamma}^{(p)}(t-\tau)}{2\tau}-\frac{\dot{\bar{u}}_{\gamma}^{(p)}(t+\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t-\tau)}{2\tau}\right)ds\right|=0.$$

In view of $e^{\mu_0}U_0\le c$ we have

$$\left|\dot{B}_{\alpha,k}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)-\dot{B}_{\alpha,k}^{(p)}(\bar{u}_1^{(1)},...,\bar{u}_3^{(2)})(t)\right|\le\left|U_{\alpha}^{(p)}(u_1^{(1)},...,u_3^{(2)})(t)-U_{\alpha}^{(p)}(\bar{u}_1^{(1)},...,\bar{u}_3^{(2)})(t)\right|+$$

$$+\frac{1}{T_0}\left|\int_{kT_0}^{(k+1)T_0}\left(U_{\alpha}^{(p)}(u_1^{(1)},...,u_3^{(2)})(s)-U_{\alpha}^{(p)}(\bar{u}_1^{(1)},...,\bar{u}_3^{(2)})(s)\right)ds\right|\le$$

$$\le\left|\sum_{\alpha=1}^{3}\frac{\partial U_{\beta}^{(p)}}{\partial \dot{u}_{\alpha}^{(q)}}\left(\dot{u}_{\alpha}^{(q)}(t-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(t-\tau_{pq})\right)\right|+\left|\sum_{\alpha=1}^{3}\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\frac{\partial U_{\beta}^{(p)}}{\partial \dot{u}_{\alpha}^{(q)}}\left(\dot{u}_{\alpha}^{(q)}(s-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(s-\tau_{pq})\right)ds\right|\le$$

$$\le\left|\sum_{\alpha=1}^{3}\sum_{\gamma=1}^{3}\frac{\partial G_{\gamma}^{(pq)}}{\partial \dot{u}_{\alpha}^{(q)}}\left(\dot{u}_{\alpha}^{(q)}(t-\tau_{pq})-\dot{\bar{u}}_{\alpha}^{(q)}(t-\tau_{pq})\right)\right|+\left|\sum_{\alpha=1}^{3}\sum_{\gamma=1}^{3}\frac{\partial G_{\alpha}^{(p)rad}}{\partial H\left(t,\tau,\dot{u}_{\gamma}^{(p)}\right)}\times\right.$$

$$\left.\times\left(\frac{\dot{u}_{\gamma}^{(p)}(t+\tau)-\dot{u}_{\gamma}^{(p)}(t-\tau)}{2\tau}-\frac{\dot{\bar{u}}_{\gamma}^{(p)}(t+\tau)-\dot{\bar{u}}_{\gamma}^{(p)}(t-\tau)}{2\tau}\right)\right|\le$$

$$\le 3\left[\frac{3\left|Q_p\right|}{(1-\beta)^3}\frac{6}{c^2 r_0}(\omega/\mu)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}\left((u_1,...,u_6),(\bar{u}_1,...,\bar{u}_6)\right)+\right.$$

$$\left.+\frac{3\left|Q_p\right|}{(1-\beta)^3}\frac{6}{c^2 r_0}(\omega/\mu)^n\left(e^{\mu_0}+1\right)\rho_{(k,\infty)}\left((u_1,...,u_6),(\bar{u}_1,...,\bar{u}_6)\right)\right]\le$$

$$\le\frac{9\left|Q_p\right|\left(e^{\mu_0}+1\right)}{(1-\beta)^3}\frac{12}{c^2 r_0}(\omega/\mu)^n\rho_{(k,\infty)}\left((u_1,...,u_6),(\bar{u}_1,...,\bar{u}_6)\right).$$

ACKNOWLEDGENTS

1) The author thanks Prof. D-r D.-A. Deckert for his kind invitation to deliver a lecture devoted to the problems of the present paper at the Seminar of International Junior Research Group "Interaction between Light and Matter", Ludwig Maximilian University, Munich, June 2015.
2) The author thanks the referee for his very useful remarks and suggestions which improved the paper.

REFERENCES

[1] Angelov, V. G., 2015, Two-body problem of classical electrodynamics with Dirac radiation terms – Derivation of equations of motion (I). Int. J. Theor. and Math. Phys., 5(5), 119-135.

[2] W. Pauli, Relativitats Theorie, Encyklopedie der Mathematischen Wissenschaften, vol. 19, 1921.

[3] Synge, J. L., 1940, On the electromagnetic two-body problem, Proc. Roy. Soc. (London), A177, 118-139.

[4] Dirac, P. A. M., 1938, Classical theory of radiating electrons, Proc. Roy. Soc. (London), A1, 148-169.

[5] V. G. Angelov, Fixed Points in Uniform Spaces and Applications. Cluj University Press, Romania, 2009.

[6] N. Bohr, Atomic Theory and Description of Nature. Cambridge University Press, Teddington, England, 1934.

[7] A. Sommerfeld, Atombau und Spektrallinien. Band I, II, Friedr. Vieweg&Sohn, Braunschwig, 1951.

[8] Wheeler, J. A. and R. P. Feynman, 1945, Interaction with absorber as the mechanism of radiation, Rev. Mod. Physics, 17(2-3), 157-179.

[9] Wheeler, J. A. and R. P. Feynman, 1949, Classical electrodynamics in terms of direct interparticle action, Rev. Mod. Physics, 21(3), 425-433.

[10] Bauer, G., D.-A. Deckert, D. Durr and G. Hinrichs, 2014, On irreversibility and radiation in classical electrodynamics of point particles, Journal of Statistical Physics, 154(1-2):610, 12 pages.

[11] D.-A. Deckert and G. Hinrichs, 2015, Electrodynamic two-body problem for prescribing initial date on the straight line, [Online]. Available: http://arxiv.org/abs/1507.04991, 26 pages.

[12] D.-A. Deckert, D. Durr and N. Vona, "Delay equations of the Wheeler-Feynman type," in Proceedings of the Sixth International Conference on Differential and Functional-Differential Equations (Moscow, August 14-21, 2011), Part 3, Journal of Contemporary Mathematics. Fundamental Directions, 47:46, 13 pages, 2013.

[13] G. Bauer, D.-A. Deckert, and D. Durr, 2012, On the existence of dynamics in Wheeler-Feynman electromagnetism, Journal of Applied Mathematics and Physics, 64(4), 1087-1124.

[14] de Luca, J., 2013, Variational electrodynamics of atoms. Progress in Electromagnetism Research, B, July, 1-11.

[15] Angelov, V. G., 1990, On the Synge equations in a three-dimensional two-body problem of classical electrodynamics, J. Math. Anal. Appl., 151, 488-511.

[16] Angelov, V. G., 2000, Escape trajectories of J. L. Synge equations, J. Nonlinear Analysis, RWA, 1, 189-204.

[17] V. G. Angelov, A Method for Analysis of Transmission Lines Terminated by Nonlinear Loads, Nova Science, New York, 2014.

[18] M. I. Korsunskii, Optics, Atomic Structure, Atomic Nucleus. Fizmatgiz, Moscow, 1962 (in Russian).

[19] R. W. Pohl, Optik und Atomphysik. Springer Verlag, 1963.

[20] I. I. Goldenblat and S. W. Ulianov, Introduction to the Raltivity Theory and its Applications to the New Technique. Moscow, 1979 (in Russian).