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Two-Body Problem of Classical Electrodynamics with Radiation Terms-Derivation of Equations (I)

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Abstract This paper is the first part of our investigations devoted to the two-body problem of classical electrodynamics. The primary purpose of this first part is to derive equations of motion describing two moving charged mass particles taking into account the radiation. We proceed from the suggestions given by J. L. Synge [1]. He has proposed a formulation of the of relativistic two-body problem with usually accepted Dirac's radiation terms [2] containing second derivatives of the velocities: $m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2} F_{rs}^{(2)} \lambda_s^{(1)} - \frac{2}{3}\frac{e_1^2}{c^2}\left(\ddot{\lambda}_r^{(1)} - \lambda_r^{(2)} \dot{\lambda}_s^{(1)} \dot{\lambda}_s^{(1)}\right)$, $m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2} F_{rs}^{(1)} \lambda_s^{(2)} - \frac{2}{3}\frac{e_2^2}{c^2}\left(\ddot{\lambda}_r^{(2)} - \lambda_r^{(1)} \dot{\lambda}_s^{(2)} \dot{\lambda}_s^{(2)}\right)$. Here we propose a general approach to introduce new equations of motion based on the same Dirac's physical assumptions from [2]. Instead of the above system of eight equations of motion we derive consider an analogous system $m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2}\left(F_{rs}^{(2)} \lambda_1^{(1)} + F_{rs}^{(1)rad} \lambda_s^{(1)}\right)$, $m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2}\left(F_{rs}^{(1)} \lambda_s^{(2)} + F_{rs}^{(2)rad} \lambda_s^{(2)}\right)$ where the classical Lorentz-Dirac radiation terms are replaced by newly derived ones. We show that two equations are consequences of the rest ones and so we have to solve a system of six equations for six unknown velocities, issue that has not been discussed in the literature. Instead of second order differential system (with respect to the unknown velocities) we obtain a first order neutral system with both retarded and advanced arguments depending on the unknown trajectories. In the second part we solve the system and so we give a method for overcoming the singularities arising in Dirac radiation terms.

Keywords Classical electrodynamics, Two-Body problem, Dirac-Lorentz radiation term, Neutral equations with both delay and advanced arguments, Fixed point theorem

1. Introduction

In [1] J. L. Synge has formulated a two-body problem in the frame of classical electrodynamics and has mentioned at the very end of [1] the possibility to generalize the model including Dirac radiation terms [2]. The Synge's considerations [1] of two-body problem are based on the Lorentz ponder-motive force derived in a relativistic form by W. Pauli [3] via Lienard-Wiechert retarded potentials and adding the Dirac's radiation terms. Our goal is, following [1], to give a unified approach of derivation of equations of motion for two-body problem with radiation terms. We propose a new mathematical formulation of the Dirac's idea based on retarded and advanced potentials. Let us note that Dirac's derivations lead to the second order differential equations with respect to the unknown velocities which by necessity needs prescribing of initial accelerations.

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A lot of papers investigate equations with usually accepted Dirac-Lorentz radiation terms [4]-[27]. In contrast of these papers following the Synge's formalism [1] here we derive a new form of the radiation term leading to the first order neutral equations with respect to unknown velocities and introduce retarded and advanced arguments depending on unknown trajectories [28]. We note that this new term is applied in the one-dimensional case to overcome P. Ehrenfest paradox [29] and in three-dimensional case for correction of the Lorentz-Dirac equation [30].

We point out that the following difficulty arises that has not been discussed in the literature. The relativistic Lorentz – Dirac equations in the Minkowski's space are four in number for three unknown functions. In [30] we have proved that 4-th equation is a consequence of the first three ones. An analogous problems arises in the two-body problem.

The equations of motion are eight in number for six un-known functions velocities. Here we prove that the 4-th and the 8-th equation are consequence of the rest ones. In this manner we overcome a mathematical problem generated by overdetermined system. From the physical point of view, solving this system substituting the found functions in the

4-th and 8-th equations we obtain the energy balance of the moving particles.

In the second part we prove an existence-uniqueness of a periodic solution of the system mentioned which means an existence of a closed orbit of two-body problem. In this way, we show that the Bohr-Sommerfeld stationary states (cf. [31], [32]) are rather implicated by the classical electrodynamics than contradict it.

Here we use the technique introduced in [33], [34] and obtain a system of eight equations of motion with new form of the radiation terms:

$$m_1 \frac{d\lambda_n^{(1)}}{ds_1} = \frac{e_1}{c^2}\left(F_{ns}^{(2)}\lambda_1^{(1)} + F_{ns}^{(1)\text{rad}}\lambda_s^{(1)}\right),$$

$$m_2 \frac{d\lambda_n^{(2)}}{ds_2} = \frac{e_2}{c^2}\left(F_{ns}^{(1)}\lambda_s^{(2)} + F_{ns}^{(2)\text{rad}}\lambda_s^{(2)}\right).$$

Main results are given in Section 2. Subsection 2.1 is devoted to the strict mathematical formulation of the original Dirac's assumptions. In fact, we compare the relative and absolute times assuming that the past and future instants depend only on the present instant. In Subsection 2.2 we show that the 4-th equation is a consequence of the first three ones and the 8-th equation is a consequence of the 5-th, 6-th and 7-th ones. So we obtain a system of six equations for six unknown functions − the velocities of the moving particles. This system is of neutral type with respect to the unknown velocities with both retarded and advanced arguments depending on the unknown trajectories.

Section 3 is Conclusion.

Some cumbersome calculations are separated in Supplement 1 and Supplement 2.

2. Main Results

2.1. Derivation of Equations of Motion for Two-Body Problem with New Dirac Radiation Term

The considerations are in the Minkowski's space [3]. Roman suffixes run over 1, 2, 3, 4 while Greek – 1, 2, 3 with Einstein summation convention. We use denotations from [1]. By $\langle .,. \rangle_4$ we denote the dot product in the Minkowski space, and by $\langle .,. \rangle$ – the dot product in three-dimensional Euclidean subspace. The space-time coordinates of the moving particles are $\left(x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)} = ict\right)$, $(p = 1,2)$. Quantities relating to the particles are denoted in the following way: L_p – world lines; m_p – proper masses; e_p – charges. The components of unit tangent vectors to world lines are

$$\left(\lambda_1^{(p)}, \lambda_2^{(p)}, \lambda_3^{(p)}, \lambda_4^{(p)}\right) = \left(\frac{\gamma_p u_\alpha^{(p)}(t)}{c}, \frac{\gamma_p u_\alpha^{(p)}(t)}{c}, \frac{\gamma_p u_\alpha^{(p)}(t)}{c}, i\gamma_p\right),$$

$$\gamma_p(t) = 1 \Big/ \sqrt{1 - \frac{1}{c^2}\left\langle u^{(p)}(t), u^{(p)}(t)\right\rangle},$$

$$\frac{d}{ds_p} = \frac{\gamma_p}{c}\frac{d}{dt} \; (p = 1,2).$$

Then

$$\lambda_\alpha^{(p)} = \frac{u_\alpha^{(p)}(t)}{c\sqrt{1 - \frac{1}{c^2}\left\langle u^{(p)}, u^{(p)}\right\rangle}} = \frac{u_\alpha^{(p)}(t)}{\sqrt{c^2 - \left\langle u^{(p)}, u^{(p)}\right\rangle}} \equiv \frac{u_\alpha^{(p)}(t)}{\Delta_p},$$

$$\lambda_4^{(p)} = \frac{i}{\sqrt{1 - \frac{1}{c^2}\left\langle u^{(p)}, u^{(p)}\right\rangle}} = \frac{ic}{\sqrt{c^2 - \left\langle u^{(p)}, u^{(p)}\right\rangle}} \equiv \frac{ic}{\Delta_p}$$

where

$$\Delta_p = \sqrt{c^2 - \left\langle u^{(p)}, u^{(p)}\right\rangle} \equiv \sqrt{c^2 - \sum_{\alpha=1}^{3}\left(u_\alpha^{(p)}(t)\right)^2}, \quad (p = 1,2),$$

c is the vacuum speed of light and

$$u^{(p)} = \left\{u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t)\right\} = \left\{\dot{x}_1^{(p)}(t), \dot{x}_2^{(p)}(t), \dot{x}_3^{(p)}(t)\right\}$$

are velocities of the moving particles. The components of the accelerations are

$$\frac{d\lambda^{(p)}}{ds_p} = \left\{\begin{array}{l} \frac{\gamma_p}{c}\frac{d}{dt}\left(\frac{\gamma_p \lambda_1^{(p)}}{c}\right), \frac{\gamma_p}{c}\frac{d}{dt}\left(\frac{\gamma_p \lambda_2^{(p)}}{c}\right), \\ \frac{\gamma_p}{c}\frac{d}{dt}\left(\frac{\gamma_p \lambda_3^{(p)}}{c}\right), i\frac{\gamma_p}{c}\frac{d\gamma_p}{dt} \end{array}\right\}$$

or

$$\frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{1}{\Delta_p}\frac{d}{dt}\left(\frac{u_\alpha^{(p)}}{\Delta_p}\right)(\alpha = 1,2,3), \; \frac{d\lambda_4^{(p)}}{ds_p} = \frac{ic}{\Delta_p}\frac{d}{dt}\left(\frac{1}{\Delta_p}\right).$$

Let e_1 be a charge describing any curve L_1 in space-time (cf. [1]). Let A_2 be any event and let A_1 be an intersection of L_1 with the null-cone drawn into the past from A_2. Let $\lambda_r^{(1)}$ be the unit tangent vector to L_1 at A_1, and let $\xi_r^{(21)}$ be the null-vector $A_1 A_2$. Then, by hypothesis from [1], the field at A_2 due to L_1 is given by the retarded potential 4-vector

$$\Phi_r^{(1)} = -\frac{e_1 \lambda_r^{(1)}}{\left\langle \lambda^{(1)}, \xi^{(21)}\right\rangle_4} \tag{1}$$

and the corresponding electromagnetic tensor is

$$F_{kn}^{(1)} = \frac{\partial \Phi_n^{(1)}}{\partial x_k^{(2)}} - \frac{\partial \Phi_k^{(1)}}{\partial x_n^{(2)}} = e_1\left(P_k^{(1)}\xi_n^{(21)} - P_n^{(1)}\xi_k^{(21)}\right) \tag{2}$$

where

$$P_k^{(1)} = -\frac{\lambda_k^{(1)}}{\left\langle \lambda^{(1)}, \xi^{(21)} \right\rangle_4^3} \left[1 + \left\langle \xi^{(21)}, \frac{d\lambda^{(1)}}{ds_1} \right\rangle \right] + \frac{1}{\left\langle \lambda^{(1)}, \xi^{(21)} \right\rangle_4^2} \frac{d\lambda_k^{(1)}}{ds_1} \tag{3}$$

and

$$\xi^{(21)} = \left(\xi_1^{(21)}, \xi_2^{(21)}, \xi_3^{(21)}, \xi_4^{(21)} \right) = \left(x_1^{(2)}(t) - x_1^{(1)}(t-\tau_{21}), x_2^{(2)}(t) - x_2^{(1)}(t-\tau_{21}), x_3^{(2)}(t) - x_3^{(1)}(t-\tau_{21}), ic\tau_{21} \right)$$

is an isotropic vector, i.e. it lies on the light-cone [1]. The world-line L_2 of the charge e_2, passing through A_2, satisfies the equations of motion

$$m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2} F_{rs}^{(1)} \lambda_s^{(2)} .$$

We have to add however the radiation field caused by the particle itself. Then equations of motion for the second particle are the following four ones:

$$m_2 \frac{d\lambda_k^{(2)}}{ds_2} = \frac{e_2}{c^2} \left(F_{kn}^{(1)} \lambda_n^{(2)} + F_{kn}^{(2)rad} \lambda_n^{(2)} \right).$$

If we interchange the roles of the two world-lines we have the following equations of motion for the first particle:

$$m_1 \frac{d\lambda_k^{(1)}}{ds_1} = \frac{e_1}{c^2} \left(F_{kn}^{(2)} \lambda_n^{(1)} + F_{kn}^{(1)rad} \lambda_n^{(1)} \right) \tag{4}$$

or

$$\frac{d\lambda_k^{(1)}}{ds_1} = \frac{e_1 e_2}{m_1 c^2} \left[P_k^{(2)} \left\langle \lambda^{(1)}, \xi^{(12)} \right\rangle_4 - \xi_k^{(12)} \left\langle \lambda^{(1)}, P^{(2)} \right\rangle_4 \right] + \frac{e_1}{m_1 c^2} F_{kn}^{(1)rad} \lambda_n^{(1)} \tag{5}$$

where

$$P_k^{(2)} = -\frac{\lambda_k^{(2)}}{\left\langle \lambda^{(2)}, \xi^{(12)} \right\rangle_4^3} \left[1 + \left\langle \xi^{(12)}, \frac{d\lambda^{(2)}}{ds_2} \right\rangle \right] + \frac{1}{\left\langle \lambda^{(2)}, \xi^{(12)} \right\rangle_4^2} \frac{d\lambda_k^{(2)}}{ds_2} .$$

So we obtain eight equations of motion for two-body problem with radiation terms:

$$m_p \frac{d\lambda_k^{(p)}}{ds_p} = \frac{e_p}{c^2} \left(F_{kn}^{(q)} \lambda_n^{(p)} + F_{kn}^{(p)rad} \lambda_n^{(p)} \right),$$

$$(pq) = (12), (21) \, (k = 1,2,3,4).$$

We recall that derivation of the radiation term is based on the physical assumptions from [2].

Consider a charge e_p $(p = 1,2)$ describing any curve L_p in the space-time. Let

$$A_p \, (x_1^{(p)}(t), x_2^{(p)}(t), x_4^{(p)}(t), ict)$$

be any event,

$$A^{(p)ret} (x_1^{(p)}(\check{t}), \, x_2^{(p)}(\check{t}), \, x_3^{(p)}(\check{t}), \, ic\check{t}_p), \, \check{t}_p < t$$

be the intersection of L_p with the null-cone drawn into the past from A_p, and

$$A^{(p)adv} (x_1^{(p)}(\hat{t}), x_2^{(p)}(\hat{t}), x_3^{(p)}(\hat{t}), ic\,\hat{t}_p), \; t < \hat{t}_p$$

be the intersection of L_p with the null-cone drawn into the future from A_p.

The components of the velocity (tangent) vector to the world-line L_p at $A^{(p)ret}$ are

$$\lambda^{(p)r} = \left(\lambda_1^{(p)r}, \lambda_2^{(p)r}, \lambda_3^{(p)r}, \lambda_4^{(p)r} \right) = \left(\frac{u_1^{(p)}(\check{t}_p)}{\Delta_{(p)r}}, \frac{u_2^{(p)}(\check{t}_p)}{\Delta_{(p)r}}, \frac{u_3^{(p)}(\check{t}_p)}{\Delta_{(p)r}}, \frac{ic}{\Delta_{(p)r}} \right)$$

where $\Delta_{(p)r} = \sqrt{c^2 - \left\langle u^{(p)}(\check{t}_p), u^{(p)}(\check{t}_p) \right\rangle}$, and let $A^{(p)ret} A_p$ be isotropic vector $\xi^{(p)r} = \left(\xi_1^{(p)r}, \xi_2^{(p)r}, \xi_3^{(p)r}, \xi_4^{(p)r} \right)$,

where

$$\xi_\alpha^{(p)r} = x_\alpha^{(p)}(t) - x_\alpha^{(p)}(\check{t}_p) (\alpha = 1,2,3),$$

$$\xi_4^{(p)r} = ic(t - \check{t}_p), \; \check{t}_p < t .$$

Similarly the components of the velocity (tangent) vector to the world-line L_p at $A^{(p)adv}$ are

$$\lambda^{(p)a} = \left(\lambda_1^{(p)a}, \lambda_2^{(p)a}, \lambda_3^{(p)a}, \lambda_4^{(p)a} \right) = \left(\frac{u_1^{(p)}(\hat{t}_p)}{\Delta_{(p)a}}, \frac{u_2^{(p)}(\hat{t}_p)}{\Delta_{(p)a}}, \frac{u_3^{(p)}(\hat{t}_p)}{\Delta_{(p)a}}, \frac{ic}{\Delta_{(p)a}} \right)$$

where $\Delta_{(p)a} = \sqrt{c^2 - \left\langle u^{(p)}(\hat{t}_p), u^{(p)}(\hat{t}_p) \right\rangle}$, and let $A^{(p)adv} A_p$ be isotropic vector $\xi^{(p)a} = \left(\xi_1^{(p)a}, \xi_2^{(p)a}, \xi_3^{(p)a}, \xi_4^{(p)a}\right)$ where $\xi_\alpha^{(p)a} = x_\alpha^{(p)}(\hat{t}_p) - x_\alpha^{(p)}(t)\ (\alpha = 1,2,3),\ \xi_4^{(p)r} = ic(\hat{t}_p - t),\ \hat{t}_p > t$.

In accordance with Dirac assumptions [2] the radiation term is defined as a half of the difference between both retarded and advanced potentials, that is,

$$F_{kn}^{(p)\,rad} = \frac{1}{2}\left[\left(\frac{\partial A_n^{(p)ret}}{\partial x_k^{(p)ret}} - \frac{\partial A_\kappa^{(p)ret}}{\partial x_n^{(p)ret}}\right) - \left(\frac{\partial A_n^{(p)adv}}{\partial x_k^{(p)adv}} - \frac{\partial A_\kappa^{(p)adv}}{\partial x_n^{(p)adv}}\right)\right]$$

where

$$A_n^{(p)ret} = -\frac{e_p \lambda_n^{(p)r}}{\langle \lambda^{(p)r}, \xi^{(p)r}\rangle_4},\ A_n^{(p)adv} = -\frac{e_p \lambda_n^{(p)a}}{\langle \lambda^{(p)a}, \xi^{(p)a}\rangle_4}.$$

So in view of (3) we obtain the following eight equations:

$$\frac{d\lambda_k^{(p)}}{ds_p} = \frac{e_p e_q}{m_p c^2}\left[P_k^{(q)}\left\langle \lambda^{(p)}, \xi^{(pq)}\right\rangle_4 - \xi_k^{(pq)}\left\langle \lambda^{(p)}, P^{(q)}\right\rangle_4\right] + \frac{e_p}{m_p c^2} F_{kn}^{(p)\,rad} \lambda_n^{(p)},$$

or

$$\frac{d\lambda_k^{(p)}}{ds_p} = \frac{e_p e_q}{m_p c^2}\left[P_k^{(q)}\left\langle \lambda^{(p)}, \xi^{(pq)}\right\rangle_4 - \xi_k^{(pq)}\left\langle \lambda^{(p)}, P^{(q)}\right\rangle_4\right] + \frac{e_p^2}{2m_p c^2}\left[\left(\frac{\partial A_n^{(p)\,ret}}{\partial x_k^{(p)\,ret}} - \frac{\partial A_\kappa^{(p)\,ret}}{\partial x_n^{(p)\,ret}}\right) - \left(\frac{\partial A_n^{(p)\,adv}}{\partial x_k^{(p)\,adv}} - \frac{\partial A_\kappa^{(p)\,adv}}{\partial x_n^{(p)\,adv}}\right)\right]$$

$(k = 1,2,3,4),\ (pq) =, (12), (21))$.

Now follow [33] and [34] we obtain

$$\frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{e_p e_q}{m_p c^2}\left[\frac{\xi_\alpha^{(pq)}\left\langle \lambda^{(p)}, \lambda^{(q)}\right\rangle_4 - \lambda_\alpha^{(q)}\left\langle \lambda^{(p)}, \xi^{(pq)}\right\rangle_4}{\left\langle \lambda^{(q)}, \xi^{(pq)}\right\rangle_4^3} \times \left(1 + \left\langle \frac{d\lambda^{(q)}}{ds_q}, \xi^{(pq)}\right\rangle_4\right) + \right.$$
$$\left. + \frac{1}{\left\langle \lambda^{(q)}, \xi^{(pq)}\right\rangle_4^2}\left(\frac{d\lambda_\alpha^{(q)}}{ds_q}\langle \lambda^{(p)}, \xi^{(pq)}\rangle_4 - \xi_\alpha^{(pq)}\left\langle \lambda^{(p)}, \frac{d\lambda^{(q)}}{ds_q}\right\rangle_4\right)\right] +$$
$$+ \frac{e_p^2}{2m_p c^2}\left[\frac{\xi_\alpha^{(p)r}\langle \lambda^{(p)}, \lambda^{(p)r}\rangle_4 - \lambda_\alpha^{(p)r}\langle \xi^{(p)r}, \lambda^{(p)}\rangle_4}{\langle \lambda^{(p)r}, \xi^{(p)r}\rangle_4^3} \times \left(1 + \left\langle \frac{d\lambda^{(p)r}}{ds_r}, \xi^{(p)r}\right\rangle_4\right) + \frac{1}{\langle \lambda^{(p)r}, \xi^{(p)r}\rangle_4^2} \times \right. \qquad (6.p.\alpha)$$

$$\left. \times \left(\frac{d\lambda_\alpha^{(p)r}}{ds_r}\langle \xi^{(p)r}, \lambda^{(p)}\rangle_4 - \xi_\alpha^{(p)r}\left\langle \lambda^{(p)}, \frac{d\lambda^{(p)r}}{ds_r}\right\rangle_4\right)\right] - \frac{e_p^2}{2m_p c^2}\left[\frac{\xi_\alpha^{(p)a}\langle \lambda^{(p)}, \lambda^{(p)a}\rangle_4 - \lambda_\alpha^{(p)a}\langle \xi^{(p)a}, \lambda^{(p)}\rangle_4}{\langle \lambda^{(p)a}, \xi^{(p)a}\rangle_4^3}\right.$$
$$\left. \times \left(1 + \left\langle \frac{d\lambda^{(p)a}}{ds_a}, \xi^{(p)a}\right\rangle_4\right) + \frac{1}{\langle \lambda^{(p)a}, \xi^{(p)a}\rangle_4^2} \times \left(\frac{d\lambda_\alpha^{(p)a}}{ds_a}\langle \xi^{(p)a}, \lambda^{(p)}\rangle_4 - \xi_\alpha^{(p)a}\left\langle \lambda^{(p)}, \frac{d\lambda^{(p)a}}{ds_a}\right\rangle_4\right)\right]$$

$(\alpha = 1,2,3), (p,q) = (2,1), (1,2)$

$$\frac{d\lambda_4^{(p)}}{ds_p}=\frac{e_pe_q}{m_pc^2}\Biggl[\frac{\xi_4^{(pq)}\langle\lambda^{(p)},\lambda^{(q)}\rangle_4-\lambda_4^{(q)}\langle\lambda^{(p)},\xi^{(pq)}\rangle_4}{\left\langle\lambda^{(q)},\xi^{(pq)}\right\rangle_4^3}\times\left(1+\left\langle\frac{d\lambda^{(q)}}{ds_q},\xi^{(pq)}\right\rangle_4\right)+\frac{1}{\langle\lambda^{(q)},\xi^{(pq)}\rangle_4^2}\times$$

$$\times\left(\frac{d\lambda_4^{(q)}}{ds_q}\langle\lambda^{(p)},\xi^{(pq)}\rangle_4-\xi_4^{(pq)}\left\langle\lambda^{(p)},\frac{d\lambda^{(q)}}{ds_q}\right\rangle_4\right)\Biggr]+\frac{e_p^2}{2m_pc^2}\Biggl[\frac{\xi_4^{(p)r}\langle\lambda^{(p)},\lambda^{(p)r}\rangle_4-\lambda_4^{(p)r}\langle\xi^{(p)r},\lambda^{(p)}\rangle_4}{\langle\lambda^{(p)r},\xi^{(p)r}\rangle_4^3}\times$$

$$\times\left(1+\left\langle\xi^{(p)r},\frac{d\lambda^{(p)r}}{ds_{ret}}\right\rangle_4\right)+\frac{1}{\langle\lambda^{(p)r},\xi^{(p)r}\rangle_4^2}\times\left(\langle\xi^{(p)r},\lambda^{(p)}\rangle_4\frac{d\lambda_4^{(p)r}}{ds_{ret}}-\left\langle\lambda^{(p)},\frac{d\lambda^{(p)r}}{ds_{ret}}\right\rangle_4\xi_4^{(p)r}\right)\Biggr]-$$

$$-\frac{e_p^2}{2m_pc^2}\Biggl[\frac{\xi_\alpha^{(p)a}\langle\lambda^{(p)},\lambda^{(p)a}\rangle_4-\lambda_\alpha^{(p)a}\langle\xi^{(p)a},\lambda^{(p)}\rangle_4}{\langle\lambda^{(p)a},\xi^{(p)a}\rangle_4^3}\times\left(1+\left\langle\xi^{(p)a},\frac{d\lambda^{(p)a}}{ds_{adv}}\right\rangle_4\right)+\frac{1}{\langle\lambda^{(p)a},\xi^{(p)a}\rangle_4^2}$$

$$\times\left(\langle\xi^{(p)a},\lambda^{(p)}\rangle_4\frac{d\lambda_4^{(p)a}}{ds_{adv}}-\left\langle\lambda^{(p)},\frac{d\lambda^{(p)a}}{ds_{adv}}\right\rangle_4\xi_4^{(p)}\right)\Biggr]\Biggr]$$

$$(p,q)=(2,1),(1,2)\,. \qquad (6.p.4)$$

2.2. The Fourth Equation Is a Consequence of the First Three Ones

The system (S.$p.\alpha$), ($S.p.4$) obtained in Supplement 1 can be rewritten as $\left(\dot{u}_\alpha(t)\equiv du_\alpha(t)/dt\right)$

$$\dot{u}_\alpha^{(p)}+\frac{u_\alpha^{(p)}}{\Delta_p^2}\langle u^{(p)},\dot{u}^{(p)}\rangle=\frac{\Delta_pe_pe_q}{m_pc^2}\times\Biggl\{\Biggl[\frac{c^2-\langle u^{(p)},u^{(q)}\rangle}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}\xi_\alpha^{(pq)}-\frac{c^2\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}u_\alpha^{(q)}\Biggr]H_{pq}+$$

$$+D_{pq}\frac{\langle\xi^{(pq)},u^{(p)}\rangle-c^2\tau_{pq}}{\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}\dot{u}_\alpha^{(q)}+\frac{e_p^2\Delta_p}{2m_pc^2}\Biggl\{\Biggl[\frac{\langle u^{(p)},u^{(p)r}\rangle-c^2}{\left(\langle u^{(p)r},\xi^{(p)r}\rangle-c^2\tau^{(p)r}\right)^3}\xi_\alpha^{(p)r}$$

$$-\frac{\langle u^{(p)},\xi^{(p)r}\rangle-c^2\tau^{(p)r}}{\left(\langle u^{(p)r},\xi^{(p)r}\rangle-c^2\tau^{(p)r}\right)^3}u_\alpha^{(p)r}\Biggr]H_{(p)r}+D_{(p)r}\frac{\langle\xi^{(p)r},u^{(p)}\rangle-c^2\tau^{(p)r}}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}\dot{u}_\alpha^{(p)r}+$$

$$+\frac{D_{(p)r}\left(\langle\xi^{(p)r},u^{(p)}-c^2\tau^{(p)r}\rangle\right)\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}u_\alpha^{(p)r}-\frac{D_{(p)r}}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}\times$$

$$\times\left(\frac{\Delta_{(p)r}^2\langle u^{(p)},\dot{u}^{(p)r}\rangle}{\Delta_{(p)r}^2}+\frac{\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta_{(p)r}^2}\right)\xi_\alpha^{(p)r}\Biggr\}-\frac{e_p^2\Delta_p}{2m_pc^2}\Biggl\{\Biggl[\frac{\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)}{\left(\langle u^{(p)a},\xi^{(p)a}\rangle-c^2\tau^{(p)a}\right)^3}\xi_\alpha^{(p)a}-$$

$$-\frac{\left(\langle u^{(p)},\xi^{(p)a}\rangle-c^2\tau^{(p)a}\right)}{\left(\langle u^{(p)a},\xi^{(p)a}\rangle-c^2\tau^{(p)a}\right)^3}u_\alpha^{(p)a}\Biggr]H_{(p)a}+\frac{D_{(p)a}\left(\langle\xi^{(p)a},u^{(p)}\rangle-c^2\tau^{(p)a}\right)\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}\dot{u}_\alpha^{(p)a}+$$

$$-\frac{D_{(p)a}}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}\times\left(\frac{\Delta_{(p)a}^2\langle u^{(p)},\dot{u}^{(p)a}\rangle}{\Delta_{(p)a}^2}+\frac{\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta_{(p)a}^2}\right)\xi_\alpha^{(p)a}\Biggr\}\equiv$$

$$\equiv G_\alpha^{(p)}\quad(\alpha=1,2,3);$$

$$\frac{1}{\Delta_p^2}\langle u^{(p)},\dot u^{(p)}\rangle=\frac{e_pe_q\Delta_p}{m_pc^2}\left\{\frac{\langle\xi^{(pq)},u^{(p)}\rangle-\tau_{pq}\langle u^{(p)},u^{(q)}\rangle}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}H_{pq}+\right.$$

$$\left.+D_{pq}\frac{\left\langle u^{(q)},\dot u^{(q)}\right\rangle\left(\langle\xi^{(pq)},u^{(p)}\rangle-\tau_{pq}\langle u^{(p)},u^{(q)}\rangle\right)-\Delta_{pq}^2\langle u^{(p)},\dot u^{(q)}\rangle\tau_{pq}}{\Delta_{pq}^2\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}\right\}+$$

$$+\frac{e_p^2\Delta_p}{2m_pc^2}\left\{\frac{\langle u^{(p)},\xi^{(p)r}\rangle-\langle u^{(p)},u^{(p)r}\rangle\tau^{(p)r}}{\left(c^2\tau^{(p)r}-\langle u^{(p)r},\xi^{(p)r}\rangle\right)^3}H_{(p)r}+\frac{D_{(p)r}\langle u^{(p)r},\dot u^{(p)r}\rangle\left(\langle\xi^{(p)r},u^{(p)}\rangle-\tau^{(p)r}\langle u^{(p)},u^{(p)r}\rangle\right)}{\Delta_{(p)r}^2\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}-\right.$$

$$\left.-\frac{D_{(p)r}\langle u^{(p)},\dot u^{(p)r}\rangle\tau^{(p)r}}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}\right\}-\frac{e_p^2\Delta_p}{2m_pc^2}\left\{\frac{\langle u^{(p)},\xi^{(p)a}\rangle-\langle u^{(p)},u^{(p)a}\rangle\tau^{(p)a}}{\left(c^2\tau^{(p)a}-\langle u^{(p)a},\xi^{(p)a}\rangle\right)^3}H_{(p)a}+\right.$$

$$\left.+\frac{D_{(p)a}\left(\langle\xi^{(p)a},u^{(p)}\rangle-\tau^{(p)a}\langle u^{(p)},u^{(p)a}\rangle\right)\langle u^{(p)a},\dot u^{(p)a}\rangle}{\Delta_{(p)a}^2\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}-\frac{D_{(p)a}\langle u^{(p)},\dot u^{(p)a}\rangle\tau^{(p)a}}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}\right\}. \qquad (7.p.4)$$

Proceeding as in [30], [34] we multiply $(7.p.\alpha)$ by $u_\alpha(t)$, summing up in α and dividing into c^2 we obtain $(7.p.4)$.

In other words the fourth equation is a consequence of the first three ones and the eighth equation is a consequence of the previous three ones. In this way we obtain 6 equations for 6 unknown functions.

The system from Supplement 2 can be written in the following form $(\alpha=1,2,3)$; $(pq)=(12),(21)$:

$$\dot u_\alpha^{(p)}+\frac{u_\alpha^{(p)}}{\Delta_p^2}\left\langle u^{(p)},\dot u^{(p)}\right\rangle=\frac{e_pe_q\Delta_p}{m_pc^2}\left(A_{pq}\xi_\alpha^{(pq)}+B_{pq}u_\alpha^{(q)}+C_{pq}\dot u_\alpha^{(q)}\right)+\frac{e_p^2\Delta_p}{m_pc^2}\frac{A_{(p)r}\xi_\alpha^{(p)r}-A_{(p)a}\xi_\alpha^{(p)a}}{2}$$

$$+\frac{e_p^2\Delta_p}{m_pc^2}\frac{B_{(p)r}u_\alpha^{(p)r}-B_{(p)a}u_\alpha^{(p)a}}{2}+\frac{e_p^2\Delta_p}{m_pc^2}\frac{C_{(p)r}\dot u_\alpha^{(p)r}-C_{(p)a}\dot u_\alpha^{(p)a}}{2}\equiv G_\alpha^{(p)}. \qquad (8.p.\alpha)$$

Denoting by $G_\alpha^{(p)}$ the right-hand sides of $(8.p.\alpha)$ we have to solve the following system with respect to $\dot u_1(t),\dot u_2(t),\dot u_3(t)$:

$$\left(1+\frac{\left(u_1^{(p)}(t)\right)^2}{\Delta_p^2}\right)\dot u_1^{(p)}(t)+\frac{u_1^{(p)}(t)u_2^{(p)}(t)}{\Delta_p^2}\dot u_2^{(p)}(t)+\frac{u_1^{(p)}(t)u_3^{(p)}(t)}{\Delta_p^2}\dot u_3^{(p)}(t)=G_1^{(p)}$$

$$\frac{u_1^{(p)}(t)u_2^{(p)}(t)}{\Delta_p^2}\dot u_1^{(p)}(t)+\left(1+\frac{\left(u_2^{(p)}(t)\right)^2}{\Delta_p^2}\right)\dot u_2^{(p)}(t)+\frac{u_2^{(p)}(t)u_3^{(p)}(t)}{\Delta_p^2}\dot u_3^{(p)}(t)=G_2^{(p)}$$

$$\frac{u_1^{(p)}(t)u_3^{(p)}(t)}{\Delta_p^2}\dot u_1^{(p)}(t)+\frac{u_2^{(p)}(t)u_3(t)}{\Delta_p^2}\dot u_2^{(p)}(t)+\left(1+\frac{\left(u_3^{(p)}(t)\right)^2}{\Delta_p^2}\right)\dot u_3^{(p)}(t)=G_3^{(p)}.$$

We make the following:

Assumption (C): All velocities satisfy the inequalities

$$\left|u_\alpha^{(p)}(t)\right|\le\sqrt{\left\langle u^{(p)},u^{(p)}\right\rangle}\le\overline{c}<c$$

and then $c^2-\left\langle u^{(p)},u^{(p)}\right\rangle\ge c^2-\overline{c}^2>0$. Therefore, the determinant of the above system is $\delta_p=c^2/\Delta_p^2>0$ and consequently we reach the system:

$$\dot{u}_1^{(p)}(t)=\frac{c^2-(u_1^{(p)}(t))^2}{c^2}G_1^{(p)}-\frac{u_1^{(p)}(t)u_2^{(p)}(t)}{c^2}G_2^{(p)}-\frac{u_1^{(p)}(t)u_3^{(p)}(t)}{c^2}G_3^{(p)}$$

$$\dot{u}_2^{(p)}(t)=\frac{-u_1^{(p)}(t)u_2^{(p)}(t)}{c^2}G_1^{(p)}+\left(\frac{c^2-(u_2^{(p)}(t))^2}{c^2}\right)G_2^{(p)}-\frac{u_2^{(p)}(t)u_3^{(p)}(t)}{c^2}G_3^{(p)}$$

$$\dot{u}_3^{(p)}(t)=-\frac{u_1^{(p)}(t)u_3^{(p)}(t)}{c^2}G_1^{(p)}-\frac{u_2^{(p)}(t)u_3^{(p)}(t)}{c^2}G_2^{(p)}+\frac{c^2-(u_3^{(p)}(t))^2}{c^2}G_3^{(p)}$$

$(pq)=(12),(21)$.

The right-hand side of $(8.p.\alpha)$ $G_\alpha^{(p)}$ is a sum of two terms

$G_\alpha^{(p)}=G_\alpha^{(pq)}+G_\alpha^{(p)rad}$:

Lorentz term –

$$G_\alpha^{(pq)}=\frac{e_pe_q\Delta_p}{m_pc^2}\left(A_{pq}\xi_\alpha^{(pq)}+B_{pq}u_\alpha^{(q)}+C_{pq}\dot{u}_\alpha^{(q)}\right)$$

and radiation term –

$$G_\alpha^{(p)rad}=\frac{e_p^2\Delta_p}{m_pc^2}\left(\frac{A_{(p)r}\xi_\alpha^{(p)r}-A_{(p)a}\xi_\alpha^{(p)a}}{2}+\frac{B_{(p)r}u_\alpha^{(p)r}-B_{(p)a}u_\alpha^{(p)a}}{2}+\frac{C_{(p)r}\dot{u}_\alpha^{(p)r}-C_{(p)a}\dot{u}_\alpha^{(p)a}}{2}\right).$$

3. Conclusions

Here we introduce a unified approach to derive two-body problem with radiation terms. The system obtained is of neutral type containing both retarded and advanced arguments. The unknown functions are velocities of the moving particles. The deviating arguments depend on the unknown trajectories. We would like to point that we follow physical reasoning due to Dirac [2] which leads to assumption

$$\tau^{(p)r}(t)=\tau^{(p)a}(t)=\tau=const.$$

But with accordance of special relativity theory we have to consider radiation time in the form $\tau=\tau_0\sqrt{1-\beta^2}$ where $\beta=\overline{c}/c<1$. In view of $\tau_0=r_e/c\approx 9,1.10^{-24}$ sec and $\tau\to 0$ as $\beta\to 1$ then the parameter τ should be consider as an infinitely small parameter. Extending the technique from [30] and [34] we replace the usually accepted second order Dirac's system of ordinary differential equations by a first order system of neutral equations with both retarded and advanced arguments.

We would like to comment our basic assumption **(C)**: $\sqrt{\left\langle u^{(p)},u^{(p)}\right\rangle}\le\overline{c}<c$ (cf. also [30]). In the Newton theory − the speed of propagation of the interaction is ∞ , but anybody cannot reach this speed ∞ . Here the role of ∞ is played by *c*.

Supplement 1

In 2.1 we have obtained system (6.p.α), (6.p.4). In order to solve it we have to transform it in a suitable form. First of all we have to find a relation between the relativistic and absolute time. Indeed, following [19], [33], and [34] we assume that

$$t-t_{pq}=\tau_{pq}(t),\ t-\check{t}_p=\tau^{(p)r}(t),\hat{t}_p-t=\tau^{(p)a}(t). \quad \text{(AR)}$$

Since $A^{(p)ret}$ and $A^{(p)adv}$ lie on the trajectory L_p we put

$$x^{(p)r}=\left(x_1^{(p)}(\check{t}_p),x_2^{(p)}(\check{t}_p),x_3^{(p)}(\check{t}_p),ic\check{t}_p\right)=\left(x_1^{(p)}(t-\tau^{(p)r}(t)),x_2^{(p)}(t-\tau^{(p)r}(t)),x_3^{(p)}(t-\tau^{(p)r}(t)),\ ic\check{t}_p\right),$$

$$x^{(p)a}=\left(x_1^{(p)}(\hat{t}_p),x_1^{(p)}(\hat{t}_p),x_1^{(p)}(\hat{t}_p),ic\hat{t}_p\right)=\left(x_1(t+\tau^{(p)a}(t)),x_2(t+\tau^{(p)a}(t)),x_3(t+\tau^{(p)a}(t)),\ ic\hat{t}_p\right),$$

$$\lambda^{(p)r}=\left(\frac{u_1^{(p)}(t-\tau^{(p)r})}{\Delta_{(p)r}},\frac{u_2^{(p)}(t-\tau^{(p)r})}{\Delta_{(p)r}},\frac{u_3^{(p)}(t-\tau^{(p)r})}{\Delta_{(p)r}},\frac{ic}{\Delta_{(p)r}}\right)$$

where

$$\Delta_{(p)r}=\sqrt{c^2-\left\langle u^{(p)r},u^{(p)r}\right\rangle}=\sqrt{c^2-\left\langle u^{(p)}(t-\tau^{(p)r}),u^{(p)}(t-\tau^{(p)r})\right\rangle},$$

$$\lambda^{(p)a}=\left(\frac{u_1^{(p)}(t+\tau^{(p)a})}{\Delta_{(p)a}},\frac{u_2^{(p)}(t+\tau^{(p)a})}{\Delta_{(p)a}},\frac{u_3^{(p)}(t+\tau^{(p)a})}{\Delta_{(p)a}},\frac{ic}{\Delta_{(p)a}}\right)$$

where

$$\Delta_{(p)a}=\sqrt{c^2-\left\langle u^{(p)a},u^{(p)a}\right\rangle}=\sqrt{c^2-\left\langle u^{(p)}(t+\tau^{(p)a}),u^{(p)}(t+\tau^{(p)a})\right\rangle}.$$

Since the isotropic vectors

$$\xi^{(pq)}=\left(x_\alpha^{(p)}(t)-x_\alpha^{(q)}(t-\tau_{pq}),ic\tau_{pq}\right),$$

$$\xi^{(p)ret}=\left(x_\alpha^{(p)}(t)-x_\alpha^{(p)}\left(\check{t}_p\right),\ ic\left(t-\check{t}_p\right)\right),$$

$$\xi^{(p)adv}=\left(x_\alpha^{(p)}\left(t+\hat{t}_p\right)-x_\alpha^{(p)}(t),\ ic\left(t+\hat{t}_p\right)\right),$$

$$(pq)=(21),(12),\alpha=1,2,3$$

have lengths 0 we obtain

$$t-t_{pq}=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(q)}(t_{pq})\right]^2},$$

$$t-\check{t}_p=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(p)}(\check{t}_p)\right]^2},$$

$$\hat{t}_p-t=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(\hat{t}_p)-x_\gamma^{(p)}(t)\right]^2}.$$

We put

$$\tau^{(pq)}(t)=t-t_{pq},\ \tau^{(p)r}(t)=t-\hat{t}_p,\tau^{(p)a}(t)=\hat{t}_p-t$$

and then

$$\tau^{(pq)}(t),\ \tau^{(p)r}(t),\tau^{(p)a}(t),$$

can be obtained as solutions of the equations

$$\left\langle\xi^{(pq)},\xi^{(pq)}\right\rangle_4=0,\ \left\langle\xi^{(p)r},\xi^{(p)r}\right\rangle_4=0,\ \left\langle\xi^{(p)a},\xi^{(p)a}\right\rangle_4=0$$

or

$$\tau_{pq}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(q)}(t-\tau_{pq}(t)\right]^2}=\frac{1}{c}\sqrt{\left\langle \xi^{(pq)},\xi^{(pq)}\right\rangle},$$

$$\tau^{(p)r}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(p)}(t-\tau^{(p)r}(t)\right]^2}=\frac{1}{c}\sqrt{\left\langle \xi^{(p)r},\xi^{(p)r}\right\rangle},$$

$$\tau^{(p)a}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t+\tau^{(p)a}(t))-x_\gamma^{(p)}(t)\right]^2}=\frac{1}{c}\sqrt{\left\langle \xi^{(p)a},\xi^{(p)a}\right\rangle}.$$

$$\lambda^{(q)}=\left(\frac{u_1^{(q)}(t-\tau_{pq})}{\Delta_{pq}},\frac{u_2^{(q)}(t-\tau_{pq})}{\Delta_{pq}},\frac{u_3^{(q)}(t-\tau_{pq})}{\Delta_{pq}},\frac{ic}{\Delta_{pq}}\right),$$

$$\frac{d\lambda_\alpha^{(p)}}{ds_p}=\frac{1}{\Delta_p}\frac{d}{dt}\left(\frac{u_\alpha^{(p)}}{\Delta_p}\right)=\frac{1}{\Delta_p^2}\dot{u}_\alpha^{(p)}+\frac{u_\alpha^{(p)}}{\Delta_p^4}\left\langle u^{(p)},\dot{u}^{(p)}\right\rangle,$$

$$\frac{d\lambda_4^{(p)}}{ds_p}=\frac{ic}{\Delta_p}\frac{d}{dt}\left(\frac{1}{\Delta_p}\right)=\frac{ic}{\Delta_p^4}\left\langle u^{(p)},\dot{u}^{(p)}\right\rangle,$$

$$\frac{d}{ds_p}=\frac{1}{\Delta_p}\frac{d}{dt},\ \frac{d}{ds_q}=\frac{1}{\Delta_{pq}}\frac{d}{dt_{pq}}=\frac{1}{\Delta_{pq}}\frac{dt}{dt_{pq}}\frac{d}{dt},\ \frac{d}{dt_{pq}}\equiv D_{pq}.$$

We have to find relations between the derivatives at past, present and future instants. Indeed, extending reasoning from [19], differentiating the relations

$$t-t_{pq}=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(q)}(t_{pq})\right]^2}$$

and solving with respect to $\dfrac{dt}{dt_{pq}}$ we obtain

$$\frac{dt}{dt_{pq}}=\frac{c\sqrt{\langle\xi^{(pq)},\xi^{(pq)}\rangle}-\langle\xi^{(pq)},u^{(q)}\rangle}{c\sqrt{\langle\xi^{(pq)},\xi^{(pq)}\rangle}-\langle\xi^{(pq)},u^{(p)}\rangle}=\frac{c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle}{c^2\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle}\equiv D_{pq}.$$

Therefore

$$\frac{d}{ds_q}=\frac{1}{\Delta_{pq}}\frac{d}{dt_{pq}}=\frac{1}{\Delta_{pq}}\frac{dt}{dt_{pq}}\frac{d}{dt}=\frac{D_{pq}}{\Delta_{pq}}\frac{d}{dt},$$

$$\Delta_{pq}=\sqrt{c^2-\left\langle u^{(q)},u^{(q)}\right\rangle}$$

$$\frac{d\lambda_\alpha^{(q)}}{ds_q}=D_{pq}\left[\frac{\dot{u}_\alpha^{(q)}}{\Delta_{pq}^2}+\frac{u_\alpha^{(q)}}{\Delta_{pq}^4}\langle u^{(q)},\dot{u}^{(q)}\rangle\right],$$

$$\frac{d\lambda_4^{(q)}}{ds_q}=\frac{icD_{pq}}{\Delta_{pq}^4}\langle u^{(q)},\dot{u}^{(q)}\rangle,$$

$$\langle\lambda^{(p)},\lambda^{(q)}\rangle_4=\frac{\langle u^{(p)},u^{(q)}\rangle-c^2}{\Delta_p\Delta_{pq}},$$

$$\langle\lambda^{(p)},\xi^{(pq)}\rangle_4=\frac{\langle u^{(p)},\xi^{(pq)}\rangle-c^2\tau_{pq}}{\Delta_p},$$

$$\langle\lambda^{(q)},\xi^{(pq)}\rangle_4=\frac{\langle u^{(q)},\xi^{(pq)}\rangle-c^2\tau_{pq}}{\Delta_{pq}},$$

$$\langle\xi^{(pq)},\frac{d\lambda^{(q)}}{ds_q}\rangle_4=D_{pq}\left[\frac{\langle\xi^{(pq)},\dot{u}^{(q)}\rangle}{\Delta_{pq}^2}+\frac{\left\langle\xi^{(pq)},u^{(q)}\right\rangle-c^2\tau_{pq}}{\Delta_{pq}^4}\langle u^{(q)},\dot{u}^{(q)}\rangle\right],$$

$$\langle\lambda^{(p)},\frac{d\lambda^{(q)}}{ds_q}\rangle_4=\frac{D_{pq}}{\Delta_p}\left[\frac{\langle u^{(p)},\dot{u}^{(q)}\rangle}{\Delta_{pq}^2}+\frac{\langle u^{(p)},u^{(q)}\rangle-c^2\tau_{pq}}{\Delta_{pq}^4}\langle u^{(q)},\dot{u}^{(q)}\rangle\right].$$

In a similar way we differentiate

$$t-\overset{\vee}{t}_p=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(p)}(\overset{\vee}{t}_p)\right]^2}$$

with respect to $\overset{\vee}{t}_p$ (considering $t=t(\overset{\vee}{t}_p)$) and obtain

$$\frac{dt}{d\overset{\vee}{t}_p}-1=\frac{\sum_{\gamma=1}^{3}\left[x_\gamma^{(p)}(t)-x_\gamma^{(p)}(\overset{\vee}{t}_p)\right]\left[u_\gamma^{(p)}(t)\frac{dt}{d\overset{\vee}{t}_p}-u_\gamma^{(p)}(\overset{\vee}{t}_p)\right]}{c\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t)-x_\gamma^{(p)}(\overset{\vee}{t}_p)\right]^2}}.$$

Hence

$$D_{(p)r}\equiv\frac{dt}{d\overset{\vee}{t}_p}=\frac{c\sqrt{\langle\xi^{(p)r},\xi^{(p)r}\rangle}-\langle\xi^{(p)r},u^{(p)r}\rangle}{c\sqrt{\langle\xi^{(p)r},\xi^{(p)r}\rangle}-\langle\xi^{(p)r},u^{(p)}\rangle}=\frac{c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)}(t-\tau^{(p)r})\rangle}{c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)}(t)\rangle},$$

and similarly

$$D_{(p)a}\equiv\frac{dt}{d\hat{t}_p}=\frac{c\sqrt{\langle\xi^{(p)a},\xi^{(p)a}\rangle}-\langle\xi^{(p)a},u^{(p)a}\rangle}{c\sqrt{\langle\xi^{(p)a},\xi^{(p)a}\rangle}-\langle\xi^{(p)a},u^{(p)}\rangle}=\frac{c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)}(t+\tau^{(p)a})\rangle}{c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)}(t)\rangle}.$$

Further on we have

$$\left\langle\lambda^{(p)},\lambda^{(p)r}\right\rangle_4=\frac{\left\langle u^{(p)}(t),u^{(p)}(t-\tau^{(p)r})\right\rangle-c^2}{\Delta_p\Delta_{(p)r}};\quad\left\langle\lambda^{(p)},\lambda^{(p)a}\right\rangle_4=\frac{\left\langle u^{(p)}(t),u^{(p)}(t+\tau^{(p)a})\right\rangle-c^2}{\Delta_p\Delta_{(p)a}};$$

$$\left\langle\xi^{(p)r},\lambda^{(p)}\right\rangle_4=\frac{\left\langle\xi^{(p)r},u^{(p)}(t)\right\rangle-c^2\tau^{(p)r}}{\Delta_p};\ \left\langle\xi^{(p)a},\lambda^{(p)}\right\rangle_4=\frac{\left\langle\xi^{(p)a},u^{(p)}(t)\right\rangle-c^2\tau^{(p)a}}{\Delta_p};$$

$$\left\langle\xi^{(p)r},\lambda^{(p)r}\right\rangle_4=\frac{\left\langle\xi^{(p)r},u^{(p)r}\right\rangle-c^2\tau^{(p)r}}{\Delta_{(p)r}};\ \left\langle\xi^{(p)a},\lambda^{(p)a}\right\rangle_4=\frac{\left\langle\xi^{(p)a},u^{(p)a}\right\rangle-c^2\tau^{(p)a}}{\Delta_{(p)a}};$$

$$\frac{d}{ds_{ret}}=\frac{1}{\Delta_{(p)r}}\frac{d}{d\overset{\vee}{t}_p}=\frac{1}{\Delta_{(p)r}}\frac{dt}{d\overset{\vee}{t}_p}\frac{d}{dt}=\frac{1}{\Delta_{(p)r}}D_{(p)r}\frac{d}{dt};\ \frac{d}{ds_{adv}}=\frac{1}{\Delta_{(p)a}}\frac{d}{d\hat{t}_p}=\frac{1}{\Delta_{(p)a}}\frac{dt}{d\hat{t}_p}\frac{d}{dt}=\frac{1}{\Delta_{(p)a}}D_{(p)a}\frac{d}{dt};$$

$$\frac{d\lambda_\alpha^{(p)r}}{ds_{ret}}=D_{(p)r}\times\left[\frac{c\dot{u}_\alpha^{(p)}(t-\tau^{(p)r})}{\Delta_{(p)r}^2}+\frac{u_\alpha^{(p)}(t-\tau^{(p)r})\left\langle u^{(p)}(t-\tau^{(p)r}),\dot{u}^{(p)}(t-\tau^{(p)r})\right\rangle}{\Delta_{(p)r}^4}\right];$$

$$\frac{d\lambda_4^{(p)r}}{ds_{ret}}=\frac{icD_{(p)r}\left\langle u^{(p)}(t-\tau^{(p)r}),\dot{u}^{(p)}(t-\tau^{(p)r})\right\rangle}{\Delta_{(p)r}^4};$$

$$\frac{d\lambda_\alpha^{(p)a}}{ds_{adv}}=D_{(p)a}\times\left[\frac{\dot{u}_\alpha^{(p)}(t+\tau^{(p)a})}{\Delta_{(p)a}^2}+\frac{cu_\alpha^{(p)}(t+\tau^{(p)a})\left\langle u^{(p)}(t+\tau^{(p)a}),\dot{u}^{(p)}(t+\tau^{(p)a})\right\rangle}{\Delta_{(p)a}^4}\right];$$

$$\frac{d\lambda_4^{(p)a}}{ds_{adv}}=\frac{icD_{(p)a}\left\langle u^{(p)}(t+\tau^{(p)a}),\dot{u}^{(p)}(t+\tau^{(p)a})\right\rangle}{\Delta_{(p)a}^4};$$

$$\left\langle \xi^{(p)r},\frac{d\lambda_\alpha^{(p)r}}{ds_{ret}}\right\rangle_4=D_{(p)r}\left[\frac{\left\langle \xi^{(p)r},\dot{u}^{(p)}(t-\tau^{(p)r})\right\rangle}{\Delta_{(p)r}^2}+\frac{\left(\left\langle \xi^{(p)r},u(t-\tau^{(p)r})\right\rangle-c^2\tau^{(p)r}\right)\left\langle u(t-\tau^{(p)r}),\dot{u}(t-\tau^{(p)r})\right\rangle}{\Delta_{(p)r}^4}\right];$$

$$\langle\lambda^{(p)},\frac{d\lambda^{(p)r}}{ds_{ret}}\rangle_4=\frac{D_{(p)r}}{\Delta_p}\left[\frac{\langle u^{(p)}(t),\dot{u}^{(p)}(t-\tau^{(p)r})\rangle}{\Delta_{(p)r}^2}+\frac{\left(\langle u^{(p)}(t),u^{(p)}(t-\tau^{(p)r})\rangle-c^2\tau^{(p)r}\right)\langle u^{(p)}(t-\tau^{(p)r}),\dot{u}^{(p)}(t-\tau^{(p)r})\rangle}{\Delta_{(p)r}^4}\right];$$

$$\langle\xi^{(p)a},\frac{d\lambda^{(p)a}}{ds_{adv}}\rangle_4=D_{(p)a}\left[\frac{\langle \xi^{(p)a},\dot{u}^{(p)}(t+\tau^{(p)a})\rangle}{\Delta_{(p)a}^2}+\frac{\left(\langle \xi^{(p)a},u^{(p)}(t+\tau^{(p)a})\rangle-c^2\tau^{(p)a}\right)\langle u^{(p)}(t+\tau^{(p)a}),\dot{u}^{(p)}(t+\tau^{(p)a})\rangle}{\Delta_{(p)a}^4}\right];$$

$$\langle\lambda^{(p)},\frac{d\lambda^{(p)a}}{ds_{adv}}\rangle_4=\frac{D_{(p)a}}{\Delta_p}\left[\frac{\langle u^{(p)}(t),\dot{u}^{(p)}(t+\tau^{(p)a})\rangle}{\Delta_{(p)a}^2}+\frac{\left(\langle u^{(p)}(t),u^{(p)}(t+\tau^{(p)a})\rangle-c^2\tau^{(p)a}\right)\langle u^{(p)}(t+\tau^{(p)a}),\dot{u}^{(p)}(t+\tau^{(p)a})\rangle}{\Delta_{(p)a}^4}\right].$$

Substituting the expressions obtained we reach the system

$$\frac{1}{\Delta_p^2}\dot{u}_\alpha^{(p)}+\frac{u_\alpha^{(p)}}{\Delta_p^4}\left\langle u^{(p)},\dot{u}^{(p)}\right\rangle=\frac{e_pe_q}{m_pc^2}\left\{\frac{\frac{\left(c^2-\left\langle u^{(p)},u^{(q)}\right\rangle\right)}{\Delta_p\Delta_{pq}}\xi_\alpha^{(pq)}-\frac{u_\alpha^{(q)}}{\Delta_{pq}}\frac{\left(c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(p)}\right\rangle\right)}{\Delta_p}}{\left(\frac{c^2\tau_{pq}-\left\langle \xi^{(pq)},u^{(q)}\right\rangle}{\Delta_{pq}}\right)^3}\times\right.$$

$$\times\left[1+D_{pq}\left(\frac{\left\langle \xi^{(pq)},\dot{u}^{(q)}\right\rangle}{\Delta_{pq}^2}+\frac{\left(\left\langle \xi^{(pq)},u^{(q)}\right\rangle-c^2\tau_{pq}\right)\left\langle u^{(q)},\dot{u}^{(q)}\right\rangle}{\Delta_{pq}^4}\right)\right]+$$

$$+\Delta_{pq}^2\left[\frac{\frac{\left(\left\langle \xi^{(pq)},u^{(p)}\right\rangle-c^2\tau_{pq}\right)}{\Delta_p}D_{pq}\left(\frac{\dot{u}_\alpha^{(q)}}{\Delta_{pq}^2}+\frac{u_\alpha^{(q)}\left\langle u^{(q)},\dot{u}^{(q)}\right\rangle}{\Delta_{pq}^4}\right)}{\left(\left\langle \xi^{(pq)},u^{(q)}\right\rangle-c^2\tau_{pq}\right)^2}\right.$$

$$\left.\left.-\frac{\xi_\alpha^{(pq)}\frac{D_{pq}}{\Delta_p}\left(\frac{\langle u^{(p)},\dot{u}^{(q)}\rangle}{\Delta_{pq}^2}+\frac{\left(\langle u^{(p)},u^{(q)}\rangle-c^2\right)\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta_{pq}^4}\right)}{\left(\langle \xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}\right]\right\}$$

$$
+\frac{e_p^2}{2m_p c^2}\left\{\frac{\xi_\alpha^{(p)r}\dfrac{\langle u^{(p)},u^{(p)r}\rangle-c^2}{\Delta_p\Delta_{(p)r}}-u_\alpha^{(p)r}\dfrac{\langle u^{(p)},\xi^{(p)r}\rangle-c^2\tau^{(p)r}}{\Delta_p\Delta_{(p)r}}}{\left(\dfrac{\langle u^{(p)r},\xi^{(p)r}\rangle-c^2\tau^{(p)r}}{\Delta_{(p)r}}\right)^3}\right.
$$

$$
\times\left[1+D_{(p)r}\left(\frac{\langle\xi^{(p)r},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^2}+\frac{\langle\xi^{(p)r},u^{(p)r}\rangle-c^2\tau^{(p)r}}{\Delta_{(p)r}^4}\Big\langle\langle u^{(p)r},\dot u^{(p)r}\rangle\Big\rangle\right)\right]+
$$

$$
+\Delta_{(p)r}^2\left[\frac{\dfrac{\left(\langle\xi^{(p)r},u^{(p)}\rangle-c^2\tau^{(p)r}\right)}{\Delta_p}D_{(p)r}\left(\dfrac{\dot u_\alpha^{(p)r}}{\Delta_{(p)r}^2}+\dfrac{u_\alpha^{(p)r}\langle u^{(p)r},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^4}\right)}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}-\right.
$$

$$
\left.\left.\left.-\frac{\xi_\alpha^{(p)r}\dfrac{D_{(p)r}}{\Delta_p}\left(\dfrac{\langle u^{(p)},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^2}+\dfrac{\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)\langle u^{(p)},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^4}\right)}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}\right]\right]\right\}-
$$

$$
-\frac{e_p^2}{2m_p c^2}\left\{\frac{\xi_\alpha^{(p)a}\dfrac{\langle u^{(p)},u^{(p)a}\rangle-c^2}{\Delta_p\Delta_{(p)a}}-\dfrac{u_\alpha^{(p)a}}{\Delta_{(p)a}}\dfrac{\langle u^{(p)},\xi^{(p)a}\rangle-c^2\tau^{(p)a}}{\Delta_p}}{\left(\dfrac{\langle u^{(p)a},\xi^{(p)a}\rangle-c^2\tau^{(p)a}}{\Delta_{(p)a}}\right)^3}\right.\times
$$

$$
\times\left[1+D_{(p)a}\left(\frac{\langle\xi^{(p)a},\dot u^{(p)a}\rangle}{\Delta_{(p)a}^2}+\frac{\langle\xi^{(p)a},u^{(p)a}\rangle-c^2\tau^{(p)a}}{\Delta_{(p)a}^4}\langle u^{(p)a},\dot u^{(p)a}\rangle\right)\right]+
$$

$$
+\Delta_{(p)a}^2\left[\frac{\dfrac{\langle\xi^{(p)a},u^{(p)}\rangle-c^2\tau^{(p)a}}{\Delta_p}D_{(p)a}\left(\dfrac{\dot u_\alpha^{(p)a}}{\Delta_{(p)a}^2}+\dfrac{u_\alpha^{(p)a}\langle u^{(p)a},\dot u^{(p)a}\rangle}{\Delta_{(p)a}^4}\right)}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}-\right.
$$

$$
\left.-\frac{\xi_\alpha^{(p)a}\dfrac{D_{(p)a}}{\Delta_p}\left(\dfrac{\langle u^{(p)},\dot u^{(p)a}\rangle}{\Delta_{(p)a}^2}+\dfrac{\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)\langle u^{(p)},\dot u^{(p)a}\rangle}{\Delta_{(p)a}^4}\right)}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}\right]
$$

$$
(\alpha=1,2,3); \qquad (S.p.\alpha)
$$

$$\frac{ic}{\Delta_p^4}\langle u^{(p)},\dot u^{(p)}\rangle = \frac{e_p e_q}{m_p c^2}\left\{ \frac{\frac{\left(c^2-\langle u^{(p)},u^{(q)}\rangle\right)}{\Delta_p\Delta_{pq}} ic\tau_{pq} - \frac{ic}{\Delta_{pq}}\frac{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle\right)}{\Delta_p}}{\left(\frac{c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle}{\Delta_{pq}}\right)^3}\times\right.$$

$$\times\left[1+D_{pq}\left(\frac{\langle\xi^{(pq)},\dot u^{(q)}\rangle}{\Delta_{pq}^2}+\frac{\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)\langle u^{(q)},\dot u^{(q)}\rangle}{\Delta_{pq}^4}\right)\right]+$$

$$+\Delta_{pq}^2\left[\frac{\frac{\left(\langle\xi^{(pq)},u^{(p)}\rangle-c^2\tau_{pq}\right)}{\Delta_p}\frac{icD_{pq}}{\Delta_{pq}^4}\langle u^{(q)},\dot u^{(q)}\rangle}{\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}-\right.$$

$$\left.\left.\left.-\frac{ic\tau_{pq}\frac{D_{pq}}{\Delta_p}\left(\frac{\langle u^{(p)},\dot u^{(q)}\rangle}{\Delta_{pq}^2}+\frac{\left(\langle u^{(p)},u^{(q)}\rangle-c^2\right)\langle u^{(q)},\dot u^{(q)}\rangle}{\Delta_{pq}^4}\right)}{\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}\right]\right]\right\}+$$

$$+\frac{e_p^2}{2m_pc^2}\times\left\{\frac{ic\tau^{(p)r}(t)\frac{\langle u^{(p)},u^{(p)r}\rangle-c^2}{\Delta_p\Delta_{(p)r}}-\frac{ic}{\Delta_{(p)r}}\frac{\langle u^{(p)},\xi^{(p)r}\rangle-c^2\tau^{(p)r}}{\Delta_p}}{\left(\frac{\langle u^{(p)r},\xi^{(p)r}\rangle-c^2\tau^{(p)r}}{\Delta_{(p)r}}\right)^3}\times\right.$$

$$\times\left[1+D_{(p)r}\left(\frac{\langle\xi^{(p)r},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^2}+\frac{\langle\xi^{(p)r},u^{(p)r}\rangle-c^2\tau^{(p)r}}{\Delta_{(p)r}^4}\langle u^{(p)r},\dot u^{(p)r}\rangle\right)\right]+$$

$$+\Delta_{(p)r}^2\left[\frac{\left[\frac{\left(\langle\xi^{(p)r},u^{(p)}\rangle-c^2\tau^{(p)r}\right)}{\Delta_p}\frac{icD_{(p)r}\langle u^{(p)r},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^4}\right.}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}-ic\tau^{(p)r}(t)\frac{D_{(p)r}}{\Delta_p}\times\right.$$

$$\left.\left.\left.\times\frac{\left(\frac{\langle u^{(p)},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^2}+\frac{\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)\langle u^{(p)},\dot u^{(p)r}\rangle}{\Delta_{(p)r}^4}\right)}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}\right]\right]\right\}-\frac{e_p^2}{2m_pc^2}\times$$

$$
\times\left\{\frac{ic\tau^{(p)a}(t)\dfrac{\langle u^{(p)},u^{(p)a}\rangle-c^2}{\Delta_p\Delta_{(p)a}}-\dfrac{ic}{\Delta_{(p)a}}\dfrac{\langle u^{(p)},\xi^{(p)a}\rangle-c^2\tau^{(p)a}}{\Delta_p}}{\left(\dfrac{\langle u^{(p)a},\xi^{(p)a}\rangle-c^2\tau^{(p)a}}{\Delta_{(p)a}}\right)^3}\times\right.
$$

$$
\times\left[1+D_{(p)a}\left(\frac{\langle\xi^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)a}}+\frac{\langle\xi^{(p)a},u^{(p)a}\rangle-c^2\tau^{(p)a}}{\Delta^4_{(p)a}}\langle u^{(p)a},\dot{u}^{(p)a}\rangle\right)\right]+
$$

$$
+\Delta^2_{(p)a}\left[\frac{\left[\dfrac{\left(\langle\xi^{(p)a},u^{(p)}\rangle-c^2\tau^{(p)a}\right)}{\Delta_p}\dfrac{icD_{(p)a}\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^4_{(p)a}}\right.}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}-\right.
$$

$$
\left.\left.\left.-\frac{ic\tau^{(p)a}(t)\dfrac{D_{(p)a}}{\Delta_p}\left(\dfrac{\langle u^{(p)},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)a}}+\dfrac{\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)\langle u^{(p)},\dot{u}^{(p)a}\rangle}{\Delta^4_{(p)a}}\right)}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}\right\rangle\right]\right]\right\}. \qquad (S.p.4)
$$

The last equation should be divided into ic .

Supplement 2

Introduce denotations

$$
A_{pq}=\frac{H_{pq}\left(c^2-\langle u^{(p)},u^{(q)}\rangle\right)}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}-\frac{\Delta^2_{pq}\langle u^{(p)},\dot{u}^{(q)}\rangle+\left(\langle u^{(p)},u^{(q)}\rangle-c^2\right)\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle\right)},
$$

$$
B_{pq}=\frac{\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)}-\frac{H_{pq}\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle\right)}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3},
$$

$$
C_{pq}=-\frac{1}{c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle},
$$

$$
A_{(p)r}=\frac{H_{(p)r}\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)}{\left(\langle u^{(p)r},\xi^{(p)r}\rangle-c^2\tau^{(p)r}\right)^3}-\frac{\Delta^2_{(p)r}\langle u^{(p)},\dot{u}^{(p)r}\rangle+\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta^2_{(p)r}\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)}\rangle\right)},
$$

$$
B_{(p)r}=\frac{\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta^2_{(p)r}\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)}-\frac{H_{(p)r}\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)}\rangle\right)}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^3},
$$

$$
C_{(p)r}=-\frac{1}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)},
$$

$$
A_{(p)a}=\frac{H_{(p)a}\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)}{\left(\langle u^{(p)a},\xi^{(p)a}\rangle-c^2\tau^{(p)a}\right)^3}-\frac{\Delta^2_{(p)a}\langle u^{(p)},\dot{u}^{(p)a}\rangle+\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)a}\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)}\rangle\right)},
$$

$$B_{(p)a}=\frac{\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)a}\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)}-\frac{H_{(p)a}\left(c^2\tau^{(p)a}-\langle u^{(p)},\xi^{(p)a}\rangle\right)}{\left(c^2\tau^{(p)a}-\langle u^{(p)a},\xi^{(p)a}\rangle\right)^3},$$

$$C_{(p)a}=-\frac{1}{c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle},$$

$$H_{pq}=\Delta^2_{pq}+D_{pq}\left(\langle\xi^{(pq)},\dot{u}^{(q)}\rangle+\frac{\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}}\right),$$

$$H_{(p)r}=\Delta^2_{(p)r}+D_{(p)r}\left(\langle\xi^{(p)r},\dot{u}^{(p)r}\rangle+\frac{\left(\langle\xi^{(p)r},u^{(p)r}\rangle-c^2\tau^{(p)r}\right)\left\langle u^{(p)r},\dot{u}^{(p)r}\right\rangle}{\Delta^2_{(p)r}}\right)$$

$$H_{(p)ar}=\Delta^2_{(p)a}+D_{(p)a}\left(\langle\xi^{(p)a},\dot{u}^{(p)a}\rangle+\frac{\left(\langle\xi^{(p)a},u^{(p)a}\rangle-c^2\tau^{(p)a}\right)\left\langle u^{(p)a},\dot{u}^{(p)a}\right\rangle}{\Delta^2_{(p)a}}\right)$$

Regrouping the terms in the right-hand sides of $(7.p.\alpha)$ we obtain the system in a vector form:

$$\dot{u}^{(p)}_\alpha+\frac{u^{(p)}_\alpha}{\Delta^2_p}\langle u^{(p)},\dot{u}^{(p)}\rangle=\frac{e_pe_q\Delta_p}{m_pc^2}\left\{\left[\frac{H_{pq}\left(c^2-\langle u^{(p)},u^{(q)}\rangle\right)}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}-D_{pq}\frac{\Delta^2_{pq}\langle u^{(p)},\dot{u}^{(q)}\rangle+\left(\langle u^{(p)},u^{(q)}\rangle-c^2\right)\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}\right]\xi^{(pq)}_\alpha\right.$$

$$\left.-\left[\frac{\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)}+\frac{H_{pq}\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(p)}\rangle\right)}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}\right]u^{(q)}_\alpha-\frac{1}{c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle}\dot{u}^{(q)}_\alpha\right\}$$

$$+\frac{e_p^2\Delta_p}{2m_pc^2}\left\{\left[\frac{H_{(p)r}\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)}{\left(\langle u^{(p)r},\xi^{(p)r}\rangle-c^2\tau^{(p)r}\right)^3}-\frac{\Delta^2_{(p)r}\langle u^{(p)},\dot{u}^{(p)r}\rangle+\left(\langle u^{(p)},u^{(p)r}\rangle-c^2\right)\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta^2_{(p)r}\left(\langle\xi^{(p)r},u^{(p)r}\rangle-c^2\tau^{(p)r}\right)\left(\langle\xi^{(p)r},u^{(p)}\rangle-c^2\tau^{(p)r}\right)}\right]\xi^{(p)r}_\alpha-\right.$$

$$\left.-\left[\frac{\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta^2_{(p)r}\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(q)}\rangle\right)}+\frac{H_{(p)r}\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)}\rangle\right)}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^3}\right]u^{(p)r}_\alpha-\frac{1}{c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle}\dot{u}^{(p)r}_\alpha\right\}$$

$$-\frac{e_p^2\Delta_p}{2m_pc^2}\left\{\left[\frac{H_{(p)a}\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)}{\left(\langle u^{(p)a},\xi^{(p)a}\rangle-c^2\tau^{(p)a}\right)^3}-\frac{\Delta^2_{(p)a}\langle u^{(p)},\dot{u}^{(p)a}\rangle+\left(\langle u^{(p)},u^{(p)a}\rangle-c^2\right)\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)a}\left(\langle\xi^{(p)a},u^{(p)a}\rangle-c^2\tau^{(p)a}\right)\left(\langle\xi^{(p)a},u^{(p)}\rangle-c^2\tau^{(p)a}\right)}\right]\xi^{(p)a}_\alpha-\right.$$

$$\left.-\left[\frac{\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)a}\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)}+\frac{H_{(p)a}\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)}\rangle\right)}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^3}\right]u^{(p)a}_\alpha-\frac{1}{c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle}\dot{u}^{(p)a}_\alpha\right\}\equiv G^{(p)}_\alpha,$$

$$\frac{\langle u^{(p)},\dot{u}^{(p)}\rangle}{\Delta^2_p}=\frac{e_pe_q\Delta_p}{m_pc^2}\left\{\frac{\langle\xi^{(pq)},u^{(p)}\rangle-\tau_{pq}\langle u^{(p)},u^{(q)}\rangle}{\left(c^2\tau_{pq}-\langle\xi^{(pq)},u^{(q)}\rangle\right)^3}H_{pq}+D_{pq}\left[\frac{\left(\langle\xi^{(pq)},u^{(p)}\rangle-\tau_{pq}\langle u^{(p)},u^{(q)}\rangle\right)\langle u^{(q)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}-\right.\right.$$

$$\left.\left.-\frac{\Delta^2_{pq}\tau_{pq}\langle u^{(p)},\dot{u}^{(q)}\rangle}{\Delta^2_{pq}\left(\langle\xi^{(pq)},u^{(q)}\rangle-c^2\tau_{pq}\right)^2}\right]\right\}++\frac{e_p^2\Delta_p}{2m_pc^2}\left\{\frac{\langle u^{(p)},\xi^{(p)\mathrm{r}}\rangle-\tau^{(p)r}\langle u^{(p)},u^{(p)r}\rangle}{\left(c^2\tau^{(p)r}-\langle u^{(p)r},\xi^{(p)\mathrm{r}}\rangle\right)^3}H_{(p)r}+\right.$$

$$+\frac{D_{(p)r}\left(\langle\xi^{(p)r},u^{(p)}\rangle-\tau^{(p)r}\langle u^{(p)},u^{(p)r}\rangle\right)\langle u^{(p)r},\dot{u}^{(p)r}\rangle}{\Delta^2_{(p)\mathrm{r}}\left(c^2\tau^{(p)r}-\langle\xi^{(p)\mathrm{r}},u^{(p)r}\rangle\right)^2}-\frac{\tau^{(p)r}D_{(p)r}\langle u^{(p)},\dot{u}^{(p)r}\rangle}{\left(c^2\tau^{(p)r}-\langle\xi^{(p)r},u^{(p)r}\rangle\right)^2}\Bigg\}-$$

$$-\frac{e_p^2\Delta_p}{2m_pc^2}\Bigg\{\frac{\langle u^{(p)},\xi^{(p)a}\rangle-\tau^{(p)a}\langle u^{(p)},u^{(p)a}\rangle}{\left(c^2\tau^{(p)a}-\langle u^{(p)a},\xi^{(p)\mathrm{a}}\rangle\right)^3}H_{(p)a}+\frac{D_{(p)a}\left(\langle\xi^{(p)a},u^{(p)}\rangle-\tau^{(p)a}\langle u^{(p)},u^{(p)a}\rangle\right)\langle u^{(p)a},\dot{u}^{(p)a}\rangle}{\Delta^2_{(p)\mathrm{a}}\left(c^2\tau^{(p)a}-\langle\xi^{(p)\mathrm{a}},u^{(p)a}\rangle\right)^2}$$

$$-\frac{\tau^{(p)a}D_{(p)a}\langle u^{(p)},\dot{u}^{(p)a}\rangle}{\left(c^2\tau^{(p)a}-\langle\xi^{(p)a},u^{(p)a}\rangle\right)^2}\Bigg\}.$$

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