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Ring of Large Number Mutually Coupled Oscillators – Periodic Solutions

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Abstract The paper is devoted to the investigation of periodic regimes of a ring of a large number mutually coupled oscillators with active elements characterized by a simple symmetric cubic nonlinearity. In contrast to the usually accepted approach leading to Van der Pol equations we reduce the original system describing the ring oscillator to a first order integro-differential one. Introducing a suitable function space we define an operator acting in this space and by fixed point method we prove the basic result: an existence-uniqueness of T_0-periodic solution of the obtained integro-differential system. Finally we give a numerical example.

Keywords Oscillators, Coupled oscillators, Ring of oscillators, Van der Pol differential equations, Integro-differential system, Periodic solutions, Fixed-point theorem

1. Introduction

Oscillatory behavior is ubiquitous in all physical systems in different disciplines ranging from biology and chemistry to engineering and physics, especially in electronic and optical systems. In radio frequency and lightwave communication systems, oscillators are used for transformation signals and channel selection. Oscillators are also present in all digital electronic systems, which require a time reference. Coupled oscillators are oscillators connected in such a way that energy can be transferred between them. Since about 1960, mathematical biologists have been studying simplified models of coupled oscillators that retain the essence of their biological prototypes: pacemaker cells in the heart, insulin-secreting cells in the pancreas; and neural networks in the brain and spinal cord that control such rhythmic behavior as breathing, running and chewing. Ring of coupled oscillators is cascaded combination of delay stages, connected in a close loop chain and it has a number of applications in communication systems, especially in anatomic – organs such as the heart, intestine and ureter consist of many cellular oscillators coupled together. An ideal oscillator would provide a perfect time reference, i.e., a periodic signal. However all physical oscillators are corrupted by undesired perturbation noise. Hence signals, generated by physical oscillators, are not perfectly periodic.

Mathematical basis of coupled oscillators has been established in [1], [2] and mutual synchronization of a large number of oscillators has been investigated in [3]. Very interesting applications can be found in [4] and [5]. In [6] the authors have investigated a ring array of van der Pol oscillators and clarified each mode structure. Their results are based on the method of equivalent linearization of the nonlinear terms using Krylov-Bogoliubov method. An active element is characterized by a simple symmetric cubic nonlinearity, that is, with *V-I* characteristic

$$I_n(u_n) = -g_1^{(n)} u_n + g_3^{(n)} u_n^3, \quad \left(g_1^{(n)}, g_3^{(n)} > 0 \right) (n = 1, \dots, N).$$

A ring oscillator is described by the following integro-differential system:

$$\begin{aligned} u_n(t) - u_{n+1}(t) &= \frac{1}{L_0} \frac{di_n(t)}{dt} \\ i_{n-1}(t) - i_n(t) &= C \frac{du_n(t)}{dt} + \frac{1}{L} \int_0^t u_n(s) ds + I_n(u_n(t)), \end{aligned} \tag{1}$$

$$t \geq 0.$$

The usually accepted approach (cf. [6]) is to exclude the current functions after differentiation and to obtain a second order system of van der Pol differential equations

$$\frac{d^2 u_n(t)}{dt^2} - \frac{g_1^{(n)}}{C} \left(1 - \frac{3 g_3^{(n)}}{g_1^{(n)}} u_n^2 \right) \frac{du_n(t)}{dt} + \left(\frac{1}{CL} + \frac{2}{CL_0} \right) u_n(t)$$
$$- \frac{1}{CL_0} u_{n-1}(t) - \frac{1}{CL_0} u_{n+1}(t) = 0 \ \ (n = 1, 2, \dots, N).$$

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Here we consider the original integro-differential system (1) and exclude current functions without differentiation. So we reach the following first order (instead of second order) integro-differential system:

$$\frac{1}{L_0}\int_0^t \left(u_{n-1}(s)-2u_n(s)+u_{n+1}(s)\right)ds = C\frac{du_n(t)}{dt}+\frac{1}{L}\int_0^t u_n(s)ds - g_1^{(n)}u_n(t)+g_3^{(n)}u_n^3(t),\ t\geq 0$$

$(n=1,2,...,N)$ or

$$\frac{du_n(t)}{dt}=\frac{1}{C}\left(\frac{1}{L_0}\int_0^t u_{n-1}(s)ds-\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_0^t u_n(s)ds+\frac{1}{L_0}\int_0^t u_{n+1}(s)ds+g_1^{(n)}u_n(t)-g_3^{(n)}u_n^3(t)\right)$$

$$\equiv U(u_{n-1},u_n,u_{n+1})(t)$$

$(n=1,2,...,N)$.

In view of the boundary conditions for the ring connection

$$u_N(t)=u_0(t),\ u_{N+1}(t)=u_1(t).$$

the above system yields the following system of *N* equations for *N* unknown functions.

$$\frac{du_1(t)}{dt}=\frac{1}{C}\left(\frac{1}{L_0}\int_0^t u_N(s)\,ds-\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_0^t u_1(s)\,ds+\frac{1}{L_0}\int_0^t u_2(s)ds+g_1^{(1)}u_1(t)-g_3^{(1)}u_1^3(t)\right)$$

$$\equiv U(u_0,u_1,u_2)$$

....

$$\frac{du_n(t)}{dt}=\frac{1}{C}\left(\frac{1}{L_0}\int_0^t u_{n-1}(s)\,ds-\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_0^t u_n(s)\,ds+\frac{1}{L_0}\int_0^t u_{n+1}(s)\,ds+g_1^{(n)}u_n(t)-g_3^{(n)}u_n^3(t)\right) \quad (2)$$

$$\equiv U(u_{n-1},u_n,u_{n+1})$$

....

$$\frac{du_N(t)}{dt}=\frac{1}{C}\left(\frac{1}{L_0}\int_0^t u_{N-1}(s)\,ds-\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_0^t u_N(s)\,ds+\frac{1}{L_0}\int_0^t u_1(s)\,ds+g_1^{(N)}u_N(t)-g_3^{(N)}u_N^3(t)\right)$$

$$\equiv U(u_{N-1},u_N,u_1)$$

satisfying the initial conditions

$$u_1(0)=u_2(0)=...=u_N(0)=0.$$

We define an operator acting on suitable function space and its fixed point is a periodic solution of the above system. The advantages of our method is its simpler technique and obtaining of successive approximations beginning with simple initial functions.

We formulate a periodic problem: to find a periodic solution $\left(u_1(t),...,\ u_N(t)\right)$ on $[0,T]$ of the system (2). To solve the periodic problem we use the method from [7].

By $C^1_{T_0}[0,T]$ we mean the space of all differentiable T_0 -periodic functions with continuous derivatives.

First we introduce the sets (assuming $T=mT_0$):

$$M_U=\left\{u(.)\in C^1_{T_0}[0,T]:\ \int_{kT_0}^{(k+1)T_0} u(t)dt=0\ (k=0,1,2,3,...,m-1)\right\},$$

$$M_U^* = \left\{ u(.) \in M_U : \left| u(t) \right| \le U_0 e^{\mu(t-kT_0)}, t \in [kT_0, (k+1)T_0] \right\}$$

$$(k = 0,1,2,3,...,m-1)$$

The set $M^* = \underbrace{M_U^* \times M_U^* \times ... \times M_U^*}_{N \text{ times}}$ turns out into a complete metric space with respect to the metric:

$$\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N)) = \max\left\{ \rho_\mu^{(k)}(u_1,\overline{u}_1), \rho_\mu^{(k)}(\dot{u}_1,\dot{\overline{u}}_1),...,\rho_\mu^{(k)}(u_N,\overline{u}_N), \rho_\mu^{(k)}(\dot{u}_N,\dot{\overline{u}}_N) : k = 0,1,2,3,...,m-1 \right\}$$

where

$$\rho_\mu^{(k)}(u_n,\overline{u}_n) = \max\left\{ e^{-\mu(t-kT_0)} \left| u_n(t) - \overline{u}_n(t) \right| : t \in [kT_0,\ (k+1)T_0] \right\},$$

$$\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n) = \max\left\{ e^{-\mu(t-kT_0)} \left| \dot{u}_n(t) - \dot{\overline{u}}_n(t) \right| : t \in [kT_0,(k+1)T_0] \right\}.$$

It is easy to see $\rho^{(k)}(u,\overline{u}) \le e^{\mu_0} \rho_\mu^{(k)}(u,\overline{u}) \le e^{\mu_0} \dfrac{\rho_\mu^{(k)}(\dot{u},\dot{\overline{u}})}{\mu}$.

Introduce the operator B as a n-tuple

$$B(u_1,...,u_N)(t) = \left(B_1(u_1,...,u_N)(t),\ B_2(u_1,...,u_N)(t),..., B_n(u_1,...,u_N)(t) \right)$$
$$= \left(B_1(u_0,u_1,u_2)(t),\ B_2(u_1,u_2,u_3)(t),..., B_n(u_{N-1},u_N,u_{N+1})(t) \right)$$
$$t \in [0,T] (u_N = u_0,\ u_{N+1} = u_1)$$

defined on every interval $[kT_0,\ (k+1)T_0], (k = 0,1,...,m-1)$ by the expressions $u_n(0) = u_n(kT_0) = 0$;

$$B_n^{(k)}(u_{n-1},u_n,u_{n+1})(t) := \int_{kT_0}^{t} U(u_{n-1},u_n,u_{n+1})(s)ds - \left(\frac{t-kT_0}{T_0} - \frac{1}{2} \right)$$
$$\times \int_{kT_0}^{(k+1)T_0} U(u_{n-1},u_n,u_{n+1})(s)ds - \frac{1}{T_0} \int_{kT_0}^{(k+1)T_0} \int_{kT_0}^{t} U(u_{n-1},u_n,u_{n+1})(s)dsdt,$$
$$t \in [kT_0,\ (k+1)T_0], (k = 0,1,...,m-1),\ (n = 1,2,...,N),$$

where

$$U(u_{n-1},u_n,u_{n+1})$$
$$= \frac{1}{CL_0} \int_{kT_0}^{t} u_{n-1}(s)ds - \frac{1}{C}\left(\frac{2}{L_0} + \frac{1}{L} \right) \int_{kT_0}^{t} u_n(s)ds + \frac{1}{CL_0} \int_{kT_0}^{t} u_{n+1}(s)ds + \frac{g_1^{(n)}}{C} u_n(t) - \frac{g_3^{(n)}}{C} u_n^3(t).$$

2. Preliminary Results

Now we formulate some useful preliminary assertions for our investigation.

Lemma 1. If $(u_1,...,u_N) \in M^*$ then $F_1(t) = \int_0^t u_1(\tau)d\tau,..., \quad F_N(t) = \int_0^t u_N(\tau)d\tau$ are T_0-periodic functions.

Proof: Indeed, we have

$$F_n(t+T_0)=\int_{kT_0}^{t+T_0}u_n(\tau)d\tau=\int_{kT_0}^{t}u_n(\tau)d\tau+\int_{t}^{t+T_0}u_n(\tau)d\tau$$

$$=F_n(t)+\int_{t}^{kT_0}u_n(\tau)d\tau+\int_{kT_0}^{(k+1)T_0}u_n(\tau)d\tau+\int_{(k+1)T_0}^{t+T_0}u_n(\tau)d\tau$$

$$=F_n(t)+\int_{t}^{kT_0}u_n(\tau)d\tau+\int_{kT_0}^{t}u_n(\theta)d\theta=F_n(t).$$

Lemma 1 is thus proved.

Lemma 2. If $(u_1,...,u_N)\in M^*$ then $U(u_{n-1},u_n,u_{n+1})(t)$ are T_0 -periodic functions.

The proof is straightforward based on the previous Lemma 1.

Lemma 3. For every

$$(u_1,...,u_N)\in\underbrace{M_U^*\times M_U^*\times...\times M_U^*}_{N-times}$$

it follows

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta\, ds=\int_{(k+1)T_0}^{(k+2)T_0}\int_{(k+1)T_0}^{s}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta\, ds\ \ (k=0,1,2,...).$$

Proof:

We notice that in view of Lemma 2 the functions $U(u_{n-1},u_n,u_{n+1})(t)$ are T_0 -periodic. Let us define the functions

$$\Gamma_k(s)=\int_{kT_0}^{s}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta \text{ and } \Gamma_{k+1}(s)=\int_{(k+1)T_0}^{s}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta$$

and rewrite them in the form

$$\Gamma_k(\eta)=\int_{kT_0}^{\eta+kT_0}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta\,,\Gamma_{k+1}(\eta)=\int_{(k+1)T_0}^{\eta+(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta\,.$$

Changing the variable $\tau=\theta-T_0$ we obtain

$$\Gamma_{k+1}(\eta)=\int_{(k+1)T_0}^{(k+1)T_0+\eta}U(u_{n-1},u_n,u_{n+1})(\theta)d\theta=\int_{kT_0}^{kT_0+\eta}U(u_{n-1},u_n,u_{n+1})(\tau+T_0)d\tau$$

$$=\int_{kT_0}^{kT_0+\eta}U(u_{n-1},u_n,u_{n+1})(\tau)d\tau=\Gamma_k(\eta).$$

Consequently $\displaystyle\int_{kT_0}^{(k+1)T_0}\Gamma_k(\eta)d\eta=\int_{(k+1)T_0}^{(k+2)T_0}\Gamma_{k+1}(\eta)d\eta\ \ (k=0,1,2,...).$

Lemma 3 is thus proved.

Lemma 3 shows that operator function

$B(u_1,...,u_N)(t)$ is T_0 - periodic one.

Lemma 4. The initial value problem (2) has a solution $(u_1,...,u_N)\in M^*$ iff the operator *B* has a fixed point $(u_1,...,u_N)\in M^*$, that is,

$$(u_1,...,u_N)=\left(B_1(u_1,...,u_N),B_2(u_1,...,u_N),...,B_n(u_1,...,u_N)\right).$$

Proof: Let $(u_1,...,u_N)\in M^*$ be a periodic solution of (2). Then integrating (2) we have:

$$u_n(t)=\int_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(t)dt \Rightarrow$$

$$0=u_n((k+1)T_0)=\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(t)dt \Rightarrow$$

$$\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(t)dt=0.$$

But

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)dsdt=\int_{kT_0}^{(k+1)T_0}u_n(t)dt=0\,.$$

Therefore $u_n(t)=\int_{T+kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)ds$ is equivalent to

$$u_n(t)=\int_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)ds-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(s)ds$$
$$-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)ds\,dt$$

and then the solution $(u_1,...,u_N)$ of (2) is a fixed point of *B*.

Conversely, let $(u_1,...,u_N)\in M^*$ be a fixed point *B*, that is, $u_n=B_n^{(k)}(u_1,...,u_N)$, $t\in[kT_0,(k+1)T_0]$. Therefore by definition of *B* we obtain

$$u_n(kT_0)=B_n^{(k)}(u_1,...,u_N)(kT_0)\,,$$

or

$$0=u_n(kT_0)=\int_{kT_0}^{kT_0}U(u_{n-1},u_n,u_{n+1})(s)ds-\left(\frac{kT_0-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(s)ds$$
$$-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{T+kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)dsdt=\frac{1}{2}\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(s)ds$$
$$-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)dsdt\,.$$

We show $\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(s)ds=0$

which implies

$$\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} U(u_{n-1},u_n,u_{n+1})(s)dsdt = 0 .$$

Indeed, put $\mu T_0 = \mu_0 = \text{const.}$ and then we have

$$\left|\int_{kT_0}^{(k+1)T_0} U(u_{n-1},u_n,u_{n+1})(t)dt\right| \le \frac{1}{CL_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\left|u_{n-1}(s)\right|ds\,dt + \frac{1}{C}\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\left|u_n(s)\right|ds\,dt$$

$$+\frac{1}{CL_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\left|u_{n+1}(s)\right|ds\,dt + \frac{\left|g_1^{(n)}\right|}{C}\int_{kT_0}^{(k+1)T_0}\left|u_n(t)\right|dt + \frac{\left|g_3^{(n)}\right|}{C}\int_{kT_0}^{(k+1)T_0}\left|u_n(t)\right|^3 dt$$

$$\le \frac{1}{C}\left(\frac{1}{L_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} U_0 e^{\mu(s-kT_0)}ds\,dt + \left(\frac{2}{L_0}+\frac{1}{L}\right)\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} U_0 e^{\mu(t-sT_0)}ds\,dt\right.$$

$$\left.+\frac{1}{L_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t} U_0 e^{\mu(s-kT_0)}ds\,dt + \left|g_1^{(n)}\right|\int_{kT_0}^{(k+1)T_0} U_0 e^{\mu(t-kT_0)}dt + \left|g_3^{(n)}\right|\int_{kT_0}^{(k+1)T_0}\left(U_0 e^{\mu(t-kT_0)}\right)^3 dt\right)$$

$$\le \frac{1}{C}\left(\frac{U_0}{L_0}\int_{kT_0}^{(k+1)T_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}dt + \left(\frac{2}{L_0}+\frac{1}{L}\right)U_0\int_{kT_0}^{(k+1)T_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}dt + \frac{U_0}{L_0}\int_{kT_0}^{(k+1)T_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}dt\right.$$

$$\left.+\left|g_1^{(n)}\right|U_0\frac{e^{\mu T_0}-1}{\mu} + \left|g_3^{(n)}\right|\left(U_0\right)^3 e^{2\mu T_0}\frac{e^{\mu T_0}-1}{\mu}\right)$$

$$\le \frac{1}{C}\left(\frac{2}{L_0}\frac{U_0}{\mu}\int_{kT_0}^{(k+1)T_0} e^{\mu(t-kT_0)}dt + \left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{U_0}{\mu}\int_{kT_0}^{(k+1)T_0} e^{\mu(t-kT_0)}dt\right.$$

$$\left.+\left|g_1^{(n)}\right|U_0\frac{e^{\mu T_0}-1}{\mu} + \left|g_3^{(n)}\right|\left(U_0\right)^3 e^{2\mu T_0}\frac{e^{\mu T_0}-1}{\mu}\right)$$

$$\le \frac{e^{\mu T_0}-1}{\mu}\frac{U_0}{C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu} + \left|g_1^{(n)}\right| + \left|g_3^{(n)}\right|\left(U_0\right)^2 e^{2\mu T_0}\right)$$

$$= \frac{e^{\mu_0}-1}{\mu}\frac{U_0}{C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu} + \left|g_1^{(n)}\right| + \left|g_3^{(n)}\right|\left(U_0\right)^2 e^{2\mu_0}\right) \equiv M(\mu).$$

Obviously $M(\mu)\to 0$ as $\mu\to\infty$ that implies $\displaystyle\int_{kT_0}^{(k+1)T_0} U(u_{n-1},u_n,u_{n+1})(t)dt = 0$.

Therefore the operator equation

$$u_n = B_n^{(k)}(u_1,...,u_N)\ ,\ t\in[kT_0,(k+1)T_0]$$

becomes $u_n(t)=\int\limits_{kT_0}^{t} U(u_{n-1},u_n,u_{n+1})(s)ds.$

Differentiating the last equalities we obtain (2).

Lemma 4 is thus proved.

3. Main Result

Theorem 1. Let the following assumptions be valid:

1) $u_1(0)=...=u_N(0)=0$;

2) For sufficiently large $\mu>0$

$$\frac{1}{\mu C}\left[\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2 e^{2\mu_0}\right)\right]\le 1;$$

$$\frac{1}{\mu C}\left[\frac{1}{\mu^2}\left(2+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{\left|g_1^{(n)}\right|+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right]<1;$$

$$\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu C}\left[\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|3U_0^2e^{2\mu_0}\right)\right]<1.$$

Then there exists a unique T_0 -periodic solution of (2).

Proof: First we show that

$$\left(B_1(u_1,...,u_N),\ B_2(u_1,...,u_N),...,B_N(u_1,...,u_N)(t)\right)\in M^*.$$

Indeed

$$\left|B_n^{(k)}(u_{n-1},u_n,u_{n+1})(t)\right|\le\left|\int\limits_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)ds\right|+\frac{1}{2}\left|\int\limits_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(s)ds\right|$$

$$+\frac{1}{T_0}\left|\int\limits_{kT_0}^{(k+1)T_0}\int\limits_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)dsdt\right|\equiv V_1+V_2+V_3.$$

We have

$$V_1=\left|\int\limits_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)ds\right|\le\frac{1}{C}\left(\frac{1}{L_0}\int\limits_{kT_0}^{t}\int\limits_{kT_0}^{s}\left|u_{n-1}(\theta)\right|d\theta\,ds\right.$$

$$\left.+\left(\frac{2}{L_0}+\frac{1}{L}\right)\int\limits_{kT_0}^{t}\int\limits_{kT_0}^{s}\left|u_n(\theta)\right|d\theta\,ds+\frac{1}{L_0}\int\limits_{kT_0}^{t}\int\limits_{kT_0}^{s}\left|u_{n+1}(\theta)\right|d\theta\,ds+\left|g_1^{(n)}\right|\int\limits_{kT_0}^{t}\left|u_n(s)\right|ds+\left|g_3^{(n)}\right|\int\limits_{kT_0}^{t}\left|u_n(s)\right|^3ds\right)$$

$$\le\frac{1}{C}\left(\frac{1}{L_0}\int\limits_{kT_0}^{t}\int\limits_{kT_0}^{t}U_0e^{\mu(\theta-kT_0)}d\theta\,ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\int\limits_{kT_0}^{t}\int\limits_{kT_0}^{s}U_0e^{\mu(t-\theta T_0)}d\theta\,ds+\frac{1}{L_0}\int\limits_{kT_0}^{t}\int\limits_{kT_0}^{s}U_0e^{\mu(\theta-kT_0)}d\theta\,ds\right.$$

$$+\left|g_1^{(n)}\right|\int_{kT_0}^{t}U_0e^{\mu(s-kT_0)}ds+\left|g_3^{(n)}\right|\int_{kT_0}^{t}\left(U_0e^{\mu(s-kT_0)}\right)^3ds\Bigg)\le\frac{1}{C}\left(\frac{U_0}{L_0}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds\right.$$

$$+\left(\frac{2}{L_0}+\frac{1}{L}\right)U_0\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds+\frac{U_0}{L_0}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds$$

$$\left.+\left|g_1^{(n)}\right|U_0\frac{e^{\mu(t-kT_0)}-1}{\mu}+\left|g_3^{(n)}\right|\left(U_0\right)^3e^{2\mu T_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}\right)$$

$$\le\frac{1}{C}\left(\frac{2}{L_0}\frac{U_0}{\mu}\int_{kT_0}^{t}e^{\mu(s-kT_0)}ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{U_0}{\mu}\int_{kT_0}^{t}e^{\mu(s-kT_0)}ds\right.$$

$$\left.+\left|g_1^{(n)}-g\right|U_0\frac{e^{\mu(t-kT_0)}-1}{\mu}+\left|g_3^{(n)}\right|\left(U_0\right)^3e^{2\mu T_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}\right)$$

$$\le\frac{e^{\mu(t-kT_0)}-1}{\mu}\frac{U_0}{C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu T_0}\right)$$

$$\le\frac{e^{\mu(t-kT_0)}}{\mu}\frac{U_0}{C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu_0}\right).$$

Using the estimates from Lemma 3 we obtain

$$V_2=\frac{1}{2}\left|\int_{kT_0}^{(k+1)T_0}U(u_{n-1},u_n,u_{n+1})(t)dt\right|\le\frac{e^{\mu_0}-1}{\mu}\frac{U_0}{C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu_0}\right)$$

and

$$V_3=\left|\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U(u_{n-1},u_n,u_{n+1})(s)dsdt\right|\le\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\left|\int_{kT_0}^{t}U(u_{n-1},u_n)(s)ds\right|dt$$

$$\le\frac{1}{\mu}\frac{U_0}{C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu_0}\right)\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}e^{\mu(t-kT_0)}dt$$

$$\le\frac{e^{\mu_0}-1}{\mu_0}\frac{U_0}{\mu C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu_0}\right).$$

Therefore for sufficiently large μ we have

$$\left|B_n^{(k)}(u_{n-1},u_n,u_{n+1})(t)\right|\le e^{\mu(t-kT_0)}\frac{U_0}{\mu C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu_0}\right)$$

$$+\left(e^{\mu_0}-1\right)\frac{U_0}{\mu C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2e^{2\mu_0}\right)$$

$$+\frac{e^{\mu_0}-1}{\mu_0}\frac{U_0}{\mu C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2 e^{2\mu_0}\right)$$

$$\leq e^{\mu(t-kT_0)}\frac{U_0}{\mu C}\left[\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|\left(U_0\right)^2 e^{2\mu_0}\right)\right]$$

$$\leq U_0 e^{\mu(t-kT_0)}.$$

It remains to show that $B=(B_1,B_2,...B_N)$ is a contractive operator on M^*.
Indeed,

$$\left|B_n^{(k)}(u_{n-1},u_n,u_{n+1})(t)-B_n^{(k)}(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1}))(t)\right|$$

$$\leq\left|\int_{kT_0}^{t}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)ds\right|$$

$$+\left|\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)ds\right|$$

$$+\left|\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{T+kT_0}^{t}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)dsdt\right|$$

$$\equiv P_1+P_2+P_3.$$

Then

$$P_1\leq\left|\int_{T+kT_0}^{t}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)ds\right|$$

$$\leq\frac{1}{C}\left(\frac{1}{L_0}\int_{kT_0}^{t}\int_{kT_0}^{s}\left|u_{n-1}(\theta)-\overline{u}_{n-1}(\theta)\right|d\theta\,ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_{kT_0}^{t}\int_{kT_0}^{s}\left|u_n(\theta)-\overline{u}_n(\theta)\right|d\theta\,ds\right.$$

$$\left.+\frac{1}{L_0}\int_{kT_0}^{t}\int_{kT_0}^{s}\left|u_{n+1}(\theta)-\overline{u}_{n+1}(\theta)\right|d\theta\,ds+\left|g_1^{(n)}\right|\int_{kT_0}^{t}\left|u_n(s)-\overline{u}_n(s)\right|ds+\left|g_3^{(n)}\right|\int_{kT_0}^{t}\left|u_n{}^3(s)-\overline{u}_n{}^3(s)\right|ds\right)$$

$$\leq\frac{1}{C}\left(\frac{\rho_\mu^{(k)}(u_{n-1},\overline{u}_{n-1})}{L_0}\int_{kT_0}^{t}\int_{kT_0}^{s}e^{\mu(\theta-kT_0)}d\theta\,ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{t}\int_{kT_0}^{s}e^{\mu(\theta-kT_0)}d\theta\,ds\right.$$

$$+\frac{\rho_\mu^{(k)}(u_{n+1},\overline{u}_{n+1})}{L_0}\int_{kT_0}^{t}\int_{kT_0}^{s}e^{\mu(\theta-kT_0)}d\theta\,ds+\left|g_1^{(n)}\right|\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{t}e^{\mu(s-kT_0)}ds$$

$$\left.+\left|g_3^{(n)}\right|\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{t}\left(\left|u_n(s)\right|^2+\left|\overline{u}_n(s)\right|^2+\left|u_n(s)\right|\left|\overline{u}_n(s)\right|\right)e^{\mu(s-kT_0)}ds\right)$$

$$\leq \frac{1}{C}\left(\frac{\rho_\mu^{(k)}(u_{n-1},\overline{u}_{n-1})}{L_0}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds\right.$$

$$+\frac{\rho_\mu^{(k)}(u_{n+1},\overline{u}_{n+1})}{L_0}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds+\left|g_1^{(n)}\right|\rho_\mu^{(k)}(u_n,\overline{u}_n)\frac{e^{\mu(t-kT_0)}-1}{\mu}$$

$$\left.+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{t}e^{\mu(s-kT_0)}ds\right)\leq\frac{1}{C}\left(\frac{\rho_\mu^{(k)}(\dot{u}_{n-1},\dot{\overline{u}}_{n-1})}{\mu L_0}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds\right.$$

$$+\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n)}{\mu}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds+\frac{\rho_\mu^{(k)}(\dot{u}_{n+1},\dot{\overline{u}}_{n+1})}{\mu L_0}\int_{kT_0}^{t}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds$$

$$\left.+\left|g_1^{(n)}\right|\frac{\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n)}{\mu}\frac{e^{\mu(t-kT_0)}-1}{\mu}+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|\frac{\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n)}{\mu}\frac{e^{\mu(t-kT_0)}-1}{\mu}\right)$$

$$\leq e^{\mu(t-kT_0)}\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))$$

$$\times\frac{1}{\mu C}\left(\frac{1}{\mu^2L_0}+\frac{1}{\mu^2}\left(\frac{2}{L_0}+\frac{1}{L}\right)+\frac{\left|g_1^{(n)}\right|}{\mu}+\frac{3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right);$$

$$P_2=\left|\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)ds\right|$$

$$\leq\frac{1}{C}\left(\frac{1}{L_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}\left|u_{n-1}(\theta)-\overline{u}_{n-1}(\theta)\right|d\theta\,ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}\left|u_n(\theta)-\overline{u}_n(\theta)\right|d\theta\,ds\right.$$

$$+\frac{1}{L_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}\left|u_{n+1}(\theta)-\overline{u}_{n+1}(\theta)\right|d\theta\,ds$$

$$\left.+\left|g_1^{(n)}\right|\int_{kT_0}^{(k+1)T_0}\left|u_n(s)-\overline{u}_n(s)\right|ds+\left|g_3^{(n)}\right|\int_{kT_0}^{(k+1)T_0}\left|u_n{}^3(s)-\overline{u}_n{}^3(s)\right|ds\right)$$

$$\leq\frac{1}{C}\left(\frac{\rho_\mu^{(k)}(u_{n-1},\overline{u}_{n-1})}{L_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}e^{\mu(\theta-kT_0)}d\theta\,ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}e^{\mu(\theta-kT_0)}d\theta\,ds\right.$$

$$+\frac{\rho_\mu^{(k)}(u_{n+1},\overline{u}_{n+1})}{L_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{s}e^{\mu(\theta-kT_0)}d\theta\,ds$$

$$+\left|g_1^{(n)}\right|\rho_\mu^{(k)}(u_n,\overline{u}_n)\int_{kT_0}^{(k+1)T_0} e^{\mu(s-kT_0)}ds+\left|g_3^{(n)}\right|\rho_\mu^{(k)}(u_n,\overline{u}_n)$$

$$\left.\times\int_{kT_0}^{(k+1)T_0}\left(\left|u_n(s)\right|^2+\left|\overline{u}_n(s)\right|^2+\left|u_n(s)\right|\left|\overline{u}_n(s)\right|\right)e^{\mu(s-kT_0)}ds\right)$$

$$\le\frac{1}{C}\left(\frac{\rho_\mu^{(k)}(\dot{u}_{n-1},\dot{\overline{u}}_{n-1})}{\mu L_0}\int_{kT_0}^{(k+1)T_0}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds++\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n)}{\mu}\int_{kT_0}^{(k+1)T_0}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds\right.$$

$$+\frac{\rho_\mu^{(k)}(\dot{u}_{n+1},\dot{\overline{u}}_{n+1})}{\mu L_0}\int_{kT_0}^{(k+1)T_0}\frac{e^{\mu(s-kT_0)}-1}{\mu}ds$$

$$\left.+\left|g_1^{(n)}\right|\frac{\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n)}{\mu}\frac{e^{\mu T_0}-1}{\mu}+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|\frac{\rho_\mu^{(k)}(\dot{u}_n,\dot{\overline{u}}_n)}{\mu}\frac{e^{\mu T_0}-1}{\mu}\right)$$

$$\le\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))$$

$$\times\frac{1}{C}\left(\frac{1}{\mu^2L_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}+\frac{1}{\mu^2}\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{e^{\mu(t-kT_0)}-1}{\mu}\right.$$

$$\left.+\frac{1}{\mu^2L_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}+\left|g_1^{(n)}\right|\frac{e^{\mu T_0}-1}{\mu^2}+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|\frac{e^{\mu T_0}-1}{\mu^2}\right)$$

$$\le e^{\mu(t-kT_0)}\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))$$

$$\times\frac{1}{\mu C}\left(\frac{4}{\mu^2L_0}+\frac{1}{\mu^2L}+\frac{\left|g_1^{(n)}\right|\left(e^{\mu_0}-1\right)}{\mu}+\frac{3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|\left(e^{\mu_0}-1\right)}{\mu}\right)$$

and

$$P_3\le\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\left|\int_{kT_0}^{t}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)ds\right|dt$$

$$\le\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))$$

$$\times\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}e^{\mu(t-kT_0)}dt\,\frac{1}{\mu C}\left(\frac{4}{\mu^2L_0}+\frac{1}{\mu^2L}+\frac{\left|g_1^{(n)}\right|}{\mu}+\frac{3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right)$$

$$\le\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))\times$$

$$\times\frac{e^{\mu_0}-1}{\mu_0}\frac{1}{\mu C}\left(\frac{4}{\mu^2L_0}+\frac{1}{\mu^2L}+\frac{\left|g_1^{(n)}\right|}{\mu}+\frac{3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right).$$

It follows

$$\left|B_n^{(k)}(u_{n-1},u_n,u_{n+1})(t)-B_n^{(k)}(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(t)\right|\le e^{\mu(t-kT_0)}\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))$$

$$\times\left[\frac{1}{\mu C}\left(\frac{4}{\mu^2 L_0}+\frac{1}{\mu^2 L}+\frac{\left|g_1^{(n)}\right|}{\mu}+\frac{3U_0^2 e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right)\right.$$

$$+\frac{1}{\mu C}\left(\frac{4}{\mu^2 L_0}+\frac{1}{\mu^2 L}+\frac{\left|g_1^{(n)}\right|\left(e^{\mu_0}-1\right)}{\mu}+\frac{3U_0^2 e^{2\mu_0}\left|g_3^{(n)}\right|\left(e^{\mu_0}-1\right)}{\mu}\right)$$

$$\left.+\frac{e^{\mu_0}-1}{\mu_0}\frac{1}{\mu C}\left(\frac{4}{\mu^2 L_0}+\frac{1}{\mu^2 L}+\frac{\left|g_1^{(n)}\right|}{\mu}+\frac{3U_0^2 e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right)\right]$$

$$\le\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N))\times$$

$$\times\frac{e^{\mu(t-kT_0)}}{\mu C}\left[\frac{1}{\mu^2}\left(2+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\frac{\left|g_1^{(n)}\right|}{\mu}+\frac{3U_0^2 e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right)\right]$$

$$\equiv e^{\mu(t-T-kT_0)}K_U\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N)).$$

Thus

$$\rho\left(B_n^{(k)}(u_1,\dot{u}_1,...,u_N,\dot{u}_N),B_n^{(k)}(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N)\right)\le K_U\rho_\mu((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\overline{u}_1,\dot{\overline{u}}_1,...,\overline{u}_N,\dot{\overline{u}}_N)).$$

For the derivative we obtain

$$\left|\dot{B}_n^{(k)}(u_{n-1},u_n,u_{n+1})(t)-\dot{B}_n^{(k)}(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1}))(t)\right|\le\left|U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(t)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(t)\right|$$

$$+\left|\frac{1}{T_0}\int\limits_{kT_0}^{(k+1)T_0}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\overline{u}_{n-1},\overline{u}_n,\overline{u}_{n+1})(s)\right)ds\right|\equiv Q_1+Q_2.$$

We have

$$Q_1\le\frac{1}{C}\left(\frac{1}{L_0}\int\limits_{kT_0}^{t}\left|u_{n-1}(s)-\overline{u}_{n-1}(s)\right|ds+\left(\frac{2}{L_0}+\frac{1}{L}\right)\int\limits_{kT_0}^{t}\left|u_n(s)-\overline{u}_n(s)\right|ds+\frac{1}{L_0}\int\limits_{kT_0}^{t}\left|u_{n+1}(s)-\overline{u}_{n+1}(s)\right|ds\right.$$

$$\left.+\left|g_1^{(n)}\right|\left|u_n(t)-\overline{u}_n(t)\right|+\left|g_3^{(n)}\right|\left|u_n{}^3(t)-\overline{u}_n{}^3(t)\right|\right)\le\frac{1}{C}\left(\frac{1}{L_0}\rho_\mu^{(k)}\left(u_{n-1},\overline{u}_{n-1}\right)\int\limits_{kT_0}^{t}e^{\mu(s-kT_0)}ds\right.$$

$$+\left(\frac{2}{L_0}+\frac{1}{L}\right)\rho_\mu^{(k)}\left(u_n,\overline{u}_n\right)\int\limits_{kT_0}^{t}e^{\mu(s-kT_0)}ds+\frac{1}{L_0}\rho_\mu^{(k)}\left(u_{n+1},\overline{u}_{n+1}\right)\int\limits_{kT_0}^{t}e^{\mu(s-kT_0)}ds$$

$$\left.+\left|g_1^{(n)}\right|\rho_\mu^{(k)}\left(u_n,\overline{u}_n\right)e^{\mu(t-kT_0)}+\left|g_3^{(n)}\right|3U_0^2 e^{2\mu_0}e^{\mu(t-kT_0)}\rho_\mu^{(k)}\left(u_n,\overline{u}_n\right)\right)$$

$$\leq \frac{1}{C}\left(\frac{\rho_{\mu}^{(k)}\left(\dot{u}_{n-1},\dot{\bar{u}}_{n-1}\right)}{\mu L_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}+\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{\rho_{\mu}^{(k)}\left(\dot{u}_{n},\dot{\bar{u}}_{n}\right)}{\mu}\frac{e^{\mu(t-kT_0)}-1}{\mu}\right.$$

$$\left.+\frac{\rho_{\mu}^{(k)}\left(\dot{u}_{n+1},\dot{\bar{u}}_{n+1}\right)}{\mu L_0}\frac{e^{\mu(t-kT_0)}-1}{\mu}+\left|g_1^{(n)}\right|\frac{\rho_{\mu}^{(k)}\left(\dot{u}_{n},\dot{\bar{u}}_{n}\right)}{\mu}e^{\mu(t-kT_0)}+\left|g_3^{(n)}\right|3U_0^2e^{2\mu_0}e^{\mu(t-kT_0)}\frac{\rho_{\mu}^{(k)}\left(\dot{u}_{n},\dot{\bar{u}}_{n}\right)}{\mu}\right)$$

$$\leq e^{\mu(t-kT_0)}\rho_{\mu}((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\bar{u}_1,\dot{\bar{u}}_1,...,\bar{u}_N,\dot{\bar{u}}_N))\times\frac{1}{\mu C}\left(\left(\frac{4}{L_0}+\frac{1}{L}\right)\frac{1}{\mu}+\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|3U_0^2e^{2\mu_0}\right);$$

$$Q_2 \leq \left|\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\left(U(u_{n-1},u_n,u_{n+1})(s)-U(\bar{u}_{n-1},\bar{u}_n,\bar{u}_{n+1})(s)\right)ds\right|$$

$$\leq \frac{e^{\mu_0}-1}{\mu_0}\frac{1}{\mu C}\left(\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left|g_1^{(n)}\right|+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|\right)$$

$$\times\rho_{\mu}((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\bar{u}_1,\dot{\bar{u}}_1,...,\bar{u}_N,\dot{\bar{u}}_N)).$$

Thus

$$\left|\dot{B}_n^{(k)}(u_{n-1},u_n,u_{n+1})(t)-\dot{B}_n^{(k)}(\bar{u}_{n-1},\bar{u}_n,\bar{u}_{n+1}))(t)\right|$$

$$\leq e^{\mu(t-kT_0)}\rho_{\mu}((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\bar{u}_1,\dot{\bar{u}}_1,...,\bar{u}_N,\dot{\bar{u}}_N))$$

$$\times\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu C}\left[\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|3U_0^2e^{2\mu_0}\right)\right]$$

$$\equiv e^{\mu(t-kT_0)}\dot{K}_U\rho_{\mu}((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\bar{u}_1,\dot{\bar{u}}_1,...,\bar{u}_N,\dot{\bar{u}}_N)).$$

The above inequalities imply

$$\rho_{\mu}\left(B_n(u_1,...,u_N),B_n(\bar{u}_1,...,\bar{u}_N)\right)\leq \dot{K}_U\rho_{\mu}((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\bar{u}_1,\dot{\bar{u}}_1,...,\bar{u}_N,\dot{\bar{u}}_N))$$

and then

$$\rho_{\mu}\left(\left(B_1,\dot{B}_1...,B_N,\dot{B}_N\right),\left(\bar{B}_1,\dot{\bar{B}}_1,...,\bar{B}_N,\dot{\bar{B}}_N\right)\right)\leq K\rho_{\mu}((u_1,\dot{u}_1,...,u_N,\dot{u}_N),(\bar{u}_1,\dot{\bar{u}}_1,...,\bar{u}_N,\dot{\bar{u}}_N))$$

where $K=\max\left\{K_U,\dot{K}_U\right\}$.

Theorem 1 is thus proved.

4. Numerical Example

The inequalities guaranteeing an existence-uniqueness of a periodic solution are

$$\frac{1}{\mu C}\left[\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|(U_0)^2e^{2\mu_0}\right)\right]\leq 1;$$

$$\frac{1}{\mu C}\left[\frac{1}{\mu^2}\left(2+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{\left|g_1^{(n)}\right|+3U_0^2e^{2\mu_0}\left|g_3^{(n)}\right|}{\mu}\right]<1;$$

$$\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu C}\left[\frac{1}{\mu}\left(\frac{4}{L_0}+\frac{1}{L}\right)+\left(\left|g_1^{(n)}\right|+\left|g_3^{(n)}\right|3U_0^2e^{2\mu_0}\right)\right]<1.$$

Let us consider the case when $L=44\,mH,\ \ L_0=111\,mH,\ \ C=1\ \mu F$. We assume the active elements have characteristics $I_n(u)=-0{,}12u+0{,}4u^3(n=1,2,...,N)$. Then

$$\frac{1}{\mu.10^{-6}}\left[\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{10^3}{\mu}\left(\frac{4}{111}+\frac{1}{44}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(0{,}12+0{,}4U_0^2e^{2\mu_0}\right)\right]\le 1;;$$

$$\frac{1}{\mu^2.10^{-6}}\left[\frac{1}{\mu}\left(2+\frac{e^{\mu_0}-1}{\mu_0}\right)10^3\left(\frac{4}{111}+\frac{1}{44}\right)+\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\left(0{,}12+1{,}2U_0^2e^{2\mu_0}\right)\right]<1;$$

$$\frac{1}{\mu.10^{-6}}\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\left[\frac{1}{\mu}10^3\left(\frac{4}{111}+\frac{1}{44}\right)+0{,}12+1{,}2U_0^2e^{2\mu_0}\right]<1.$$

If for instance $T_0=\frac{1}{f}=\frac{1}{10^6\,Hz}\approx 10^{-6}\sec$, and $U_0=0{,}1$. Then we have to choose $\mu=10^6$ and the second inequality can be disregarded. Then $\mu_0=\mu T_0=1$ that implies $\frac{4{,}44}{10^3}0{,}06+4{,}44.0{,}15\le 1$ and hence

$$K=2\left[10^{-3}\left((4/111)+(1/44)\right)+0{,}12+1{,}2U_0^2e^2\right]\approx 0{,}42<1.$$

5. Conclusions

Successive approximations to the solution can be obtained beginning with the following initial functions $u_n^{(0)}(t)=U_0\sin\frac{2\pi}{T_0}t=U_0\sin\omega_0 t$. It is easy to calculate the next approximation from the right-hand side of the above system:

$$u_n^{(1)}(t)=B_n^{(k)}(u_{n-1}^{(0)},u_n^{(0)},u_{n+1}^{(0)})(t):=\int_{kT_0}^{t}U(u_{n-1}^{(0)},u_n^{(0)},u_{n+1}^{(0)})(s)ds$$

$$-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\int_{kT_0}^{(k+1)T_0}U(u_{n-1}^{(0)},u_n^{(0)},u_{n+1}^{(0)})(s)ds-\frac{1}{T_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}U(u_{n-1}^{(0)},u_n^{(0)},u_{n+1}^{(0)})(s)dsdt$$

$$=\frac{U_0}{CL_0}\int_{kT_0}^{t}\int_{kT_0}^{t}\sin(\omega_0\theta)d\theta ds-\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{U_0}{C}\int_{kT_0}^{t}\int_{kT_0}^{t}\sin(\omega_0\theta)d\theta ds$$

$$+\frac{U_0}{CL_0}\int_{kT_0}^{t}\int_{kT_0}^{t}\sin(\omega_0\theta)d\theta ds+\frac{U_0g_1^{(n)}}{C}\int_{kT_0}^{t}\sin(\omega_0 s)ds-\frac{U_0^3g_3^{(n)}}{C}\int_{kT_0}^{t}\sin^3(\omega_0 s)ds$$

$$-\left(\frac{t-kT_0}{T_0}-\frac{1}{2}\right)\left(\frac{U_0}{CL_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\sin\left(\omega_0\theta\right)d\theta ds-\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{U_0}{C}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\sin\left(\omega_0\theta\right)d\theta ds\right.$$

$$\left.+\frac{U_0}{CL_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\sin\left(\omega_0\theta\right)d\theta ds+\frac{U_0 g_1^{(n)}}{C}\int_{kT_0}^{(k+1)T_0}\sin\left(\omega_0 s\right)ds-\frac{U_0^3 g_3^{(n)}}{C}\int_{kT_0}^{(k+1)T_0}\sin^3\left(\omega_0 s\right)ds\right)$$

$$-\frac{1}{T_0}\left(\frac{U_0}{CL_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\int_{kT_0}^{s}\sin\left(\omega_0\theta\right)d\theta dsdt-\left(\frac{2}{L_0}+\frac{1}{L}\right)\frac{U_0}{C}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\int_{kT_0}^{s}\sin\left(\omega_0\theta\right)d\theta dsdt\right.$$

$$+\frac{U_0}{CL_0}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\int_{kT_0}^{s}\sin\left(\omega_0\theta\right)d\theta dsdt$$

$$\left.+\frac{U_0 g_1^{(n)}}{C}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\sin\left(\omega_0 s\right)dsdt-\frac{U_0^3 g_3^{(n)}}{C}\int_{kT_0}^{(k+1)T_0}\int_{kT_0}^{t}\sin^3\left(\omega_0 s\right)dsdt\right).$$

Principal Remark. Once we have found the voltages in view of (1) we solve the equations with respect to $i_n(t)$:

$$\frac{1}{L_0}\frac{di_n(t)}{dt}=u_n(t)-u_{n+1}(t)\Rightarrow\ i_n(t)=L_0\int_0^t\left[u_n(t)-u_{n+1}(t)\right],$$

where $u_n(t),u_{n+1}(t)$ are already known functions.

The currents are periodic functions too:

$$i_n(t+T_0)=L_0\int_0^{t+T_0}\left[u_n(t)-u_{n+1}(t)\right]=L_0\int_0^t\left[u_n(t)-u_{n+1}(t)\right]+L_0\int_t^{t+T_0}\left[u_n(t)-u_{n+1}(t)\right]$$

$$=L_0\int_0^t\left[u_n(t)-u_{n+1}(t)\right]+L_0\int_0^{T_0}\left[u_n(t)-u_{n+1}(t)\right]=L_0\int_0^t\left[u_n(t)-u_{n+1}(t)\right]=i_n(t).$$

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