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Solutions of the Klein-Gordon Equation with the Hulthen Potential Using the Frobenius Method

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Abstract A generalized series is used to obtain bounded solutions of the Klein Gordon equation using the Frobenius method. For some examples used, we obtain similar results to that using Nikiforov Uvarov method (NU). Our approach is very simple and can solve many problems where the Nikiforov Uvarov method (NU) fails.

Keywords Klein Gordon Equation, Frobenius Method, Nikiforov Uvarov method, Bound State Solutions

1. Introduction

Many of the physical phenomena of nature are characterized by some basic differential equations. For example, quantum mechanical phenomena are described by Schrodinger's equation, which dictates the dynamics of some quantum systems represented by a Hamiltonian operator. One is primarily interested in finding all eigenvalues and eigenstates of such Hamiltonians. As a consequence, finding a large class of analytically exactly solvable quantum systems is an important goal and this search has already been initiated by Schrodinger using the factorization method [1, 2]. Recently, there has been renewed interest in solving simple quantum mechanical systems within the framework of the Nikiforov-Uvarov (NU) method [3]. This algebraic technique is based on solving the second-order linear differential equation which has been used successfully to solve Schrodinger, Dirac, Klein-Gordon and Duffin-Kemmer-Petiau wave equations in the presence of some well-known central and non-central potentials [4-13]. In this paper, our focus is to deal with the Klein Gordon equation with Hulthen potential.

This paper is organized as follows: after a brief introduction in section 1, we give the review of the NU method and in section 2, we present the Frobenius method. In section 3 we solve the Klein Gordon equation by the NU method and in section 4 we use the Frobenius method to solve the same equation. Finally the two solutions obtained are compared and conclusions drawn.

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2. Review of the Nikiforov-Uvarov Method (NU)

The Nikiforov-Uvarov (NU) method reduces the second order linear differential equation to a generalized equation of hypergeometric type. Using an appropriate coordinate transformation $S= S(r)$, the equation takes the form:

$$\psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \qquad (1)$$

Where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials of at most second degree and $\bar{\tau}(s)$ is a first degree polynomial.

By taking the following factorization

$$\psi(s)=\phi(s)y(s) \qquad (2)$$

Equation (1) reduces to the hypergeometric type equation in the form:

$$\sigma(s)y''(s)+\tau(s)y(s)+\lambda y(s)=0 \qquad (3)$$

Where

$$\bar{\tau}(s)=\bar{\tau}(s)+2\pi(s) \qquad (4)$$

Ans satisfies the condition

$$\tau'(s)\langle 0 \text{ and } \pi(s)=\sigma(s)\frac{d}{ds}[\ln\phi(s)] \qquad (5)$$

The parameter λ, is defined by:

$$\lambda=\lambda_n=-n\tau'(s)-[\frac{n(n-1)}{2}\sigma''(s)];n=0,1,2.... \qquad (6)$$

To determine the eigenvalues of energy, we need first to determine λ by using the first derivative of $\pi(s)$ and defining by:

$$k = \lambda - \pi'(s) \tag{7}$$

By solving the resulting quadratic equation for $\pi(s)$, we obtain the following expression:

$$\pi(s) = (\frac{\sigma'-\bar{\tau}}{2}) \pm \sqrt{(\frac{\sigma'-\bar{\tau}}{2})^2 - \bar{\sigma} + k\sigma} \tag{8}$$

The determination of k is the essential point in the calculation of $\pi(s)$ and it can be obtained by setting the discriminant of the square root to zero.

The Wave function in the relation (2) can be determined using the Rodrigue's relation:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{9}$$

Polynomial solutions $y_n(s)$ are given by:

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n}[\sigma^n(s)\rho(s)] \tag{10}$$

B_n is the normalization constant and the weight function $\rho(s)$ satisfies the following relation:

$$\frac{d}{ds}[\sigma(s)\rho(s)] = \tau(s)\rho(s) \tag{11}$$

3. Review of the Frobenius Method

This method finds the solutions of a differential equation in the form of series, either a whole series, a Laurent series, or even a series involving contribute exhibitors. The difference between these situations is the properties of regularity of the equation coefficients. To do this you must put the equation in the form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{12}$$

Suppose a regular singular point x_0, singular functions $P(x)$ and $Q(x)$ and using the Fuck's theorem, we can write the solutions of the differential equation in the form:

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r} \tag{13}$$

The indicial equation is obtained for

$$r(r-1) + P(0)r + Q(0) = 0 \tag{14}$$

For each found values r, we determine the values a_k and then the solutions of the differential equation.

4. Application of the Nikiforov-Uvarov (NU)

Let us consider the following Klein Gordon equation in polar coordinates. The three dimensional radial wave equation of the KGE is written as

$$\frac{d^2R_n(r)}{dr^2} + [(E - V(r))^2 - (M + S(r))^2]R_n(r) = 0 \tag{15}$$

Where the wave function is $\varphi(r;\theta;\alpha) = \frac{R}{r} Y_m(\theta;\alpha)$

E is the relativistic energy, M the mass of the particle's spin-zero and $\psi_n(r) = {R_n(r)}/{r}$ the wave function. The Hulthen potential is given by $V(r) = -\dfrac{Ae^{-\frac{2}{b}r}}{b^2(1-e^{-\frac{2}{b}r})}$.

Using the condition S (r) = V (r), the radial Klein-Gordon equation becomes:

$$\frac{d^2R_n(r)}{dr^2} + [(E^2 - M^2) + 2\frac{(E+M)Ae^{-\frac{2}{b}r}}{b^2(1-e^{-\frac{2}{b}r})}]R_n(r) = 0 \tag{16}$$

Taking into account the transformation $s(r) = e^{-\frac{2}{b}r}$, the equation (16) becomes:

$$\frac{d^2R_n(s)}{ds^2} + \frac{1-s}{s(1-s)}\frac{dR_n(s)}{ds} + \frac{1}{s^2(1-s)^2} \tag{17}$$
$$[-\varepsilon_n{}^2 s^2 + (-\beta^2 + 2\varepsilon_n{}^2)s + \beta^2 - \varepsilon_n{}^2]R_n(s) = 0$$

with $\varepsilon^2 = \dfrac{b^2(E^2 - M^2)}{4}$, $\beta^2 = \dfrac{2A(E+M)}{4}$ and

$$-\varepsilon_n{}^2 = \varepsilon^2 - \beta^2 \tag{18}$$

Comparing equation (17) to equation (1), we get the following polynomials:

$$\bar{\tau}(s) = 1 - s, \sigma(s) = s(1-s), \sigma^2(s) = s^2(1-s)^2,$$
$$\bar{\sigma}(s) = -\varepsilon_n{}^2 s^2 + (-\beta^2 + 2\varepsilon_n{}^2)s + \beta^2 - \varepsilon^2 \tag{19}$$

By replacing these polynomials by their values in the equation (8), one obtains:

$$k = \beta^2 \pm \sqrt{\varepsilon_n{}^2 - \beta^2} \tag{20}$$

$$\pi(s) = -\frac{s}{2} \pm \frac{1}{2}[(\sqrt{\varepsilon_n{}^2 - \beta^2} + \frac{1}{2})s - 2\sqrt{\varepsilon_n{}^2 - \beta^2}] \tag{21}$$

To check the conditions of the polynomial $\tau(s) = \bar{\tau}(s) + 2\pi(s)$, we will take as values:

$$k = \beta^2 - \sqrt{\varepsilon_n{}^2 - \beta^2} \quad (22)$$

$$\pi(s) = -\frac{s}{2} - \frac{1}{2}[(\sqrt{\varepsilon_n{}^2 - \beta^2} + \frac{1}{2})s - 2\sqrt{\varepsilon_n{}^2 - \beta^2}] \quad (23)$$

If we combine these with $\lambda = k + \pi'$ defined in equation (7), we obtain the following expressions:

$$\tau(s) = 1 - 2s - (\sqrt{\varepsilon_n{}^2 - \beta^2} + \frac{1}{2})s + 2\sqrt{\varepsilon_n{}^2 - \beta^2} \quad (24)$$

And

$$\lambda = \beta^2 - \frac{3}{4} - \frac{3}{2}\sqrt{\varepsilon_n{}^2 - \beta^2}] \quad (25)$$

Using equation (6), we have:

$$\lambda_n = n^2 + \frac{3}{2}n + n\sqrt{\varepsilon_n{}^2 - \beta^2}] \quad (26)$$

Using the condition $\lambda = \lambda_n$, i.e. tying the equations (25) and (26), one obtains the eigenvalues of ε_n as:

$$\varepsilon_n = \beta^2 + (-n + \frac{4\beta^2 - 3}{2(2n+3)})^2 \quad (27)$$

Equations (18) and (27), one obtains the eigenvalues of energy given by:

$$E = \frac{-2A(2AM - N_1) \pm \sqrt{4AMb(-bAN_2 + bN_1N_2 + AN_2M) + b^2N_2(bN_2M^2 - N_1{}^2)}}{b^2N_2 + 4A^2} \quad (28)$$

with $N_1 = 4n^2 + 6n + 3$ and $N_2 = 4n^2 + 12n + 9$

The polynomial solution of the hypergeometric function y (s), depends on the determination of the function $\rho(s)$. Using the differential equation (11) we have:

$$\rho(s) = s^{2\nu}.(1-s)^{\nu - \frac{1}{2}} \text{ where } \nu = \sqrt{\varepsilon_n{}^2 - \beta^2} \quad (29)$$

By substituting equation (29) in Rodrigue's relations given in equation (10), one obtains:

$$y(s) = s^{-2\nu}.(1-s)^{-\nu + \frac{1}{2}} B_n \frac{d^n}{ds^n}[s^{n+2\nu}x(1-s)^{n+\nu-\frac{1}{2}}] \quad (30)$$

Equation (30) may come in the form of Jacobi polynomials [23] according to the following relationship:

$$y(s) = P_n^{(2\nu, \nu - \frac{1}{2})}(s) \quad (31)$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}$ is given by [23]:

$$P_n^{(\alpha,\beta)}(s) = \frac{(\alpha+1)_n}{n!} F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-s)) \quad (32)$$

We can therefore write the equation (31) in the following form:

$$y(s) = P_n^{(2\nu,\nu-\frac{1}{2})}(s) = \frac{(2\nu+1)_n}{n!} {}_2F_1(-n, n+3\nu+\frac{1}{2}; 2\nu+1; \frac{1}{2}(1-s)) \quad (33)$$

where $(2\nu+1)_n$ is the Pochhammer symbol [23].

For the determination of $\phi(s)$ we solve equation (9), using the calculated values for $\pi(s)$ and. $\sigma(s)$. we obtain:

$$\phi(s) = s^{\nu}.(1-s)^{\frac{1}{2}(\nu - \frac{3}{2})} \quad (34)$$

Replacing $\phi(s)$ and $y_n(s)$ by their values in the equation (2), we obtain:

$$R_n(s) = \frac{(2\nu+1)_n}{n!} N_n (1 - e^{-\frac{2}{b}r})^{\chi}.e^{-\frac{2}{b}\nu.r} {}_2F_1(-n, n+3\nu+\frac{1}{2}; 2\nu+1; \frac{1}{2}(1-e^{-\frac{2}{b}r})) \quad (35)$$

With $\chi = \frac{1}{2}(\nu - \frac{3}{2})$ and N_n is the normalizing constant.

5. Application of the Frobenius Method

Consider the same Klein-Gordon equation and the same Hulthen potential given in section 2. After development, we get the following equation:

$$\frac{d^2R_n(s)}{ds^2}+\frac{1}{s}\frac{dR_n(s)}{ds}+\frac{1}{s^2}[\frac{b^2(E^2-M^2)}{4}+\frac{2A(E+M)sb^2}{4b^2(1-s)}]R_n(s)=0 \tag{36}$$

Using the condition $\varepsilon^2=\frac{b^2(E^2-M^2)}{4}$, $\beta^2=\frac{2A(E+M)}{4}$, equation (36) becomes:

$$\frac{d^2R_n(s)}{ds^2}+\frac{1}{s}\frac{dR_n(s)}{ds}+\frac{1}{s^2}[\varepsilon^2+\frac{\beta^2 s}{1-s}]R_n(s)=0 \tag{37}$$

Ask $P(s)=1$ and $Q(s)=\varepsilon^2+\frac{\beta^2 s}{1-s}$, equation (37) becomes:

$$\frac{d^2R_n(s)}{ds^2}+\frac{P(s)}{s}\frac{dR_n(s)}{ds}+\frac{Q(s)}{s^2}R_n(s)=0 \tag{38}$$

By using the Fuck's theorem, we can write:

$$R_n(s)=\sum_{k=0}^{\infty}a_k s^{k+r}\text{, with } a_0\neq 0 \tag{39}$$

Differentiation gives us:

$$R_n''(s)=\sum_{k=0}^{\infty}(k+r-1)(k+r)a_k s^{k+r-2} \text{ and } R_n'(s)=\sum_{k=0}^{\infty}(k+r)a_k s^{k+r-1} \tag{40}$$

Equation (40) in (38), one obtains:

$$\sum_{k=0}^{\infty}[(k+r)^2+\varepsilon^2]a_k s^k-\sum_{k=0}^{\infty}[(k+r)^2-(\beta^2-\varepsilon^2)]a_k s^{k+1}=0 \tag{41}$$

By effecting a change of variable, we obtain:

$$r^2+\varepsilon^2+\sum_{n=1}^{\infty}[[(n+r)^2+\varepsilon^2]a_n-[(n+r-1)^2-(\beta^2-\varepsilon^2)]a_{n-1}]s^n=0 \tag{42}$$

By solving the indicial equation $I=r^2+\varepsilon^2=0$, we obtain:

$$r=\pm\sqrt{\varepsilon_n{}^2-\beta^2}=\pm\nu \tag{43}$$

For $r=\nu$, we have:

$$a_n=\prod_{i=1}^{n}\frac{(\nu+i-1)^2-\varepsilon_n{}^2}{(\nu+i)^2+\varepsilon^2}a_0\text{, n=1, 2, 3….} \tag{44}$$

So it gets a representation of the solution.

$$a_k=\prod_{i=1}^{k}\frac{(\nu+i-1)^2-\varepsilon_n{}^2}{(\nu+i)^2+\varepsilon^2}a_0\text{, k=1, 2, 3} \tag{45}$$

Using the relations (36) and (43), we obtain the energy E eigenvalue associated with the wave function. We can express the solutions obtained based on the Jacobi polynomial: this result is more accurate. The coefficients of the solution being assessed explicitly, we seek bounded solutions. We will only retain the negative value. In one of our papers in preparation, an explicit relationship between the solutions obtained by Nikiforov and Frobenius method is established. We also establish that the coefficients of the solution by the Frobenius method may be obtained by the differential transform method (DTM).

6. Conclusions

In this paper we have constructed exact solutions of the Klein Gordon equations by the NU and the Frobenius method. The energy eigenvalues are estimated and the coefficients of the Frobenius series estimated: The exact relation between the solutions obtained by the two methods is to be established in a fort coming paper

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