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Lagrangian for the Spinor Field in the Foldy – Wouthuysen Representation

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Abstract The relativistic invariant Lagrange approach for the spinor field in the Foldy–Wouthuysen representation is constructed. In the procedure of postulating the Lagrangian, the analogy with the classical mechanics of the system with the arbitrary number degrees of freedom is used. Conservation laws in the terms of particle and antiparticle quantum-mechanical momentum-spin amplitdes are presented. 24 conserved quantities for the free spinor field are given. Ten of them are consequences of the Poincaré symmetry. 12 additional conservation laws are the consequences of the fact that in the Foldy–Wouthuysen representation spin and orbital parts of the Poincaré angular momentum are conserved independently. The conserved quantities of charge and of particle number are found as well.

Keywords Spinor Field, Foldy–Wouthuysen Representation, Lagrange Approach, Noether Theorem, Conservation Laws

1. Introduction

Among the goals of this article the following two main points can be noted.

(i) Construction of a mathematically well-defined Lagrangian, for which the Foldy–Wouthuysen(FW) equation [1] follows from the Euler–Lagrange equations.

(ii) Application of the Lagrangian under consideration to the Noether analysis of the dynamical invariants of the spinor field in the FW representation.

In spite of the fact that some advantages of the Dirac equation consideration in the FW representation are well known (see, e. g.,[1–6]), the acceptable Lagrange approach in this representation is not developed. The Lagrangian and the main conservation laws for the FW field ϕ can be found in the papers[7, 8]. However, the author of[7, 8] has suggested a non-standard formulation of the least action principle. The derivatives of the infinite order from the field functions were used. The representation of the operator $\omega = \sqrt{-\Delta + m^2}$ in the form of a series over the Laplace operator powers was used.

Below we suggest the standard formulation of the least action principle and of the Lagrange approach to a free spinor field in the FW representation. Unlike[7, 8], our consideration is based[9] on the mathematically well-defined definition of the nonlocal (pseudodifferential) operator $\omega = \sqrt{-\Delta + m^2}$ and its functions. We consider the integral form of such operators in the momentum representation of the Schwartz space $\mathrm{S}^{3,4}$.

The Noether analysis of the conservation laws for the spinor field in the Dirac theory is also a non-trivial problem. This results from the fact that the spin part of the total angular momentum generator does not commute with the Dirac Hamiltonian. Therefore, derivation of the spin conservation law in the Dirac theory requires additional non-Noether efforts even in obtaining the third component of the spin conserved quantity[10]. The special efforts in derivation of the spin conservation law can be found in[11] as well.

The important advantage of the FW representation[1–3] is commutation of the spin operator with the FW Hamiltonian itself (the operator of angular momentum commutes here with the Hamiltonian as well). Therefore, having a well-defined Lagrangian for the free spinor field in the FW representation, we are able to find not only the 10 Poincaré conservation laws, but also 12 additional conservation laws. The additional conserved quantities are the Noether consequences of the following 12 operators: 3 of spin, 3 of pure Lorentz spin, 3 of angular momentum and 3 of pure Lorentz angular momentum operators, which in the FW representation are the independent symmetry operators. The conserved quantities of charge and number of particles are found. as well.

The Lagrange approach for the spinor field under consideration is interesting for the construction of the quantum electrodynamics in the FW representation. The version of such quantum electrodynamics was suggested recently in[12–14]. In our approach to such theory, we start here from the construction of the Lagrange formalism for the free spinor field and from the Noether analysis of the corresponding dynamical invariants.

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The ***summary of our motivation*** is very simple. We consider the construction of the Lagrange approach for the spinor field in the FW representation because this problem was not solved before (even after 60 years of the introduction of the FW equation and corresponding representation). Moreover, the known Noether (based on the Noether theorem) analysis of the conservation laws for the free spinor field and for the Dirac equation is not sufficient (even in the best investigations like[10], where even 10 main conservation laws, which follow from the Poincaré symmetry, are not described completely). Below we hope to be able to fill this gap.

2. Definitions, Notations and Some Justifications

We use the standard relativistic notations. The metric tensor is given by

$$g=(g^{\mu}_{\nu})=\mathrm{diag}\, g(+---),\quad g^{\mu\nu}=g_{\mu\nu}=g^{\mu}_{\nu}. \quad (1)$$

Therefore, in the Minkowski space-time M(1,3), the Lorentz 4-vector $x\equiv(x^{\mu})\in \mathrm{M}(1,3)$ foursquare has the form

$$x^2\equiv x^{\mu}x^{\nu}g_{\mu\nu}=x^{\mu}x_{\mu}=x_0^{\ 2}-\vec{x}^2,\qquad \vec{x}\equiv(x^j)\in \mathrm{R}^3\subset \mathrm{M}(1,3). \quad (2)$$

The system of units $\hbar=c=1$ is taken. Here the Greek indices are changed in the range 0,1,2,3 ≡ 0,3, Latin — 1,3, the summation over a twice repeated index is implied. The Dirac γ^{μ} matrices in the standard Pauli–Dirac representation are used:

$$\gamma^0=\begin{vmatrix} \mathrm{I} & 0\\ 0 & -\mathrm{I}\end{vmatrix},\ \gamma^j=\begin{vmatrix} 0 & \sigma^j\\ -\sigma^j & 0\end{vmatrix}, \quad (3)$$

where σ^j are the standard Pauli matrices.

The analysis of the relativistic invariance of an arbitrary physical model demands as the first step consideration of its invariance with respect to the proper ortochronous Lorentz $\mathrm{L}^{\uparrow}_{+}=\mathrm{SO}(1,3)=\{\Lambda=(\Lambda^{\mu}_{\nu})\}$ and Poincaré $\mathrm{P}^{\uparrow}_{+}=\mathrm{T}(4)\times)\mathrm{L}^{\uparrow}_{+}\supset \mathrm{L}^{\uparrow}_{+}$ groups. This invariance in an arbitrary relativistic model is the allowance of the Einstein's relativity principle in the form of special relativity. Note that the mathematical correctness requires to consider the invariance mentioned above as that with respect to the universal coverings $\mathsf{L}=\mathrm{SL}(2,\mathrm{C})$ and $\mathsf{P}\supset\mathsf{L}$ of the groups $\mathrm{L}^{\uparrow}_{+}$ and $\mathrm{P}^{\uparrow}_{+}$, respectively.

Below we construct *a relativistic invariant Lagrange approach* for the spinor field in its canonical FW representation on the basis of the following principles. The quantum-mechanical rigged Hilbert space (both in the coordinate and momentum realizations of this space) is used. We do not appeal to the manifestly covariant conceptions and use some analogy with the Lagrange approach in the classical mechanics of systems with large (infinite) number of degrees of freedom. This analogy becomes complete in the case *of a start from the momentum* (more exactly – from the momentum-spin) realization of the state space of the FW field. The elements $\tilde{\phi}(t,\vec{k})$ of such space are linked with those $\phi(t,\vec{x})$ in the coordinate realization by the Fourier transformation:

$$\mathsf{H}^{3,4}\supset\phi(t,\vec{x})\to\tilde{\phi}(t,\vec{k})=\frac{1}{(2\pi)^{3/2}}\int d^3k e^{-i\vec{k}\vec{x}}\phi(t,\vec{x})\in\tilde{\mathsf{H}}^{3,4}, \quad (4)$$

where the square of the norm in the $\vec{k}$-realization is given by the Lebesgue measure

$$\|\tilde{\phi}\|^2=\int d^3k\tilde{\phi}^{\dagger}(t,\vec{k})\tilde{\phi}(t,\vec{k})<\infty\leftrightarrow\tilde{\phi}\in\tilde{\mathsf{H}}^{3,4}. \quad (5)$$

Here $\mathsf{H}^{3,4}$ is the 4-component quantum-mechanical Hilbert space in the coordinate or $\vec{x}$-realization

$$\mathsf{H}^{3,4}=\mathrm{L}_2(\mathrm{R}^3)\otimes\mathsf{C}^{\otimes4}=\{\phi\equiv(\phi^{\alpha}):\ \mathrm{R}^3\to\mathsf{C}^{\otimes4};\ \int d^3x|\phi(t,\vec{x})|^2<\infty\},\ \alpha=0,1,2,3, \quad (6)$$

inserted into the rigged Hilbert space

$$\mathsf{S}^{3,4}\subset\mathsf{H}^{3,4}\subset{}^{\times}\mathsf{S}^{3,4}, \quad (7)$$

and $\tilde{\mathsf{H}}^{3,4}$

$$\tilde{\mathsf{H}}^{3,4}=\{\tilde{\phi}\equiv(\tilde{\phi}^{\alpha})^4_{\alpha=1}:\ \mathrm{R}^3_{\vec{k}}\to\mathsf{C}^{\otimes4};\ \int d^3k\left|\tilde{\phi}(k)\right|^2<\infty\}, \quad (8)$$

is the corresponding Hilbert space in the momentum or $\vec{k}$-realization, inserted into *the rigged Hilbert space*

$$\tilde{\mathsf{S}}^{3,4}\subset\tilde{\mathsf{H}}^{3,4}\subset{}^{\times}\tilde{\mathsf{S}}^{3,4} \quad (9)$$

in the momentum realization.

It is necessary to appeal to the rigged Hilbert space (7), (9) due to the fact that the FW equation[1] has solutions belonging to the space of the Schwartz generalized functions ${}^{\times}\mathsf{S}^{3,4}$ (the details of the application of the rigged Hilbert space can be found in[15]). Therefore, we deal here with the following situation.

In the above formulae $\mathsf{S}^{3,4}$ is the space of the 4-component Schwartz test functions over the $\mathrm{R}^3\subset\mathrm{M}(1,3)$. The symbol «×» in ${}^{\times}\mathsf{S}^{3,4}$ means that the space of the Schwartz generalized functions ${}^{\times}\mathsf{S}^{3,4}$ is conjugated to the Schwartz test function space $\mathsf{S}^{3,4}$ by the corresponding topology (see, e. g.,[16]). Strictly speaking, the mathematical correctness of this consideration requires the calculations in the space ${}^{\times}\mathsf{S}^{3,4}$ of the generalized functions, i. e. with the application of a cumbersome functional analysis, to be made.

Nevertheless, let us take into account that the Schwartz test function space $\mathsf{S}^{3,4}$ in the triple (7) is *kernel*. This means that $\mathsf{S}^{3,4}$ is dense both in the quantum-mechanical space $\mathrm{H}^{3,4}$ and in the space of generalized functions $^{\times}\mathrm{S}^{3,4}$. Therefore, any physical state from $\mathrm{H}^{3,4}$ can be approximated with an arbitrary precision by the corresponding elements of the Cauchy sequence in $\mathrm{S}^{3,4}$, which converges to the given state in $\mathrm{H}^{3,4}$. Further, taking into account the requirement to measure the arbitrary value of the model with non-absolute precision, this means that all necessary calculations can be fulfilled within the Schwartz test function space $\mathrm{S}^{3,4}$ without any loss of generality.

Thus, we use below the Schwartz test function space $\mathrm{S}^{3,4}$.

3. Lagrangian for the Free Spinor Field in the Foldy–Wouthuysen Representation

Now therefore, the FW equation for the vectors $\tilde{\phi} \in \tilde{\mathrm{S}}^{3,4} \subset \tilde{\mathrm{H}}^{3,4}$ has the form

$$\left(i\partial_0 - \gamma^0\tilde{\omega}\right)\tilde{\phi}(t,\vec{k}) = 0,\ \tilde{\omega} \equiv \sqrt{\vec{k}^2 + m^2},\ \ \vec{k} \in \mathrm{R}^3_{\vec{k}},\quad (10)$$

Note that the space $\mathrm{S}^{3,4}$, as well as the rigged Hilbert space, is invariant with respect to the Fourier transformation. The complete analogy with the classical mechanics is as follows. The FW equation (10) is the first-order differential equation with respect to the time parameter *t* and continually infinite system of the algebraic equations with respect to the numerical dynamical variables $\vec{k}$.

Therefore, we consider the 4-component field

$$\tilde{\phi} = \tilde{\phi}(t,\vec{k}) = \left(\tilde{\phi}^{\alpha}_{\vec{k}}(t)\right) \quad (11)$$

as the field coordinate with continuous number of components with respect to the variables $\vec{k}$ and discrete number of components with respect to the variables $\alpha = \overline{1,4}$. Therefore, we construct *the Lagrange function* in the terms of corresponding numerical numbers in the form

$$\begin{aligned} L &= L(\tilde{\phi}, \tilde{\phi}^{\dagger}, \tilde{\phi}_{,0}, \tilde{\phi}^{\dagger}{}_{,0}) \\ &= \frac{i}{2}\left(\tilde{\phi}^{\dagger}(\tilde{\phi}_{,0} + i\gamma^0\tilde{\omega}\tilde{\phi}) - (\tilde{\phi}^{\dagger}{}_{,0} - i\tilde{\omega}\tilde{\phi}^{\dagger}\gamma^0)\tilde{\phi}\right) \quad (12) \\ &= \frac{i}{2}\left(\tilde{\phi}^{\dagger}\tilde{\phi}_{,0} - \tilde{\phi}^{\dagger}{}_{,0}\,\tilde{\phi} + 2i\tilde{\phi}^{\dagger}\tilde{\omega}\gamma^0\tilde{\phi}\right). \end{aligned}$$

The corresponding action (the functional $W\left[t;\tilde{\phi},\tilde{\phi}^{\dagger}\right]$ from the functions $\tilde{\phi}(t,\vec{k}),\ \tilde{\phi}^{\dagger}(t,\vec{k})$ and from the numerical variable $t \in (-\infty,\infty)$) is given by

$$W\left[t;\tilde{\phi},\tilde{\phi}^{\dagger}\right] \overset{\mathrm{df}}{=} \int d^3k L(t,\vec{k});\quad \tilde{\phi},\tilde{\phi}^{\dagger} \in \tilde{\mathrm{S}}^{3,4}. \quad (13)$$

Here $L(t,\vec{k})$ is found from the prime Lagrangian (12) by substituting the functions $\tilde{\phi}(t,\vec{k}) = (\tilde{\phi}^{\alpha}(t,\vec{k})),\ \tilde{\phi}^{\dagger}(t,\vec{k})$ and their time-derivatives $\partial_0\tilde{\phi}(t,\vec{k}),\ \partial_0\tilde{\phi}^{\dagger}(t,\vec{k})$ belonging to the $\tilde{\mathrm{S}}^{3,4}$, instead of the numerical variables $(\tilde{\phi},\ \tilde{\phi}^{\dagger},\ \tilde{\phi}_{,0},\ \tilde{\phi}^{\dagger}{}_{,0})$, respectively.

It is easy to verify that for the action (13) the Euler–Lagrange equations

$$\begin{aligned} \frac{\delta W}{\delta\tilde{\phi}^{\dagger}} &\equiv \frac{\partial L}{\partial\tilde{\phi}^{\dagger}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\tilde{\phi}^{\dagger}{}_{,0}} = 0, \\ \frac{\delta W}{\delta\tilde{\phi}} &\equiv \frac{\partial L}{\partial\tilde{\phi}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\tilde{\phi}_{,0}} = 0, \end{aligned} \quad (14)$$

coincide with the FW equation (10) and its conjugated equation for the $\tilde{\phi}^{\dagger}(t,\vec{k})$.

The Lagrange approach in any other realization of the space $\mathrm{H}^{3,4}$, which is related to the complete set of the dynamical variables in $\mathrm{H}^{3,4}$ as the diagonal operators, is constructed similarly. For example, in the $\vec{x}$-realization of the space $\mathrm{H}^{3,4}$ the corresponding Lagrangian and action are given by

$$\begin{aligned} L(t,\vec{x}) &= \frac{i}{2}[\phi^{\dagger}(t,\vec{x})\partial_0\phi(t,\vec{x}) - (\partial_0\phi^{\dagger}(t,\vec{x}))\phi(t,\vec{x}) \quad (15) \\ &+ 2i\phi^{\dagger}(t,\vec{x})\omega\gamma^0\phi(t,\vec{x})],\quad \omega = \sqrt{-\Delta + m^2}, \end{aligned}$$

$$W\left[t;\phi,\phi^{\dagger}\right] \overset{\mathrm{df}}{=} \int d^3kL(t,\vec{k});\quad \phi,\phi^{\dagger} \in \mathrm{S}^{3,4}, \quad 16)$$

where the function $(\omega\phi)(t,\vec{x})$ (the result of the action of the pseudo-differential operator $\omega = \sqrt{-\Delta + m^2}$ on the function $\phi \in \mathrm{H}^{3,4}$) is determined by the formula

$$(\omega\phi)(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\vec{k}\vec{x}}\tilde{\omega}\tilde{f}(t,\vec{k}). \quad (17)$$

On the relativistic invariance of the theory. Note that, as usually in the canonical FW representation, we deal here with a special role of the time *t* and use the non-covariant objects $d^3k,\ d^3x$, etc. In other words, the time *t* plays a specific isolated role in all the expressions and formulae (in the formula (13) as well). The time *t* is the parameter in all canonical FW-like models (see, e.g., [17, 18] for more details). Nevertheless, the Lagrange approach under consideration is the relativistic invariant in the following sense. The set $\{\phi\} \subset \mathrm{H}^{3,4}$ of extremals for the action (14) (the set $\{\phi\}$ of solutions

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^{3/2}}\int d^3k\{e^{-ikx}[a^-_+(\vec{k})\mathrm{d}_1 + a^-_-(\vec{k})\mathrm{d}_2] \\ &+ e^{ikx}[a^{*+}_-(\vec{k})\mathrm{d}_3 + a^{*+}_+(\vec{k})\mathrm{d}_4]\}, \end{aligned} \quad (18)$$

$$\mathrm{d}_1 = \begin{vmatrix}1\\0\\0\\0\end{vmatrix},\ \ \mathrm{d}_2 = \begin{vmatrix}0\\1\\0\\0\end{vmatrix},\ \ \mathrm{d}_3 = \begin{vmatrix}0\\0\\2\\0\end{vmatrix},\ \ \mathrm{d}_4 = \begin{vmatrix}0\\0\\0\\3\end{vmatrix}, \quad (19)$$

of the Euler–Lagrange equations for $\phi(t,\vec{x})$), which coincide with the FW equation

$$\left(i\partial_0 - \gamma^0\omega\right)\phi(t,\vec{x}) = 0, \ \omega \equiv \sqrt{-\Delta + m^2}, \quad (20)$$

is P -invariant. In more detail, the set $\{\phi\} \subset \mathrm{H}^{3,4}$ is invariant with respect to the unitary in the $\mathrm{H}^{3,4}$ representation of the group P determined by the Hermitian P -generators

$$\begin{cases} p_0 = \gamma^0\omega = H^{\mathrm{F}}, \ p_l = i\partial_l, \ j_{ln} = m_{ln} + s_{ln}, \\ j_{0k} = x_0 p_k + \gamma^0\left(-x_k\omega + \dfrac{ip_k}{2\omega} + \dfrac{(\vec{s}\times\vec{p})_k}{\omega+m}\right), \end{cases} \quad (21)$$

known from[2]. Here spin operators are given by

$$\vec{s} = (s^1, s^2, s^3) \equiv (s^{23}, s^{31}, s^{12}), \ s^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]. \quad (22)$$

Some more details of the P -invariance of the canonical FW-like models are considered in[17, 18].

In solution (18), the coefficients $a_+^-(\vec{k}),\ a_-^-(\vec{k}),\ a_-^+(\vec{k}),\ a_+^+(\vec{k})$ are the quantum-mechanical momentum-spin amplitudes, i. e. the amplitudes of probability distribution over the eigen values of the fermionic Hermitian stationary complete set of operators of the momentum $p^\ell = -i\partial_\ell$, spin projection s^3 and sign of the charge $g = -\gamma^0$. In $a_+^-(\vec{k}),\ a_-^-(\vec{k}),\ a_-^+(\vec{k}),\ a_+^+(\vec{k})$, the lower indices are the signs of the spin projections and the upper indices are those of the sign of the charge, respectively.

Remark 3.1. In the above mentioned stationary complete set of operators $(\vec{p},\ s^3,\ g)$ the spin projection operator is chosen in the representation

$$\hat{\vec{s}} = \frac{1}{2}\begin{vmatrix} \vec{\sigma} & 0 \\ 0 & -C\vec{\sigma}C \end{vmatrix} \rightarrow s^3 = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad (23)$$

from the relativistic canonical quantum mechanics[17, 18], in which the physical interpretation is most evident and pure. In formula (23), C is the comlex conjugation operator, i. e. the involution operator in $\mathrm{H}^{3,4}$.

4. Dynamical Invariants of the Spinor Field in the Foldy–Wouthuysen Representation

The Noether theorem for an arbitrary Hermitian operator *q* of invariance of equation (20) gives the following general formula of deriving the conservation law *Q(t)=Const*:

$$q \rightarrow Q = \int d^3x \phi^\dagger(t,\vec{x}) q \phi(t,\vec{x}) = \int d^3k \tilde{\phi}^\dagger(t,\vec{k}) \tilde{q} \tilde{\phi}(t,\vec{k}), \quad (24)$$

where $\tilde{q}$ is the Fourier transform in $\tilde{\mathrm{H}}^{3,4}$ of the operator *q* from $\tilde{\mathrm{H}}^{3,4}$. The Noether formula (24) coincides with that for the mean value in $\tilde{\mathrm{H}}^{3,4}$ (or in $\tilde{\mathrm{H}}^{3,4}$) of an arbitrary observable *q* from the algebra of observables. The details of the application of the Noether theorem for the spinor field can be found in[19].

Thus, in order to find the main conservation laws for the spinor field in the FW representation, it is sufficient to substitute the 10 generators $q = (p,\ j)$ (21), together with the solutions (18), in the formula (24) and to fullfil the necessary calculations.

Note that, contrary to the P -generators

$$\begin{cases} p_0^{\mathrm{ind}} = \gamma^0\left(\vec{\gamma}\vec{p} + m\right) \equiv H, \quad \hat{p}_k = i\partial_k, \\ j_{mn} = m_{mn} + s_{mn}, \quad j_{0k}^{\mathrm{ind}} = x_0 p_k - x_k H + s_{0k} \end{cases} \quad (25)$$

for the Dirac equation

$$(i\partial_0 - H)\psi(t,\vec{x}) = 0, \ H \equiv \gamma^0\left(\vec{\gamma}\vec{p} + m\right), \quad (26)$$

of induced P -representation (i. e. in the Pauli – Dirac representation), for the P -generators (21) in the FW representation, $\vec{x} = (x^j)$ is the natural coordinate and $\vec{s}$ (22) is the natural spin. Therefore, each element from the set of generators (21) has the natural physical sense. In the FW representation, the spin $\vec{s}$ (22) commutes with the operator of the FW equation (20) and is conserved. The orbital angular momentum is conserved as well. It is also valid both for the space angular momentum and for the proper Lorentz momentum (boost). Therefore, here we have additional, as compared to the Dirac theory, conservation laws M_{ln}, M_{0l}, S_{ln}, S_{0l} of the orbital and spin parts of the total Poincaré angular momentum.

Remark 4.1. Strictly speaking, the 12 conservation laws M_{ln}, M_{0l}, S_{ln}, S_{0l} can be found on the basis of two methods. For the Dirac theory (i. e. in the Pauli – Dirac representation), these conservation laws can be found as well. Nevertheless, in the Pauli – Dirac representation, the corresponding symmetry generators $m_{\alpha\beta}$, $s_{\alpha\beta}$ are nonlocal and are given by much more cumbersome expressions, see, e. g., expression

$$\vec{s}^{\,\mathrm{PD}} = \vec{s} - \frac{[\vec{\gamma}\times\nabla]}{2\omega} + i\frac{\nabla\times(\vec{s}\times\nabla)}{\omega(\omega+m)} \quad (27)$$

for the natural spin operator in the Pauli – Dirac representation. The spin operator (27) commutes with the operator of the Dirac equation (26), but the corresponding operator $\vec{s}$ (22), which commutes with the operator of the FW equation (20), has much simpler and more attractive form. Moreover, consideration in the FW representation in the rigorous canonical quantum mechanical principles and conceptions is given. Therefore, below we consider the dynamical variables for the spinor field in the canonical FW representation.

According to the Noether formula (24), the 10 main conservation laws for the spinor field, which are the consequences of the P -symmetry (21) of the FW equation (20), i. e., the consequences of the generators $q=\left(p_0, p_l, j_{ln}, j_{0l}\right)$ (21), have the form

$$P_0=\int d^3x\phi^\dagger(x)\gamma^0\omega\phi(x), \tag{28}$$

$$P_\ell=\int d^3x\phi^\dagger(x)i\partial_\ell\phi(x), \tag{29}$$

$$J_{mn}=\int d^3x\left[\phi^\dagger(x)(x_m p_n-x_n p_m+s_{mn})\phi(x)\right], \tag{30}$$

$$J_{0l}=\int d^3x\{\phi^\dagger(x)[x_0 p_l+\gamma_0(-x_l\omega +\frac{ip_l}{2\omega}+\frac{(\vec{s}\times\vec{p})_l}{\omega+m})]\phi(x)\}. \tag{31}$$

The conservation laws (28)–(31) are the consequences of the generators of translations and rotations. Therefore, they are the consequences of the homogeneity and isotropy of the space-time M(1,3), in which the field $\phi(t,\vec{x})$ is evolved.

Note that the conserved quantities (30) and (31) are the sums of quantities which are conserved itself. They are the sums of the orbital and spin angular momenta. Therefore, we have 12 additional conservation laws:

$$M_{mn}=\int d^3x\left[\phi^\dagger(x)(x_m p_n-x_n p_m)\phi(x)\right], \tag{32}$$

$$S_{mn}=\int d^3x\left[\phi^\dagger(x)s_{mn}\phi(x)\right], \tag{33}$$

$$M_{0l}=\int d^3x\{\phi^\dagger(x)[x_0 p_l+\gamma_0(-x_l\omega+\frac{ip_l}{2\omega})\phi(x)]\}, \tag{34}$$

$$S_{0l}=\int d^3x[\phi^\dagger(x)\gamma^0\frac{(\vec{s}\times\vec{p})_l}{\omega+m}\phi^\dagger(x)]. \tag{35}$$

The demonstrative physical and mathematical sense of the principal conservation laws (28)–(35) is most evident in the terms of the momentum-spin amplitudes $a_+^-(\vec{k}),\ a_-^-(\vec{k}),\ a_-^+(\vec{k}),\ a_+^+(\vec{k})$, which determine the general solution (18) of the FW equation (20), i. e. in the terms of quantum-mechanical particle-antiparticle (electron-positron) amplitudes.

The following theorem is valid.

Theorem 4.1. In the terms of the momentum-spin amplitudes $a_+^-(\vec{k}),\ a_-^-(\vec{k}),\ a_-^+(\vec{k}),\ a_+^+(\vec{k})$ the conservation laws (28)–(31) have the form

$$q\to Q=\int d^3k f^\dagger\tilde{q}f, \tag{36}$$

where the operators $\tilde{q}=\left(\tilde{p}_0, \tilde{p}_l, \tilde{j}_{ln}, \tilde{j}_{0l}\right)$, which act on the momentum-spin amplitudes

$$f=\begin{vmatrix} a_+^-\left(\vec{k}\right)\\ a_-^-\left(\vec{k}\right)\\ a_-^{*+}\left(\vec{k}\right)\\ a_+^{*+}\left(\vec{k}\right)\end{vmatrix},$$

$$f^\dagger=\left|a_+^{*-}\left(\vec{k}\right)\ a_-^{*-}\left(\vec{k}\right)\ a_-^{+}\left(\vec{k}\right)\ a_+^{+}\left(\vec{k}\right)\right|, \tag{37}$$

are given by

$$\tilde{q}=(\tilde{p}_0=\gamma^0\tilde{\omega},\ \tilde{p}_l=\gamma^0 k_l,\ \tilde{j}_{ln}=\tilde{x}_l k_n-\tilde{x}_n k_l+s_{ln},$$
$$\tilde{j}_{0l}=-\tfrac{1}{2}\{\ \tilde{x},\tilde{\omega}\ \}+\gamma_0\frac{(\vec{s}\times\vec{k})_l}{\tilde{\omega}+m}),\quad \tilde{x}_l\equiv -i\frac{\partial}{\partial k^l}. \tag{38}$$

Here the spin operators $s_{mn}=(\vec{s})$ are known from (22). The similar form (36) for the additional conservation laws (32)–(35) is also valid. The operators $\tilde{q}$ (38) as well as the operators

$$\tilde{q}^{\mathrm{orb}}=(\tilde{p}_0=\gamma^0\tilde{\omega},\ \tilde{p}_l=\gamma^0 k_l,$$
$$\tilde{m}_{ln}=\tilde{x}_l k_n-\tilde{x}_n k_l,\ m_{0l}=-\tfrac{1}{2}\{\tilde{x},\tilde{\omega}\}), \tag{39}$$

of the orbital set act in the Hilbert space of the quantum-mechanical momentum-spin amplitudes $\{\tilde{f}\}$ and satisfy the commutation relations

$$\left[\hat{p}_\mu,\hat{p}_\nu\right]=0,\ \left[\hat{p}_\mu,\hat{j}_{\rho\sigma}\right]=ig_{\mu\rho}\hat{p}_\sigma-ig_{\mu\sigma}\hat{p}_\rho, \tag{40}$$

$$\left[\hat{j}_{\mu\nu},\hat{j}_{\rho\sigma}\right]=$$
$$-i(g_{\mu\rho}\hat{j}_{\nu\sigma}+g_{\rho\nu}\hat{j}_{\sigma\mu}+g_{\nu\sigma}\hat{j}_{\mu\rho}+g_{\sigma\mu}\hat{j}_{\rho\nu}), \tag{41}$$

for the generators of the Lie algebra of the Poincaré group P in the manifestly covariant form.

Proof. The proof of the theorem is performed by the direct calculations of all necessary expressions and commutation relations.

Therefore, the list of the 22 conservation laws (10 principal Poincaré conserved quantities and 12 additional ones) in terms of the quantum-mechanical momentum-spin amplitudes is given by

$$P_0=\int d^3k[a_+^{*-}\left(\vec{k}\right)\tilde{\omega}a_+^-\left(\vec{k}\right)+a_-^{*-}\left(\vec{k}\right)\tilde{\omega}a_-^-\left(\vec{k}\right)$$
$$-a_-^+\left(\vec{k}\right)\tilde{\omega}a_-^{*+}\left(\vec{k}\right)-a_+^+\left(\vec{k}\right)\tilde{\omega}a_+^{*+}\left(\vec{k}\right)], \tag{42}$$

$$P_\ell=\int d^3k[a_+^{*-}\left(\vec{k}\right)k_\ell a_+^-\left(\vec{k}\right)+a_-^{*-}\left(\vec{k}\right)k_\ell a_-^-\left(\vec{k}\right)$$
$$-a_-^+\left(\vec{k}\right)k_\ell a_-^{*+}\left(\vec{k}\right)-a_+^+\left(\vec{k}\right)k_\ell a_+^{*+}\left(\vec{k}\right)], \tag{43}$$

$$M_{ln}=\int d^3k[a_+^{*-}(\vec{k})(\tilde{x}_l k_n-\tilde{x}_n k_l)a_+^-(\vec{k})$$
$$+a_-^{*-}(\vec{k})(\tilde{x}_l k_n-\tilde{x}_n k_l)a_-^-(\vec{k})$$
$$+a_-^{+}(\vec{k})(\tilde{x}_l k_n-\tilde{x}_n k_l)a_-^{*+}(\vec{k})$$
$$+a_+^{+}(\vec{k})(\tilde{x}_l k_n-\tilde{x}_n k_l)a_+^{*+}(\vec{k})], \tag{44}$$

$$S^1=S_{23}=\int d^3k\tilde{s}^1,\quad S^2=S_{31}=\int d^3k\tilde{s}^2,$$
$$S^3=S_{12}=\int d^3k\tilde{s}^3, \tag{45}$$

$$M_{0l}=\int d^3k[a_+^{*-}(\vec{k})\{\tilde{x}_l,\omega\}a_+^-(\vec{k})$$
$$+a_-^{*-}(\vec{k})\{\tilde{x}_l,\omega\}a_-^-(\vec{k})-a_-^{+}(\vec{k})\{\tilde{x}_l,\omega\}a_-^{*+}(\vec{k}) \tag{46}$$
$$-a_+^{+}(\vec{k})\{\tilde{x}_l,\omega\}a_+^{*+}(\vec{k})],$$

$$S_{0l}=-\int d^3k\frac{(\vec{\tilde{s}}\times\vec{k})_l}{\tilde{\omega}+m}, \tag{47}$$

where

$$\begin{cases}\tilde{s}^1=\tilde{s}_{23}=\frac{1}{2}[a_+^{*-}(\vec{k})a_-^-(\vec{k})+a_-^{*-}(\vec{k})a_+^-(\vec{k})\\ +a_-^{+}(\vec{k})a_+^{*+}(\vec{k})+a_+^{+}(\vec{k})a_-^{*+}(\vec{k})],\\ \tilde{s}^2=\tilde{s}_{31}=\frac{i}{2}[a_-^{*-}(\vec{k})a_+^-(\vec{k})-a_+^{*-}(\vec{k})a_-^-(\vec{k})\\ +a_+^{+}(\vec{k})a_-^{*+}(\vec{k})-a_-^{+}(\vec{k})a_+^{*+}(\vec{k})],\\ \tilde{s}^3=\tilde{s}_{12}=\frac{1}{2}[a_+^{*-}(\vec{k})a_+^-(\vec{k})-a_-^{*-}(\vec{k})a_-^-(\vec{k})\\ +a_-^{+}(\vec{k})a_-^{*+}(\vec{k})-a_+^{+}(\vec{k})a_+^{*+}(\vec{k})],\end{cases} \tag{48}$$

Certainly, the total angular momentum

$$J_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu} \tag{49}$$

is conserved as well. The quantity J (49) is the sum of the orbital and spin parts, each of them being conserved itself.

Note that this list of conserved quantities essentially differs from that given by[10] and by the authors of other publications on the subject, where the Dirac equation has been used. The difference is in conservation of the angular and spin quantities itself.

The expressions for the conservation laws for the particle doublet given above can be easily separated into the quantities for a particle and an antiparticle. Therefore, above we have also found the conservation laws for a single fermion (and a single antifermion).

The list of the principal conservation laws given above is not complete without the conserved quantities of field charge and the number of particles. Therefore, below we consider these conserved quantities as well.

We present the current in the FW representation by analogy with the current $\vec{j}^{\,\text{classic}}(t,\vec{x})=\rho(t,\vec{x})v(t,\vec{x})$ in the theory of a continuous medium. Taking into account that the velocity operator in the FW representation is given by

$$\vec{v}(\vec{x},\vec{p})\equiv i\left[\gamma^0\omega,\vec{x}\right]=\gamma^0\frac{\vec{p}}{\omega}, \tag{50}$$

we present the current vector in the form

$$j_{\mathrm{I}}^{\ell}=-i\left(\phi^{\dagger}\frac{\gamma^0}{\omega}\phi,_{\ell}-\frac{1}{\omega}\phi^{\dagger},_{\ell}\gamma^0\phi\right). \tag{51}$$

Moreover, as follows from the quantum-mechanical consideration[17, 18], the operator $\vec{p}/\omega$ is also some velocity. Therefore, in addition to (51) we have here another current vector

$$j_{\mathrm{II}}^{\ell}=-i\left(\phi^{\dagger}\frac{1}{\omega}\phi,_{\ell}-\frac{1}{\omega}\phi^{\dagger},_{\ell}\phi\right), \tag{52}$$

which corresponds to the velocity $\vec{p}/\omega$.

It is easy to show that 4-currents, which correspond to the current vectors (51), (52), obey the continuity equation

$$\partial_{\mu}j_{\mathrm{I,II}}^{\mu}=0 \tag{53}$$

if the relevant 0-components of 4-currents have the form

$$j_{\mathrm{I}}^0=\phi^{\dagger}\phi,\quad j_{\mathrm{II}}^0=\phi^{\dagger}\gamma^0\phi. \tag{54}$$

The Noether theorem formula (24) and theorem 4.1 give the results

$$Q_{\mathrm{I}}=e\int d^3x j_{\mathrm{I}}^0(x)=e\int d^3k[a_+^{*-}(\vec{k})a_+^-(\vec{k})$$
$$+a_-^{*-}(\vec{k})a_-^-(\vec{k})+a_-^{+}(\vec{k})a_-^{*+}(\vec{k})+a_+^{+}(\vec{k})a_+^{*+}(\vec{k})], \tag{55}$$

$$Q_{\mathrm{II}}=\int d^3x j_{\mathrm{II}}^0(x)=\int d^3k[a_+^{*-}(\vec{k})a_+^-(\vec{k})$$
$$+a_-^{*-}(\vec{k})a_-^-(\vec{k})-a_-^{+}(\vec{k})a_-^{*+}(\vec{k})-a_+^{+}(\vec{k})a_+^{*+}(\vec{k})], \tag{56}$$

which after the Fermi-quantization become the operators of the charge and the number of particles, respectively.

Therefore, altogether we have found 24 conservation laws for the free spinor field.

Note that the FW transformation[1], which transforms the Dirac equation and its solution ψ into the FW equation (20) and its solution ϕ (18) (and vice versa), gives $\phi^{\dagger}\phi\rightarrow\psi^{\dagger}\psi$, i.e. does not change the charge conservation law. Moreover, in the momentum representation in the terms of momentum-spin amplitudes, the FW and the Dirac conservation laws for the spinor field coincide between each other. The difference is in the procedure of their Noether calculation. In the Pauli – Dirac representation, where the symmetry operators of the orbital angular momentum and spin are much more cumbersome, this procedure is also much more cumbersome. It is evident from the comparison of the corresponding spin operators (22) and (27).

5. The Fermi-Quantization

In general, the procedure of quantization for the spinor field in the FW representation is similar to that in the standard Pauli – Dirac representation. The procedure is carried out by the standard canonical way, i.e. by applying the proposed Lagrangian (15), the field coordinate and the canonically conjugated field momentum. The requariements of the anticommutation relations for these canonically conjugated variables are reduced to the replacement of the quantum-mechanical amplitudes $a_{+}^{-}(\vec{k}),\ a_{-}^{-}(\vec{k}),\ a_{-}^{+}(\vec{k}),\ a_{+}^{+}(\vec{k})$ in the general solution (18) by the operators $\widehat{a}_{+}^{-}(\vec{k}),\ \widehat{a}_{-}^{-}(\vec{k}),\ \widehat{a}_{-}^{+}(\vec{k}),\ \widehat{a}_{+}^{+}(\vec{k})$ obeying the Fermi anticommutation relations. These operators determine the basis vectors in the Fock space $\mathrm{H}^{\mathrm{Fock}}$ of the quantized field $\widehat{\phi}$. After such replacement and redefinition as the normal products, the expressions P_{μ} (42), (43) and $J_{\mu\nu}$ (49) are transformed into the P -generators $\widehat{P}_{\mu},\widehat{J}_{\mu\nu}$ of the Lie algebra of the group P in the Fock space $\mathrm{H}^{\mathrm{Fock}}$. The P -generators $\widehat{P}_{\mu},\widehat{J}_{\mu\nu}$ in this space obey the manifestly covariant commutation relations (40), (41) and realize the unitary P -representation.

The operator of energy P_0 (42) becomes sign-determined

$$\begin{aligned}P_0=\int d^3k:[a_{+}^{*-}\left(\vec{k}\right)\tilde{\omega}a_{+}^{-}\left(\vec{k}\right)+a_{-}^{*-}\left(\vec{k}\right)\tilde{\omega}a_{-}^{-}\left(\vec{k}\right)\\-a_{-}^{+}\left(\vec{k}\right)\tilde{\omega}a_{-}^{*+}\left(\vec{k}\right)-a_{+}^{+}\left(\vec{k}\right)\tilde{\omega}a_{+}^{*+}\left(\vec{k}\right)]:=\\\int d^3k[a_{+}^{*-}\left(\vec{k}\right)\tilde{\omega}a_{+}^{-}\left(\vec{k}\right)+a_{-}^{*-}\left(\vec{k}\right)\tilde{\omega}a_{-}^{-}\left(\vec{k}\right)\\+a_{-}^{+}\left(\vec{k}\right)\tilde{\omega}a_{-}^{*+}\left(\vec{k}\right)+a_{+}^{+}\left(\vec{k}\right)\tilde{\omega}a_{+}^{*+}\left(\vec{k}\right)].\end{aligned}\tag{57}$$

Therefore, in an arbitrary state of the Fock space $\mathrm{H}^{\mathrm{Fock}}$ of the fermion-antifermion field $\widehat{\phi}$, the energy of the quantized spinor field is positive.

After the Fermi-quantization, the conserved quantities (55), (56) also become the operators in the space $\mathrm{H}^{\mathrm{Fock}}$ and are given by

$$\begin{aligned}\widehat{Q}_{\mathrm{I}}=e\int d^3k[\widehat{a}_{+}^{*-}\left(\vec{k}\right)\widehat{a}_{+}^{-}\left(\vec{k}\right)+\widehat{a}_{-}^{*-}\left(\vec{k}\right)\widehat{a}_{-}^{-}\left(\vec{k}\right)\\-\widehat{a}_{-}^{+}\left(\vec{k}\right)\widehat{a}_{-}^{*+}\left(\vec{k}\right)-\widehat{a}_{+}^{+}\left(\vec{k}\right)\widehat{a}_{+}^{*+}\left(\vec{k}\right)],\end{aligned}\tag{58}$$

$$\begin{aligned}\widehat{Q}_{\mathrm{II}}=\int d^3k[\widehat{a}_{+}^{*-}\left(\vec{k}\right)\widehat{a}_{+}^{-}\left(\vec{k}\right)+\widehat{a}_{-}^{*-}\left(\vec{k}\right)\widehat{a}_{-}^{-}\left(\vec{k}\right)\\+\widehat{a}_{-}^{+}\left(\vec{k}\right)\widehat{a}_{-}^{*+}\left(\vec{k}\right)+\widehat{a}_{+}^{+}\left(\vec{k}\right)\widehat{a}_{+}^{*+}\left(\vec{k}\right)].\end{aligned}\tag{59}$$

Therefore, $\widehat{Q}_{\mathrm{I}}$ is the operator of charge and $\widehat{Q}_{\mathrm{II}}$ is the operator of particle number.

Note that despite the non-covariant procedure of the field ϕ quantization, the resulting formalism of the quantized field $\widehat{\phi}$ is covariant and effectively coincides with the Wightman's axiomatic formulation of the quantized spinor field theory (for the modern consideration see, e. g.,[20]).

6. Conclusions

A new Lagrangian, for which the Euler–Lagrange equations and the standard variational least action principle lead to the Foldy–Wouthuysen equation, is suggested. The extended list of 24 conservation laws is found on the basis of the Noether theorem and the suggested Lagrangian.

Thus, we presented the new Lagrange approach for the free spinor field in the Foldy–Wouthuysen representation. We hope that our consideration of the main conservation laws for the free spinor field (in the framework of the Lagrange approach and on the basis of the Noether theorem) is more complete and consistent then the similar considerations of other authors. Our advantage lies in the start from the canonical Foldy–Wouthuysen representation.

The construction of the Lagrange approach and the Noether analysis of the field ϕ dynamical invariants is the first necessary step towards the construction of the quantum electrodynamics in the Foldy–Wouthuysen representation. The version of such quantum electrodynamics, which used the 8-component formalism and appealed to the negative mass of the antiparticle, was suggested recently in[12]. On the basis of the results considered above and in[6, 18] we are starting with the new approach for the quantum electrodynamics in the Foldy–Wouthuysen representation.

Contrary to the consideration of[7, 8], we are based on the standard variational formulation of the least action principle and mathematically well-defined definition of the nonlocal (pseudodifferential) operator $\omega=\sqrt{-\Delta+m^2}$ and its functions. We consider such operators as integral and well-defined in the Schwartz space $\mathrm{S}^{3,4}$.

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