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Pole Placement of Controlled Switching Linear Systems - Bond Graph Approach -

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Abstract Switching systems are very common in various engineering fields (e.g. hydraulic systems with valves,.., electric systems with diodes, relays,…, mechanical systems with clutches...). Such systems are a particular case of hybrid systems. These systems are characterized by a Finite State Automaton (FSA) and a set of dynamic systems, each one corresponding to a state of the FSA. The change of states can be either controlled or autonomous. The aim of this work is to investigate the structural controllability for controlled switching linear systems modelled by bond graph. Several concepts appeared in the last decade addressing the controllability problem of these systems: controllable sublanguage concept[1], hybrid controllability concept[2], between-block controllability concept[3]. Controlled Switching Linear Systems (CSLS) on which we focus in this work belong to the hybrid controllability concept as they address a reachability problem of hybrid states. In the other hand, the bond graph concept is an alternate representation of physical systems. Some recent works permit to highlight structural properties**.** In[13]**,** the structural controllability property is studied using simple causal manipulations on the bond graph model. The objective of this work is to extend these properties to CSLS. In this work, the structural controllability of CSLS by means of algebraic and graphical conditions is discussed. First, formal representations of controllability subspaces are given for switched bond graph. They are calculated through causal manipulations. Second, these subspaces are used to propose structured state feedback matrices in the context of pole assignment by static state feedback. Third, a simple example is given to illustrate the previous results. The proposed method, based on a bond graph theoretic approach, assumes only the knowledge of the systems structure. This result can be implemented by classical bond graph theory algorithms.

Keywords Switching Systems, Bond Graph, Controllability, Static State Feedback

1. Introduction

A broad class of hybrid systems is composed of physical processes with switching devices. Such processes are called switching systems and are very common in various engineering fields (e.g. hydraulic systems with valves,.., electric systems with diodes, relays,…, mechanical systems with clutches...). These systems are characterized by a Finite State Automaton (FSA) and a set of dynamic systems, each one corresponding to a state of the FSA. The change of states can be either controlled or autonomous. Various researchers investigated this problem using the bond graph tool [1,2,3,4,5,6]. The ideal and the non-ideal approaches are used :

- In the non-ideal approach, switches are modelled as resistive elements associated with modulated transformer. The modulation is done using a boolean variable.
- In the ideal approach, switches commutate instantaneously. Each switch is modelled as a null source: effort source for a closed switch state, and flow source for an open one. This approach is used in this work.

Lately, there have been a lot of studies on stability analysis and design[4]-[5]-[6]. (Liberzon and Morse,[5]) summarize three basic problems regarding stability and design of switched systems. They are: (i) stability for arbitrary switching sequences; (ii) stability for certain useful classes of switching sequences; (iii) construction of stabilizing switching sequences. For problem (i), finding conditions under which there exists a common Lyapunov Function forthe system is a typical approach[6]. For problem (ii), multiple Lyapunov functions method, an extension of classical Lyapunov theory, is the main tool[7]. For problem (iii), there are many results available[4]**.**

Petterson and Lennartson in[8] show that the search for Lyapunov functions can be formulated as a linear matrix inequality (LMI) problem. Xu and Antsaklis in[9] give a necessary and sufficient condition for the asymptotic stabilizability of switched systems consisting of several second-order subsystems with unstable foci. If the condition holds, an asymptotically stabilizing switching law can be obtained. Hu, Xu, Antsaklis and Michel in[4] discuss the

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robustness of this kind of stabilizing control laws.

This paper is briefly outlined as follows: The second section formulates the problem. In section three the CSLS algebraic controllability is reviewed. In section four, an asymptotically stable state feedback design algorithm is derived for such systems. In section five, the structural controllability of these systems is discussed, for which, we calculate a formal representation of controllability subspaces of switched bond graph. It allows to propose in section six, the structure of feedback matrices for the pole assignment control problem, this for all modes. Graphical procedures are proposed. Section seven contains an illustrating example. Finally, the conclusion is provided in section 8.

2. Problem Formulation

Consider a CSLS[10], described by

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \qquad (1)$$

Where $x(t) \in R^n$ is the state variable, $u(t) \in R^m$ is the input variable, $\sigma(t)$ is a piecewise constant switching function and (σ_i, x) the hybrid state. According to values of $\sigma(t)$, there exists q configurations, $\sigma_i \in \{\sigma_1 \cdots \sigma_q\}$. So $A(\sigma_i) \in R^{n\times n}$ and $B(\sigma_i) \in R^{n\times m}$.

If we consider this system in a particular mode i, the equation (1) can be written as

$$\dot{x}(t) = A_i x(t) + B_i u(t) \qquad (2)$$

With $A_i = A(\sigma_i)$ and $B_i = B(\sigma_i)$, $i \in \{1,\ldots,q\}$.

Remark 1 System (2) can be considered as a linear time invariant system (LTI).

Assumptions 1

1) We suppose that A_i and B_i matrices are constant on a time interval $[t_0, t_0+\tau)$, where $\tau \geq \tau_{\min} > 0$, and the constant $\tau_{\min}$ is an arbitrarily small and independent of mode i. For instance, suppose that the dynamics in (1) are given by (2) over the finite time interval $[t_i, t_{i+1})$. At time t_{i+1} the dynamic in interval $[t_{i+1}, t_{i+2})$ is given by $\dot{x}(t) = A_j x(t) + B_j u(t)$.

2) We assume that the state vector $x(t)$ does not jump discontinuously at t_{i+1}.

If we further assume that $u(t) = K_i x(t)$ then the following convenient representation of (2) is obtained

$$\dot{x} = \begin{cases} \overline{A}_1 x & t\in[t_0, t_1) \\ \vdots & \\ \overline{A}_q x & t\in[t_{q-1}, t_q) \end{cases} \quad \text{with } \overline{A}_i = A_i + B_i K_i \qquad (3)$$

We refer to systems (1) and (3) interchangeably as the switching systems

3. Controllability of CSLS

The controllability of (1) was defined:

Definition 1[10] Given any pair of hybrid states, denoted as (σ_0, x_0) and (σ_q, x_q), respectively, if there exists a timed mode-switching set $\{(\sigma_{i-1}, t_i, \sigma_i)\}_{i=1}^{q}$ and a corresponding piecewise continuous-finite input signal $u(t)$, such that system (1) evolving under these two distinct inputs is reachable from (σ_0, x_0) to (σ_q, x_q) within a finite time interval, then the considered system (1) is controllable, otherwise, system (1) is uncontrollable.

3.1. An Algebraic Sufficient Condition

When system (1) has only one mode, the controllability can be analyzed through the controllability matrix (4).

$$W \triangleq [B\ AB \cdots A^{n-1}B] \qquad (4)$$

For the general case, a controllability combined matrix W_C of system (1) is given by equation (5):

$$W_C \triangleq [W_1\, W_2 \cdots W_q] \triangleq [B_1\ A_1B_1 \cdots A_1^{n-1}B_1 \cdots B_q \cdots A_q^{n-1}B_q] \quad (5)$$

Theorem 1[10] The CSLS (1) with q modes is controllable, if the controllability matrix W_C is of full row rank.

Remark 2 From this theorem, we can deduce that:

1) The system (1) can be controllable, if there is only one controllable sub-system (mode).

2) However, it is possible that no sub-system is controllable but that the system (1) is controllable.

4. Piecewise Constant Controller Design

The controller design is based on placement of all poles of all modes to appropriate positions in the left hand side of the s-plane.

In[11] the following lemma is proposed:

Lemma 1[11] Given a controllable LTI system (A, B) in controller canonical form, i.e.,

$$A = \begin{bmatrix} 0 & 1 & & \\ \vdots & 0 & \ddots & \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and a scalar $h > 0$, for any $\beta > 0$, there exists a constant state feedback $u = Kx$ such that

$$\left\| \exp\left[(A+BK)h\right] \right\| < \exp(-\beta h) \qquad (6)$$

This lemma can be extended to a more general case, which is the starting point to design the piecewise constant state feedback controller.

Lemma 2[12] Given a controllable LTI system (A, B) and a scalar $h > 0$, for any $\beta > 0$, there exists a constant state feedback $u = Kx$ such that

$$\left\| \exp\left[(A+BK)t\right] \right\| < \exp(-\beta t), \forall t > h \qquad (7)$$

4.1. Asymptotic Stability

Based on Lemma 2 and if each subsystem (A_i, B_i) is controllable, Xie and al in[12] gave the following theorem:

Theorem 2 For the system (1), and for each subsystem (A_i, B_i) there exists a state feedback $u = K_i x$ such that the closed loop system $\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))K(\sigma(t))x(t)$ is asymptotically stable.

Since each subsystem (A_i, B_i) is controllable, suppose: $\det(sI - A_i) = s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_1 s + \alpha_0$. Denote

$$F_i = [A_i^{n-1}B_i + \cdots + A_i B_i \ B_i]\begin{pmatrix} 1 & & & \\ \alpha_{n-1} & 1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_1 & \cdots & \alpha_{n-1} & 1 \end{pmatrix}$$

Then F_i is nonsingular.

Using lemma 2 and theorem 2, an asymptotically stable state feedback controller design procedure can be constructed[12].

Procedure 1 For the system (1), for each subsystem (A_i, B_i), a state feedback matrix K_i such that the closed-loop system is asymptotically stable can be calculated as follows.

1) Determine the nonsingular matrix F_i such that $(\overline{A}_i, \overline{B}_i)$ is in controller canonical form, where $\overline{A}_i = F_i^{-1}A_i F_i$, $\overline{B}_i = F_i^{-1}B_i$;

2) Calculate $\varepsilon_i = 1 - \ln\left(\left\|F_i^{-1}\right\|\left\|F_i\right\|\right)$;

3) Select $\lambda_{i,1}$ such that $n\exp(\lambda_{i,1}h_i)(\lambda_{i,1}+n)^{2n-2} \le \exp(\varepsilon_i)$, moreover, let $\lambda_{i,l} = \lambda_{i,1} - l + 1$ for $l = 2, \ldots, n$;

4) According to $\lambda_{i,1}, \ldots, \lambda_{i,n}$, calculate the state feedback matrix $\overline{K}_i$;

5) Let $K_i = \overline{K}_i F_i^{-1}$.

In the next step structural controllability of CSLS modelled by bond graph is studied.

5. Bond Graph Approach

The structure junction of a switching bond graph can be represented by figure 1[14]. Five fields model the components behaviour, 4 that belong to the standard bond graph formalism: - source field which produces energy, - detector field; - R field which dissipates it, - I and C field which can store it, and the Sw field that is added for switching components. This element (Sw) is made of the power variables imposed by the switches in the chosen configuration.

Figure 1 represents the block diagram that is deduced from the causal bond graph.

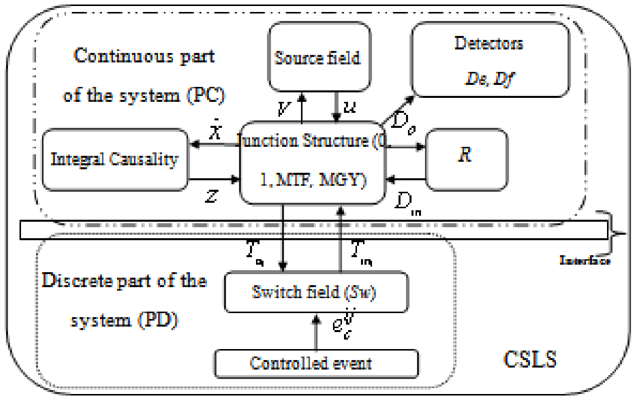

Figure 1. Structure junction

The following key variables are used :

- the state vector $x(t)$ is composed of the energy variables on the bond connected to an element in integral causality (the momenta $p = \int f dt$ on I elements and charges $q = \int e dt$ on C elements), and the complementary state vector $z(t)$ is composed of power variables (the efforts e on C elements and flows f on I elements);

- $D_{in}(t)$ and $D_o(t)$ represent the variables going out of and into the R field;

- the vector $u(t)$ is composed of the sources;

- $T_{in_i}(t)$ is composed of the zero valued variables imposed by the switches in this configuration;

- $T_{o_i}(t)$ is composed of the complementary variables in the switches;

- the vector $y(t)$ is composed of the continuous outputs.

Assumptions 2

To take into account the absence of discontinuities (*Assumption* 1), we suppose that there are no elements in derivative causality in the bond graph model in integral causality, before and after commutation.

Using this structure, the following equation is given[14] :

$$\begin{pmatrix} \dot{x} \\ D_o \\ T_{o_i} \end{pmatrix} = \begin{pmatrix} S_{11}^i & S_{13}^i & S_{14}^i & S_{15}^i \\ -S_{13}^{it} & S_{33}^i & S_{34}^i & S_{35}^i \\ -S_{14}^{it} & -S_{34}^{it} & S_{44}^i & S_{45}^i \end{pmatrix}\begin{pmatrix} z \\ D_{in} \\ T_{in_i} \\ u \end{pmatrix} \quad (8)$$

Let the constitutive law of the R field be linear: $D_{in} = L_i D_o$. L_i is a positive matrix, with $L = \begin{pmatrix} 1/R_1 & 0 \\ 0 & R_2 \end{pmatrix}$.

Let assume that $H_i = L_i(I - S_{33}^i L_i)^{-1}$ is an invertible positive matrix.

In a linear case, the law constitutive for the fields of storage I and C can be written : $z = Fx$. Where F is a symmetric positive definite matrix.

Then the second row leads to

$$D_{in} = -H_i S_{13}^{it} Fx + H_i S_{35}^i u + H_i S_{34}^i T_{in_i}$$

The third line of (8) gives:

$$T_{o_i} = (-S_{14}^{it} + S_{34}^{it} H_i S_{13}^{it}) Fx + (S_{45}^{i} - S_{34}^{it} H_i S_{35}^{i}) u \quad + (S_{44}^{i} - S_{34}^{it} H_i S_{34}^{i}) T_{in_i} \tag{9}$$

The substitution in the first line of (8) gives:

$$\dot{x} = (S_{11}^{i} - S_{13}^{i} H_i S_{13}^{it}) Fx + (S_{15}^{i} + S_{13}^{i} H_i S_{35}^{i}) u \quad + (S_{14}^{i} - S_{13}^{i} H_i S_{34}^{i}) T_{in_i} \tag{10}$$

Then, we have:

$$\dot{x} = A_i x + B_{c_i} u + B_{d_i} T_{in_i} \tag{11}$$

This system is equivalent to system (2), where $A_i = (S_{11}^{i} - S_{13}^{i} H_i S_{13}^{it}) F$, $B_{c_i} = S_{15}^{i} + S_{13}^{i} H_i S_{35}^{i}$ and $B_{d_i} = S_{14}^{i} + S_{13}^{i} H_i S_{34}^{i}$.

When the elements of commutations are in the chosen configuration (mode i for example), then $T_{in_i} = 0$.

Therefore, for N switchs, we have $2^N = q$ modes:

$$\begin{cases} \dot{x}(t) = A_1 x(t) + B_{c_1} u(t) & t \in [t_0, t_1) \\ \vdots & \\ \dot{x}(t) = A_q x(t) + B_{c_q} u(t) & t \in [t_{q-1}, t_q) \end{cases} \tag{12}$$

This system is equivalent to system (1).

5.1. Structural Controllability

The bond graph concept is an alternate representation of physical systems. Many works allow to highlight structural properties of these systems[13]-[14]. In[13], the structural controllability property is studied using simple causal manipulations on the bond graph model. This result was extended to case of CSLS[14]. Our objective is to use the latter result for to propose structured state feedback matrices in the context of pole assignment by static state feedback.

In the following we note that:

-BG: acausal (without causality) bond graph model,

-BGI: bond graph model when the preferential integral causality is affected,

-BGD: bond graph model when the preferential derivative causality is affected,

- t^i : the number of dynamical elements remaining in integral causality in the BGD of mode i.

- t_s^i : the number of dynamical elements remaining in integral causality in the BGD after the dualization of the maximum number of input sources in order to eliminate these integral causalities.

5.1.1. Graphical Sufficient Condition 1

A system (1) with q modes is controllable if only one system is controllable. This condition can be interpreted by using the result of structural controllability of LTI system.

Indeed, this result is a simple recovery of those giving the necessary and sufficient condition of structural controllability of LTI system modelled by bond graph approach.

Theorem 3[14] The CSLS system (12) is structurally state controllable if:

1- All dynamical elements in integral causality are causally connected with an input source.

2- BG-rank $[A_i \;\; B_{c_i}] = n$.

Property 1 [14] BG-rank $[A_i \;\; B_{ci}] = n - t_s^i$.

To study the controllability of system (12), it is necessary to apply this result to all modes; if one controllable mode exists, the procedure is stopped.

The case where no mode is controllable, but when the system is controllable, can be verified by formal calculation of combined matrix (4). This calculation can be formally effected by using the bond graph model in integral causality or by calculating the controllability subspace from bond graph model in derivative causality. We chose to translate the latter in the form of a second sufficient condition.

5.1.2. Graphical Sufficient Condition 2

Thereafter, formal representations of controllability subspaces, denoted as R_0, are given for bond graph models. They are calculated through causal manipulations. The bases of these subspaces are used to propose a procedure to study the controllability of system.

On the BGD_i (and dualization of input sources) there exists t_s^i elements remaining in integral causality and $n - t_s^i$ elements in derivative causality.

t_s^i algebraic equations can be written (equation 13):

$$g_k^i - \sum_r \alpha_r^{ik} g_r^i = 0 \tag{13}$$

- g_k^i is either an effort variable e_r for I-element in integral causality or a flow variable f_r for C-element in integral causality;

- g_r^i is either an effort variable e_r for I-element in derivative causality or a flow variable f_r for C- element in derivative causality;

- α_r^{ik} is the gain of the causal path between the k^{th} I or C-elements in integral causality and the r^{th} I or C-elements in derivative causality.

Let us consider the t_s^i row vectors $z_k^i \; (k = 1, \cdots, t_s^i)$ whose components are the coefficients of the variables $g_l^i \; (l = k, r)$ in the equation (13).

Property 2 The t_s^i row vectors $z_k^i \; (k = 1, \cdots, t_s^i)$ are orthogonal to the structural controllability subspace vectors of the i^{th} mode. We write $Z_i = (z_k^i)_{k=1,\cdots,t_s^i}$ and

$R_0^{i\perp} = \mathrm{Im}(Z_i)$.

Using the bond graph model in derivative causality, the uncontrollable $R_0^{i\perp}$ subspace can be calculated[14].

Procedure 2 Calculation of $R_0^{i\perp}$

1) On the $\mathrm{BGD_i}$, dualize the maximum number of input sources in order to eliminate the elements remaining in integral causality,

2) For each element remaining in integral causality, write the algebraic relations with elements in derivative causality (equation 13),

3) Write a row vector z_k^i for each algebraic relation with the causal path gains and write $Z_i = (z_k^i)_{k=1,\cdots,t_s^i}$.

In order to calculate an R_0^i basis, it is enough to find $n-t_s^i$ independent column vectors $w^{ir}(r=1,\cdots,n-t_s^i)$. These vectors are gathered in the matrix $W^i = (w^{ir})_{r=1,\cdots,n-t_s^i}$.

From the $\mathrm{BGD_i}$ (and dualization of inputs sources), $n-t_s^i$ algebraic relations can be written (14).

$$g_r^i - \sum_k \gamma_k^{ir} g_k^i = 0 \tag{14}$$

- g_r^i is either a flow variable f_r for I-element in derivative causality or an effort variable e_r for C-element in derivative causality;

- g_k^i is either a flow variable f_r for I-element in integral causality or an effort variable e_r for C-element in integral causality;

- γ_k^{ir} is the gain of the causal path between the r^{th} element in derivative causality and the k^{th} element in integral causality.

Suppose now $n-t_s^i$ column vectors w^{ir} whose components are the coefficients of g_r^i and g_k^i variables in equation (14).

Procedure 3 Calculation of R_0^i

1) On the $\mathrm{BGD_i}$, dualize the maximum number of continuous input sources in order to eliminate the elements in integral causality;

2) For each element in derivative causality, write the algebraic relations with elements in integral causality (equation 14);

3) Write a column vector w^{ir} for each algebraic relation with the causal path gains (equation 14), with $R_0^i = \mathrm{Im}(W^i)$.

Property 3[14] $n-t_s^i$ column vectors $w^{ir}(r=1,\cdots,n-t_s^i)$ compose a basis for the structural controllability subspace of i^{th} mode.

The graphical calculation of structural controllability subspaces and theorem 1 leads to theorem 4:

Theorem 4[14] If rank $[W^1 \cdots W^q]=n$, the system CSLS (12) is structurally controllable.

6. Pole Assignment

Now, we suppose a monovariable (m=1) linear sub-system (mode i), that is ($B(\sigma_i) \in R^{n\times 1}$) in equation (1). The problem is to find a state feedback law $u(t) = K_i x(t)$ to each mode i in the time interval $t \in [t_{i-1}, t_i)$ such that the closed loop state matrix $\overline{A}_i$ has the desired poles. In fact, the number of assignable poles is equal to the rank of the controllability matrix, this for each possible mode. It is deduced that the number of independent parameters in the matrix K_i is equal to the controllable subspace in mode i. The objective is to find these parameters.

We recall some relations $R_0^i = \mathrm{Im}\ \{W^i\}$ and $Z_i.W^i = 0$. We can write: $X = \mathrm{Im}\,W^i \oplus \mathrm{Im}\,Z_i$ with X the state space, and $Z_i\overline{A}_i = 0$.

Now, we calculate $\det(sI-\overline{A}_i)$. The roots of this characteristic polynomial are the closed loop roots.

First, the different matrices are decomposed. A permutation between the dynamical elements enables us to write $Z_i = [Z_{11}^i\ Z_{12}^i\ 0]$ with $\mathrm{rank}Z_i = \mathrm{rank}Z_{11}^i$ such that in the state vector, the t_s^i first variables are the non-controllable variables, which are the dynamical elements which remain in integral causality in BGD_i model. The following variables are the dynamical elements appearing in the algebraic relations after $\mathrm{BGD_i}$ and dualization.

Suppose now this new matrix : $Z_i^* = \begin{pmatrix} Z_{11}^i & Z_{12}^i & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$

With $\dim Z_i^* = n$ and I is identity matrix.

The matrices A_i, B_i and K_i are decomposed as Z_i^*.

$$A_i = \begin{pmatrix} A_{11}^i & A_{12}^i & A_{13}^i \\ A_{21}^i & A_{22}^i & A_{23}^i \\ A_{31}^i & A_{32}^i & A_{33}^i \end{pmatrix},\quad B_i = \begin{pmatrix} B_1^i \\ B_2^i \\ B_3^i \end{pmatrix} \text{ and}$$

$$K_i = \begin{pmatrix} K_1^i & K_2^i & K_3^i \end{pmatrix}$$

Then, the characteristic polynomial of $A_i + B_i K_i$ is :

$$\det(sI-\bar{A}_i) = \det\begin{pmatrix} sI-A_{11}^i-B_1^iK_1^i & -A_{12}^i-B_1^iK_2^i & -A_{13}^i-B_1^iK_3^i \\ -A_{21}^i-B_2^iK_1^i & sI-A_{22}^i-B_2^iK_2^i & -A_{23}^i-B_2^iK_3^i \\ -A_{31}^i-B_3^iK_1^i & -A_{32}^i-B_3^iK_2^i & sI-A_{33}^i-B_3^iK_3^i \end{pmatrix}$$

$$= \frac{1}{\det(Z_{11}^i)}\det(\begin{pmatrix} Z_{11}^i & Z_{12}^i & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} sI-A_{11}^i-B_1^iK_1^i & -A_{12}^i-B_1^iK_2^i & -A_{13}^i-B_1^iK_3^i \\ -A_{21}^i-B_2^iK_1^i & sI-A_{22}^i-B_2^iK_2^i & -A_{23}^i-B_2^iK_3^i \\ -A_{31}^i-B_3^iK_1^i & -A_{32}^i-B_3^iK_2^i & sI-A_{33}^i-B_3^iK_3^i \end{pmatrix})$$

From the relation $Z_i\bar{A}_i = 0$, we obtain :

$$\det(sI-\bar{A}_i)$$

$$= \frac{1}{\det(Z_i^*)}.\det\begin{pmatrix} sZ_{11}^i & sZ_{12}^i & 0 \\ -A_{21}^i-B_2^iK_1^i & sI-A_{22}^i-B_2^iK_2^i & -A_{23}^i-B_2^iK_3^i \\ -A_{31}^i-B_3^iK_1^i & -A_{32}^i-B_3^iK_2^i & sI-A_{33}^i-B_3^iK_3^i \end{pmatrix}$$

$$= \frac{1}{\det(Z_i^*)}\det(\begin{pmatrix} Z_{11}^i & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} sI & s(Z_{11}^i)^{-1}Z_{12}^i & 0 \\ -A_{21}^i-B_2^iK_1^i & sI-A_{22}^i-B_2^iK_2^i & -A_{23}^i-B_2^iK_3^i \\ -A_{31}^i-B_3^iK_1^i & -A_{32}^i-B_3^iK_2^i & sI-A_{33}^i-B_3^iK_3^i \end{pmatrix})$$

After manipulations, we have

$$\det(sI-\bar{A}_i)$$

$$= \det\begin{pmatrix} sI & 0 & 0 \\ -A_{21}^i-B_2^iK_1^i & sI-A_{22}^i+A_{21}^i(Z_{11}^i)^{-1}Z_{12}^i-B_2^iK_2^{i*} & -A_{23}^i-B_2^iK_3^i \\ -A_{31}^i-B_3^iK_1^i & -A_{32}^i+A_{31}^i(Z_{11}^i)^{-1}Z_{12}^i-B_3^iK_2^{i*} & sI-A_{33}^i-B_3^iK_3^i \end{pmatrix}$$

With $K_2^{i*} = K_2^i - K_1^i(Z_{11}^i)^{-1}Z_{12}^i$

Then K_2^i becomes K_2^{i*}. The pole assignment problem consists in calculating K_2^{i*} and K_3^i, with $K_1^i = 0$. It comes $K_i = \begin{pmatrix} 0 & K_2^{i*} & K_3^i \end{pmatrix}$.

The t_s^i non-controllable poles are equal to zero, because they correspond to zero eigenvalues of the state matrix.

Proposition 4 For each mode, the independent parameters of the closed loop characteristic polynomial $\det(sI-\bar{A}_i)$ are the parameters of the two matrices K_2^{i*} and K_3^i.

Now we write $W^i = (W_{11}^i \quad W_{21}^i \quad 0)^t$ and we calculate K_2^{i*} directly from the controllability space matrix R_0^i.

It is possible to write R_0^i as : $R_0^i = \mathrm{Im}\begin{pmatrix} W_{11}^i & 0 \\ W_{21}^i & 0 \\ 0 & I \end{pmatrix}$

From the relation $Z_iW^i = 0$. Then we have $W_{11}^i = -(Z_{11}^i)^{-1}Z_{12}^iW_{21}^i$ and

$K_iW^i = \left[(K_2^i - K_1^i(Z_{11}^i)^{-1}Z_{12}^i)W_{21}^i \;\vdots K_3^i\right]$, W_{21}^i is a square matrix and can be chosen invertible, and equal to the unity matrix, because R_0^i has maximal rank.

In fact, it is enough to keep only the minimum number of independent parameters in the matrix $(K_2^i - K_1^i(Z_{11}^i)^{-1}Z_{12}^i)W_{21}^i$.

7. Example

Let us consider the following acausal BG model :

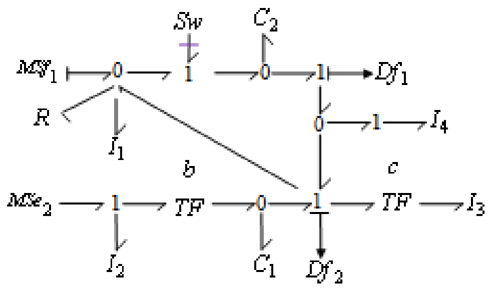

Figure 2. The acausal Bond Graph

This model contains one switch, then we have 2 possible configurations (mode F: Sw : $Se = 0$, mode E: Sw : $Sf = 0$).

■ The BGIi of these modes are shown in figure 3.

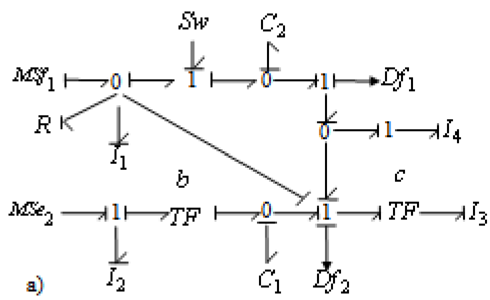

Figure 3. a) BGI of mode F

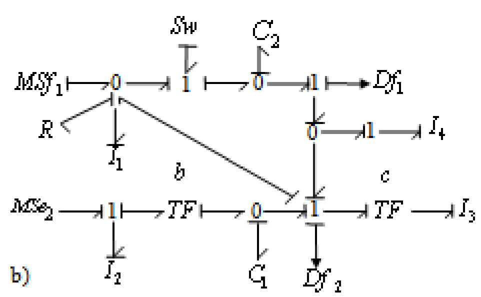

Figure 3. b) BGI of mode E

There are six state variables P_i on I_i, q_j on C_j ($i = 1,\cdots 4$; $j = 1,2$). The dimension of the system is $n = 6$. For models BGI_1 and BGI_2 all state variables are causally connected with the sources, and are in integral causality. There is no storing element in derivative causality in these configurations, so the state variables are given by : $x=\begin{pmatrix}P_{I_1} & P_{I_2} & P_{I_3} & P_{I_4} & q_{C_1} & q_{C_2}\end{pmatrix}^t$.

■ The BGDi and dualization (figure 4).

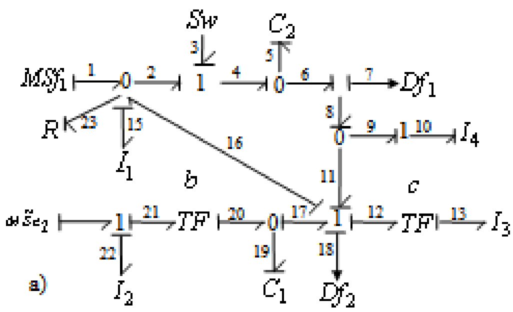

Figure 4. a) BGD of mode F

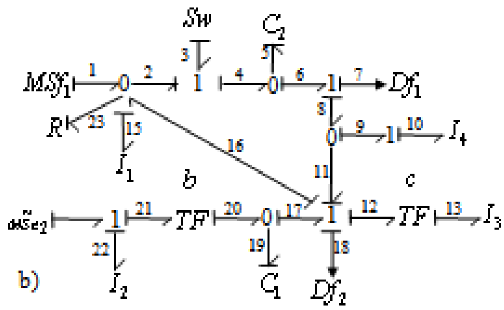

Figure 4. b) BGD of mode E

• For mode F

The element I_4 is in integral causality, we can write $e_{I_1}-e_{I_4}=0$, thus $z_1^1=\begin{pmatrix}1 & 0 & 0 & -1 & 0 & 0\end{pmatrix}$.

The four dynamical elements C_1, C_2, I_2 and I_3 are not causally connected with I_4, we can write $e_{c_1}=e_{c_2}=f_{I_2}=f_{I_3}=0$. The four corresponding vectors are $w^{12}=\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0\end{pmatrix}^t$, $w^{13}=\begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}^t$, $w^{14}=\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}^t$ and $w^{15}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^t$. The algebraic equation corresponding to the element I_1 is given by: $f_{I_1}+f_{I_4}=0$. Then $w^{11}=\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0\end{pmatrix}^t$ and $R_0^1=\mathrm{Im}\left\{w^{11},w^{12},w^{13},w^{14},w^{15}\right\}$.

We have $\mathrm{BG}-\mathrm{rank}[A_1\vdots B_1]=5$ $\mathrm{BG}-\mathrm{rank}[A_1\vdots B_1]=5$, this mode is not controllable.

• For mode E

After commutation, we pass in mode E and we have $\mathrm{BG}-\mathrm{rank}[A_2\vdots B_2]=6$. The mode E is controllable by two inputs, then this system is controllable.

Case : m=1 (Monovariable system)

In this part, we eliminate the second source and we affect the derivative causality (and dualization) on the BG model.

■ The corresponding BG models are drawn on figure 5.

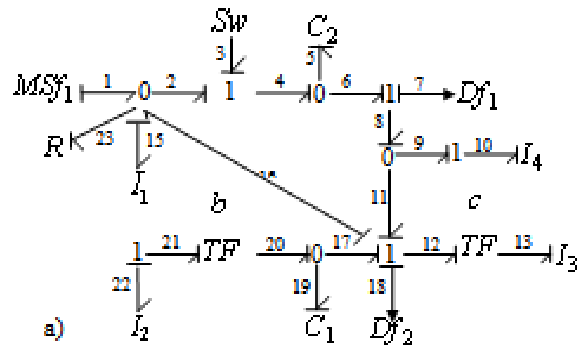

Figure 5. a) BGD of mode F

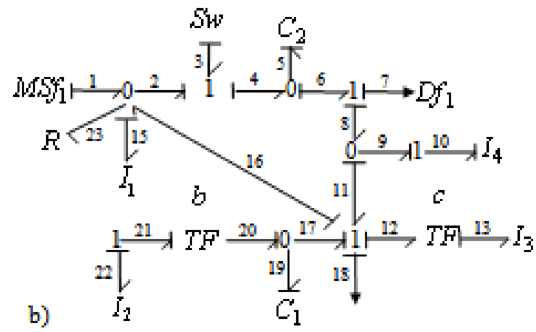

Figure 5. b) BGD of mode E

We have $\mathrm{BG}-\mathrm{rank}[A_1\vdots B_1]=4$ and $\mathrm{BG}-\mathrm{rank}[A_2\vdots B_2]=5$, these modes are not controllable.

• For mode F

The elements I_3 and I_4 are in integral causality, we have $z_1^1=\begin{pmatrix}1 & 0 & 0 & -1 & 0 & 0\end{pmatrix}$ and $z_2^1=\begin{pmatrix}0 & \frac{1}{b} & c & 0 & 0 & 0\end{pmatrix}$.

The two dynamical elements C_1 and C_2 are not causally connected with I_3 and I_4, we can write $e_{c_1}=e_{c_2}=0$, the two corresponding vectors are $w^{13}=\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}^t$ and $w^{14}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^t$.

The algebraic equations corresponding to the elements I_1 and I_2 are given by:

$$bf_{I_2}-\frac{1}{c}f_{I_3}=0 \Rightarrow w^{11}=\begin{pmatrix}0 & b & -\frac{1}{c} & 0 & 0 & 0\end{pmatrix}^t \text{ and}$$

$$f_{I_1}+f_{I_4}=0 \Rightarrow w^{12}=\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0\end{pmatrix}^t \text{ and}$$

$$R_0^1=\mathrm{Im}(\left\{w^{11},w^{12},w^{13},w^{14}\right\}.$$

• For mode E

The element I_4 is in integral causality, thus we have $z_1^2=(1 \quad \frac{1}{b} \quad c \quad -1 \quad 0 \quad 0)$ and $e_{c_1}=e_{c_2}=0$. The two corresponding vectors are $w^{24}=\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}^t$ and $w^{25}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^t$. The algebraic equations corresponding to the elements I_1, I_2 and I_3 are given by:

$bf_{I_2} + f_{I_4} - f_{Sw} = 0$, $\frac{1}{c} f_{I_3} + f_{I_4} - f_{Sw} = 0$ and $f_{I_1} + f_{I_4} = 0$; then $w^{21} = \begin{pmatrix} 0 & b & 0 & 1 & 0 & 0 \end{pmatrix}^t$, $w^{22} = \begin{pmatrix} 0 & 0 & \frac{1}{c} & 1 & 0 & 0 \end{pmatrix}^t$, $w^{23} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^t$

and $R_0^2 = \operatorname{Im}\left\{ w^{21}, w^{22}, w^{23}, w^{24}, w^{25} \right\}$.

We apply theorem 4, we have $\operatorname{rank}\begin{pmatrix} w^{11} & w^{12} & w^{13} & w^{14} & w^{21} & w^{22} & w^{23} & w^{24} & w^{25} \end{pmatrix} = 6$, this system is controllable.

The studied system has two modes, Suppose $K_1 = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \end{pmatrix}$ and $K_2 = \begin{pmatrix} k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \end{pmatrix}$.

For mode F there are two uncontrollable states variables associated the dynamical elements I_3 and I_4 , and for mode E there is one uncontrollable state variable associated the dynamical element I_3. We conclude that for mode F, k_{13} and k_{14} can be arbitrarily chosen, from the same manner for the variable k_{24} in mode E.

The four (respectively five) independent coefficients of the state feedback matrix are highlighted

$K_1 W^1 = \left(bk_{12} - \frac{1}{c} k_{13}, k_{11} + k_{14}, k_{15}, k_{16} \right)$ and

$K_2 W^2 = \left(bk_{22} + k_{24}, \frac{1}{c} k_{23} + k_{24}, k_{21} + k_{24}, k_{25}, k_{26} \right)$.

These coefficients are the unknown parameters for the pole assignment problem relating to each mode. They are the parameters of the characteristic polynomial of the state matrices

$\bar{A}_i$, $i \in \{1,2\}$.

Case : m>1 (Multivariable system)

The minimum number of parameters in the state feedback matrices for the pole assignment problem is *n*. In case of multivariable systems, the choice is not unique even for controllable systems.

Suppose now the BG model (mode F and E) (figure 3).

Suppose $K_1 = \begin{pmatrix} k_{11}^1 & k_{12}^1 & 0 & k_{14}^1 & k_{15}^1 & k_{16}^1 \\ 0 & 0 & k_{23}^1 & 0 & 0 & 0 \end{pmatrix}$ and

$$K_2 = \begin{pmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & 0 & k_{15}^2 & k_{16}^2 \\ 0 & 0 & 0 & k_{24}^2 & 0 & 0 \end{pmatrix}.$$

For mode F there is one uncontrollable variable associated the dynamical element I_4 and for the mode E all the variables are controllable. We conclude that for mode F, k_{14}^1 can be arbitrarily chosen,

The state feedback matrices can then be

$$K_1 W^1 = \begin{pmatrix} k_{11}^1 + k_{14}^1 & k_{12}^1 & 0 & k_{15}^1 & k_{16}^1 \\ 0 & 0 & k_{23}^1 & 0 & 0 \end{pmatrix}$$ and

$$K_2 W^2 = \begin{pmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & 0 & k_{15}^2 & k_{16}^2 \\ 0 & 0 & 0 & k_{24}^2 & 0 & 0 \end{pmatrix}.$$

8. Conclusions

This paper has studied the controllability property of a class of switched linear systems with the aid of simple causal manipulations on the bond graph model. Thus, formal calculation enables us to know the reachable variables, its checking is immediate on the BGI; on the other hand the BGD enables us to characterize from a graphic point of view the whole of the subspaces that are controllable with respect to each mode. While employing these subspaces, we have proposed a simplified state feedback matrices for the pole assignment problem, this for all the possible configurations of the system. The application was made on an example. The problem is now to highlight more structural information in order to solve other current questions from a structural point of view. It will be done in a future work.

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