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Proper Curvature Collineations in Non - Static Spatially Homogeneous Rotating Spacetimes by using Lorentzian Metric

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Abstract In this paper we have investigated the curvature collineations of non- static spatially homogeneous rotating spacetimes by using the rank of 6×6 Riemann matrix and direct integration techniques. The above investigation reveals that there are thirty two cases in which the non-static spatially homogeneous rotating spacetimes admits the proper curvature collineations. It is also found that when the above spacetimes admit proper curvature collineations they form an infinite dimensional vector space.

Keywords Riemann Curvature Tensor, Curvature Collineations, Bivectors, Infinite Dimensional Vector Space

1. Introduction

The general theory of relativity is the final form of an elegant restructuring of classical mechanics; it is basically the theory of gravitation, which is described by Einstein field equations. The Einstein field equations are, in general, second order highly non-linear partial differential equations. Symmetry restriction on the spacetime metric is a way to find the exact solution of Einstein field equations[9]. It is found that the existence of certain geometric symmetries which are definable in terms of Lie derivatives lead to conservation laws[13]. These symmetries can be expressed in terms of *Killing vector fields* (KVF), *homothetic vector fields* (HVF), *Ricci collineations* (RCS) and *Curvature collineations* (CCS).

Einstein's theory of relativity tells us that the space and time are all parts of a single physical entity, spacetime continuum. The aim of this paper is to investigate the existence of proper curvature collineations in non static rotating spatially homogeneous spacetime. Spatially homogeneous spacetime posses a group of isometries whose orbits are spacelike hypersurfaces which foliate the spacetime[23]. A spacetime is called static if it is stationary and timelike Killing vector field is orthogonal to the hypersurface, otherwise the spacetime is said to be non-static[24]. The curvature collineations, in general relativity, signifies not only the geometrical symmetry of spacetime, but also implies that the gravitational properties of the field are preserved along the curvature collineations vector[1]. It is therefore important to study the curvature collineations. Katzin *et al* lead the way for carrying out the detailed study of CCS[1]. G. S.

Hall and J. da. Costa provide a systematic way for finding the CCS[5,6]. The proper curvature collineations in various non-static spacetimes has been comprehensively discussed by M. Ramzan.[14]. Different approaches[1-19] are used to study the curvature collineations, but here we shall make use of rank of 6×6 Riemann matrix.

Let M be a four dimensional connected Hausdorff manifold with Lorentz metric g of signature $(-,+,+,+)$.

The components of the curvature tensor associated with the metric g through Levi- Civita connection are denoted by R^{a}_{bcd}. The covariant derivative, partial derivative and Lie derivative are denoted by a semi-colon, a comma and L respectively. Round and square brackets denotes the usual symmetrization and anti-symmetrization respectively.

The covariant derivative of a differentiable global vector field X on M can be decomposed as

$$X_{a;b}=\frac{1}{2}h_{ab}+F_{ab} \qquad (h_{ab}=h_{ba}=L_X g_{ab},\ \ F_{ab}=-F_{ba}) \tag{1}$$

Where h and F are second order symmetric and skew symmetric tensors on M. If $h_{ab;c}=0$, then X is *affine* (*proper affine* if $h_{ab}\not\propto g_{ab}$). If X is *affine* and $h_{ab}=cg_{ab}$, with c is a constant, then X is called *homothetic* (*proper homothetic* if $c\neq0$ and *Killing* if $c=0$). A vector field X is said to be curvature collineations if

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$$L_X R^a{}_{bcd} = 0 \quad . \qquad (2)$$

Or equivalently

$$R^a{}_{bcd;e}X^e + R^a{}_{ecd}X^e_{;b} + R^a_{bed}X^e_{;c} + R^a{}_{bce}X^e_{;d} - R^e{}_{bcd}X^a_{;e} = 0$$

Finally, X is said to be proper curvature collineations if it is not affine[6].

2. Classification of Riemann Curvature Tensor

In this section we shall present the classification of Riemann curvature tensor in terms of its rank and bivector decomposition. The rank of the Riemann tensor can be regarded as the rank of a 6×6 symmetric matrix, derived in a well known way[6]. The rank of the Riemann tensor is the rank of the linear map f which maps the vector space of all bivectors F_{ab} at $p \in M$ to itself defined by

$$f : F^{ab} \to R^{ab}{}_{cd}F^{cd} .$$

Define the subspace S_p of the tangent space V_p consisting of those members $k \in V_p$, which satisfy the relation

$$R_{abcd}k^d = 0 \qquad (3)$$

Then the Riemann tensor at p is known to satisfy exactly one of the following conditions[6].

Class B

The rank is 2 and the range of f is spanned by the dual pair of non-null simple bivectors and dim $S_p = 0$. The Riemann tensor at p takes the form

$$R_{abcd} = \alpha F_{ab}F_{cd} + \beta \overset{*}{F}_{ab}\overset{*}{F}_{cd} \qquad (4)$$

where $\alpha, \beta \in \mathbb{R}$ and F_{ab} and its dual $\overset{*}{F}_{ab}$ are the unique (upto scaling) simple non-null spacelike and timelike bivectors in the range of f respectively.

Class C

The rank is 2 or 3 and there exist a unique (up to scaling) solution say k of (3) (and so $\dim S_p = 1$). The Riemann tensor at point p takes the form

$$R_{abcd} = \sum_{i,j=1}^{3} \alpha_{ij} F^i_{ab} F^j_{cd} \qquad (5)$$

where $\alpha_{ij} \in \mathbb{R}$ for all i, j and $F^i_{ab}k^b = 0$ for each of the bivectors F^i which span the range of f.

Class D

Here the rank of the curvature matrix is 1. The range of the map f is spanned by a single bivector F_{ab}, say, which has to be simple because the symmetry of the Riemann tensor $R_{a[bcd]} = 0$ means $F_{a[b}F_{cd]} = 0$ which, together with a standard result implies that F_{ab} is simple. The curvature tensor admits exactly two independent solutions k, u of (3) so that $\dim S_p = 2$. The Riemann tensor at p takes the form

$$R_{abcd} = \alpha F_{ab}F_{cd} \qquad (6)$$

where $\alpha \in R$ and F_{ab} is a simple bivector with blade orthogonal to k and u.

Class O

The rank of the curvature matrix is 0 (so that $R_{abcd} = 0$) and $\dim S_p = 4$ (which is the trivial case)

Class A

The Riemann tensor is said to b of class A at p if it is not of class B, C, D or O. Here always $\dim S_p = 0$.

3. Main Results

Consider the Non-Static Spatially Homogeneous Rotating Spacetime in the cylindrical coordinate system (t, r, θ, z) (labeled by (x^0, x^1, x^2, x^3)) with line element[16]

$$ds^2 = -dt^2 + dr^2 + U(t,r)d\theta^2 + dz^2 - 2V(t,r)dtd\theta \qquad (7)$$

where $U(t,r)$ and $V(t,r)$ are nowhere zero functions of t and r. The above spacetime admits two independent Killing vector fields which are:

$$\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z} \qquad (8)$$

The non-zero independent components of the Riemann tensor in lower indices are

$$R_{0101} = \frac{V_r^2}{4(U+V^2)} = \alpha_1$$

$$R_{0102} = \frac{1}{4\left(U+V^2\right)}\begin{bmatrix} -2V_{tr}U - 2V_{tr}V^2 + 2VV_tV_r - \\ U_tV_r + 2V_tU_r \end{bmatrix} = \alpha_2$$

$$R_{0112} = \frac{1}{4\left(U+V^2\right)}\begin{bmatrix} -2V_{rr}U - 2V_{rr}V^2 + VV_r^2 \\ +V_rU_r \end{bmatrix} = \alpha_3$$

$$R_{0202} = -\frac{1}{4\left(U+V^2\right)}\begin{bmatrix} 2U_{tt}U + 2U_{tt}V^2 - V_r^2U \\ -V_r^2V^2 - 2VV_tU_t - U_t^2 \end{bmatrix} = \alpha_4$$

$$R_{0212} = \frac{1}{4\left(U+V^2\right)}\begin{bmatrix} -2U_{tr}U - 2U_{tr}V^2 \\ +2VU_tV_r + U_tU_r \end{bmatrix} = \alpha_5$$

$$R_{1212} = \frac{1}{4\left(U+V^2\right)}\begin{bmatrix} -2U_{rr}U - 2U_{rr}V^2 - V_r^2U \\ +2VV_rU_r + U_r^2 \end{bmatrix} = \alpha_6$$

The curvature tensor with components R_{abcd} at p can be written as a 6×6 symmetric matrix in a well known way[6]

$$R_{abcd}=\begin{pmatrix}\alpha_1 & \alpha_2 & 0 & \alpha_3 & 0 & 0\\ \alpha_2 & \alpha_4 & 0 & \alpha_5 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \alpha_3 & \alpha_5 & 0 & \alpha_6 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \quad (9)$$

Here, we are only interested in those cases where the rank of the 6×6 Riemann matrix is less than or equal to three. Since we know from theorem[6] that when the rank of the 6×6 Riemann matrix is greater than three, there exists no proper CCS. There are altogether, forty-one cases for the rank of 6×6 Riemann matrix to be (≤ 3). Since out of six rows of (9) only three rows are non-zero, so only one case for rank 3, three cases for rank 2 and three cases for rank 1 survive. Now suppose the rank of the 6×6 Riemann matrix is three, then there are three non-zero rows or columns in matrix (9). There are fifty six different possible ways for which the rank of the Riemann matrix remains three. Out of which twenty nine yield contradiction and twenty seven survive. For example the case when the rank of 6×6 Riemann matrix is three, i.e. $\alpha_1=\alpha_2=\alpha_4=0$, and $\alpha_3\neq 0,\alpha_5\neq 0,\alpha_6\neq 0$. The constraints imply that $U_t(t,r)=0,U_r(t,r)=0,V_r(t,r)=0$, substitution of these information in components of Riemann curvature tensor, we get $\alpha_3=0,\alpha_5=0,\alpha_6=0$, which gives contradiction because we have assumed that $\alpha_3\neq 0,\alpha_5\neq 0,\alpha_6\neq 0$. Therefore, this case is not possible. By similar analysis we have the following surviving possibilities, when the rank of the 6×6 Riemann matrix is three or less. The detail about finding the possibilities for the rank of Riemann matrix to be less than or equal to three can be found in[4]. It is important to note that we are only considering the non static cases, the CCS in the static cases can be found in[25].Thus there exist the following non-static possibilities:

(A1) Rank=3, $U_t(t,r)=0,V_r(t,r)=0,U_r(t,r)\neq 0,$ $V_t(t,r)\neq 0,\ U_{rr}(t,r)\neq 0$.

(A2) Rank=3, $U_t(t,r)=0,V_r(t,r)=0,U_{rr}(t,r)=0,$ $U_r(t,r)\neq 0,V_t(t,r)\neq 0,$

(A3) Rank=3, $U_t(t,r)\neq 0,\ U_r(t,r)\neq 0,\ V_t(t,r)\neq 0,$ $V_r(t,r)=0,U_{tt}(t,r)=0,\ U_{rr}(t,r)\neq 0,U_{tr}(t,r)\neq 0$ $2V(t,r)V_t(t,r)+U_t(t,r)=0.$

(A4) Rank=3, $U_t(t,r)\neq 0,\ U_r(t,r)\neq 0,\ V_t(t,r)\neq 0,$ $V_r(t,r)=0,\ U_{rr}(t,r)\neq 0,U_{tr}(t,r)\neq 0,$ $U_r^2(t,r)-2U_{rr}(t,r)U(t,r)-2U_{rr}(t,r)V^2(t,r)=0$.

(A5) Rank=3, $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)=0,\ U_{rr}(t,r)\neq 0,\ U_{tt}(t,r)\neq 0,U_{tr}(t,r)\neq 0.$

(A6) Rank=3 $U_t(t,r)\neq 0,\ U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)=0,\ U_{tt}(t,r)=0,\ U_{tr}(t,r)\neq 0,,U_{rr}(t,r)\neq 0.$

(A7) Rank=3, $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_r(t,r)=0,$ $V_t(t,r)\neq 0,\ U_{rr}(t,r)=0,\ U_{tt}(t,r)\neq 0,\ U_{tr}(t,r)\neq 0.$

(A8) Rank=3, $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_r(t,r)=0,$ $U_{tr}(t,r)=0,V_t(t,r)\neq 0,U_{rr}(t,r)\neq 0,\ U_{tt}(t,r)\neq 0.$

(A9) Rank=3, $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)=0,U_{rr}(t,r)=0,U_{tt}(t,r)=0,\ U_{tt}(t,r)=0,$ $U_{tr}(t,r)\neq 0.$

(A10) Rank=3, $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)=0,U_{tt}(t,r)=0,U_{tr}(t,r)=0,\ U_{rr}(t,r)\neq 0.$

(A11) Rank=3 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)=0,U_{rr}(t,r)=0,U_{tr}(t,r)=0,\ U_{tt}(t,r)\neq 0.$

(A12) Rank=3 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)=0,U_{rr}(t,r)=0,U_{tt}(t,r)=0,\ U_{tr}(t,r)=0.$

(A13) Rank=3 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,\ V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,V_{rr}(t,r)=0,\ U_{rr}(t,r)\neq 0,\ V_{tr}(t,r)\neq 0,$ $U_{tr}(t,r)\neq 0,U_{tt}(t,r)\neq 0,V(t,r)V_r(t,r)+U_r(t,r)=0.$

(A14)Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,\ U_{rr}(t,r)\neq 0,\ V_{rr}(t,r)\neq 0,\ V_{tr}(t,r)\neq 0.$

(A15) Rank=3, $U_t(t,r)=0,U_r(t,r)=0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,\ V_{rr}(t,r)\neq 0,\ V_{tr}(t,r)\neq 0.$

(A16) Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,U_{rr}(t,r)=0,\ V_{rr}(t,r)\neq 0,V_{tr}(t,r)\neq 0.$

(A17) Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,U_{rr}(t,r)=0,V_{rr}(t,r)=0,V_{tr}(t,r)\neq 0.$

(A18) Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,V_{rr}(t,r)=0,U_{rr}(t,r)\neq 0,V_{tr}(t,r)\neq 0.$

(A19) Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,V_{tr}(t,r)=0,U_{rr}(t,r)\neq 0,V_{rr}(t,r)\neq 0.$

(A20) Rank=3, $U_t(t,r)=0,U_r(t,r)=0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0V_{rr}(t,r)=0,V_{tr}(t,r)\neq 0.$

(A21) Rank=3, $U_t(t,r)=0,U_r(t,r)=0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,V_{tr}(t,r)=0,V_{rr}(t,r)\neq 0.$

(A22) Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,U_{rr}(t,r)=0,V_{tr}(t,r)=0,V_{rr}(t,r)\neq 0,$

(A23) Rank=3, $U_t(t,r)=0,U_r(t,r)\neq 0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,V_{rr}(t,r)=0,V_{tr}(t,r)=0,U_{rr}(t,r)\neq 0.$

(A24) Rank=3 $U_t(t,r)=0,U_r(t,r)=0,V_t(t,r)\neq 0,$ $V_r(t,r)\neq 0,V_{rr}(t,r)=0,V_{tr}(t,r)=0.$

(B1) Rank=2 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)=0,$ $V_r(t,r)=0,\ U_{tt}(t,r)\neq 0,U_{tr}(t,r)\neq 0.$

(B2) Rank=2 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)=0,$ $V_r(t,r)=0,U_{tt}(t,r)=0,U_{tr}(t,r)\neq 0.$

(B3) Rank=2 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)=0,$ $V_r(t,r)=0,U_{tr}(t,r)=0,U_{tt}(t,r)\neq 0.$

(B4) Rank=2 $U_t(t,r)\neq 0,U_r(t,r)\neq 0,V_t(t,r)=0,$ $V_r(t,r)=0,U_{tr}(t,r)=0,U_{tt}(t,r)=0.$

(C1) Rank=1 $U_t(t,r)\neq 0, U_r(t,r)=0, V_t(t,r)\neq 0,$
$V_r(t,r)=0, U_{tt}(t,r)\neq 0.$
(C2) Rank=1 $U_t(t,r)\neq 0, U_r(t,r)=0, V_t(t,r)\neq 0,$
$V_r(t,r)=0, U_{tt}(t,r)=0.$
(C3) Rank=1 $U_t(t,r)\neq 0, U_r(t,r)=0, V_t(t,r)=0,$
$V_r(t,r)=0, U_{tt}(t,r)\neq 0.$
(C4) Rank=1 $U_r(t,r)=0, V_r(t,r)=0, V_t(t,r)=0,$
$U_{tt}(t,r)=0, U_t(t,r)\neq 0$

4. Discussion

It is to be noted that the spacetime (7) is 1+3 decomposable, so all the possible CCS will lie in curvature class C or D.

To illustrate the methodology for calculating CCS, we discuss three cases among the thirty two cases.

Case A1

In this case $U_t(t,r)=0,\ V_r(t,r)=0,\ U_r(t,r)\neq 0,$ $U_{rr}(t,r)\neq 0$. The above constraint equations imply that $U(t,r)=k^1(r)$, $V(t,r)=k^2(t)$, where $k^1(r), k^2(t)$ are nowhere zero functions of integration. Here the rank of the Riemann matrix is three and there exists a covariantly constant vector field z_a which is a unique solution (up to the multiple) of equation $R_{abcd}k^d=0$ i.e. $z_{a;b}=0$ and consequently the Ricci identity implies that $R_{abcd}z^d=0$, meaning that $\dim S_p=1$. The line element can be written as:

$$ds^2=-dt^2+dr^2+k^1(r)d\theta^2+dz^2-2k^2(t)dtd\theta \quad (10)$$

The space-time is clearly 1+3 decomposable. The CCS in this case[6] is of the form

$$X=f(z)\frac{\partial}{\partial z}+X' \quad (11)$$

Where $f(z)$ is an arbitrary function of z and X' is the homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant z. The completion of this case requires finding the homothetic vector fields in the induced geometry on the submanifolds of constant z. The induced metric $g_{\alpha\beta}$ (where $\alpha,\beta=0,1,2$) has non-zero component given by

$$g_{00}=-1, g_{11}=1, g_{22}=k^1(r), g_{02}=g_{20}=-k^2(t) \quad (12)$$

A vector field X is called homothetic vector field if it satisfies

$$L_X g_{\alpha\beta}=2cg_{\alpha\beta},\ c\in R \quad (13)$$

We can expand (13) by using (12) and get

$$X^0_{,0}+k^2(t)X^2_{,0}=c \quad (14)$$

$$-X^0_{,1}+X^1_{,o}-k^2(t)X^2_{,1}=0 \quad (15)$$

$$-\dot{k}^2(t)X^0-k^2(t)X^0_{,0}-X^0_{,2}+ k^1(r)X^2_{,0}-k^2(t)X^2_{,2}=-2ck^2(t) \quad (16)$$

$$X^1_{,1}=c \quad (17)$$

$$-k^2(t)X^0_{,1}+X^1_{,2}+k^1(r)X^2_{,1}=0 \quad (18)$$

$$-2k^2(t)X^0_{,2}+k^1_r(r)X^1+2k^1(r)X^2_{,2}=2ck^1(r) \quad (19)$$

From equation (17), (14) and (15), we have

$$\left.\begin{aligned} X^0&=E^1_t(t,\theta)r-\frac{k^2(t)}{\dot{k}^2(t)}E^1_{tt}(t,\theta)+E^3(t,\theta)\\ X^1&=cr+E^1(t,\theta)\\ X^2&=-\frac{1}{\dot{k}^2(t)}E^1_{tt}(t,\theta)+E^2(t,\theta)\end{aligned}\right\} \quad (20)$$

Now consider (18) and using (20) we have

$$\begin{aligned}&-k^2(t)\left[E^1_t(t,\theta)-\frac{k^2(t)}{\dot{k}^2(t)}E^1_{tt}(t,\theta)\right]+\\ &E^1_\theta(t,\theta)+\frac{k^1(r)}{\dot{k}^2(t)}E^1_{tt}(t,\theta)=0\end{aligned} \quad (21)$$

Differentiate with respect to 'r', we get $\frac{k^1_r(r)}{\dot{k}^2(t)}E^1_{tt}(t,\theta)=0$ .Since $k^1_r(r)\neq 0$. $\Rightarrow E^1_{tt}(t,\theta)=0$, therefore , $E^1(t,\theta)=F^1(\theta)t+F^2(\theta)$. Using result in (21) and differentiating with respect to 't' twice. We get

$\ddot{k}^2(t)F^1(\theta)=0$

There are three possibilities:

$J_1:\ F^1(\theta)=0,\ \ddot{k}^2(t)\neq 0$

$J_2:\ F^1(\theta)=0,\ \ddot{k}^2(t)=0$

$J_3:\ F^1(\theta)\neq 0,\ \ddot{k}^2(t)=0$

Case J_1:

$F^1(\theta)=0,\ \ddot{k}^2(t)\neq 0$, By backward substitution we get $F^2(\theta)=d_1$ Thus the system (20) becomes

$$\left.\begin{aligned}X^0&=E^3(t,\theta)\\ X^1&=cr+d_1\\ X^2&=E^2(t,\theta)\end{aligned}\right\} \quad (22)$$

Now consider (16) and using (22) then differentiating with respect to 'r', we get, $F^1_r(r)E^2_t(t,\theta)=0$, where $F^1_r(r)\neq 0$ $\Rightarrow E^2(t,\theta)=F^3(\theta)$.

Consider (14) and using (22) we get, $E^3(t,\theta)=ct+F^4(\theta)$

The system (22) takes the form:

$$\left.\begin{aligned}X^0&=ct+F^4(\theta)\\ X^1&=cr+d_1\\ X^2&=F^2(\theta)\end{aligned}\right\} \quad (23)$$

Now consider equation (19) and using above system and differentiating with respect to 't' and then 'θ' we have , $F^4(\theta)=d_2$ and $F^3(\theta)=d_3\theta+d_4$.

Equation (23) reduces to:

$$\left.\begin{aligned} X^0 &= ct + d_2 \\ X^1 &= cr + d_1 \\ X^2 &= d_3\theta + d_4 \\ X^3 &= f(z) \end{aligned}\right\} \quad (24)$$

where $k^1(r) = \left(cr + d_1\right)^{2(1-d_3/c)}$, $k^2(t) = \left(ct + d_2\right)^{(1-d_3/c)}$, $\ddot{k}^2(t) \neq 0$ and $d_1, d_2, d_3, d_4 \in R$. One can write the proper CCS after subtracting the homothetic vector fields as

$$X = \left(0,0,0,f(z)\right) \quad (25)$$

Clearly, in this case the proper CCS forms an infinite dimensional vector space.

Case J_2:

In this sub case $F^1(\theta) = 0,\ \ddot{k}^2(t) = 0 \Rightarrow k^2(t) = b_1 t + b_2$,

Using this information in equation (21) and simplifying we get,

$$\left.\begin{aligned} X^0 &= ct + d_2 \\ X^1 &= cr + d_1 \\ X^2 &= d_4, X^3 = f(z) \end{aligned}\right\} \quad (26)$$

where $k^1(r) = \left(cr + d_1\right)^2$, $k^2(t) = \left(b_1 t + b_2\right) \Rightarrow b_1 = c, d_2 = b_2$ and $d_1, d_2, d_4 \in R$. The proper CCS for this case are given in equation (25).

Case J_3:

In this case $F^1(\theta) \neq 0,\ \ddot{k}^2(t) = 0$, this sub case gives contradiction because in this case we assume that $F^1(\theta) \neq 0$ but after straight forward calculation it gives $F^1(\theta) = 0$.

Cases (A2) to (A24) are precisely same.

Case B1

In this case we have, $V_t(t,r) = 0, V_r(t,r) = 0, U_t(t,r) \neq 0,$ $U_r(t,r) \neq 0, U_{tt}(t,r) \neq 0,\ U_{tr}(t,r) \neq 0,$ These conditions imply that $V(t,r) = b$, where $b \in R - \{0\}$ is the constant of integration. Here the rank of the 6×6 Riemann matrix is two and there exists a spacelike covariantly constant vector field z_a which is a unique solution (up to the multiple) of equation $R_{abcd}k^d = 0$, $z_{a;b} = 0$ and consequently the Ricci identity implies that $R_{abcd}z^d = 0$, meaning that $\dim S_p = 1$. The line element can be written as:

$$ds^2 = -dt^2 + dr^2 + U(t,r)d\theta^2 + dz^2 - 2bdtd\theta \quad (27)$$

The space-time is clearly 1+3 decomposable. The CCS in this case[6] are of the form

$$X = f(z)\frac{\partial}{\partial z} + X' \quad (28)$$

Where $f(z)$ is an arbitrary function of z and X' is the homothetic vector field in the induced geometry on each of three dimensional submanifolds of constant z. The completion of this case requires finding the homothetic vector fields in the induced geometry of submanifolds of constant z. The induced metric $g_{\alpha\beta}$ (where $\alpha, \beta = 0,1,2$) has non zero components given by

$$g_{00} = -1, g_{11} = 1, g_{22} = U(t,r), g_{02} = g_{20} = -b. \quad (29)$$

A vector field X' is called homothetic vector field if it satisfies

$$L_x g_{\alpha\beta} = 2cg_{\alpha\beta},\quad c \in R \quad (30)$$

we can expand (30) by using (29) and get

$$X^0_{,0} + bX^2_{,0} = c \quad (31)$$

$$-X^0_{,1} + X^1_{,o} - bX^2_{,1} = 0 \quad (32)$$

$$-bX^0_{,0} - X^0_{,2} + U(t,r)X^2_{,0} - bX^2_{,2} = -2bc \quad (33)$$

$$X^1_{,1} = c \quad (34)$$

$$-bX^0_{,1} + X^1_{,2} + U(t,r)X^2_{,1} = 0 \quad (35)$$

$$\begin{aligned} &\dot{U}(t,r)X^0 - 2bX^0_{,2} + U_{,r}(t,r)X^1 + 2U(t,r)X^2_{,2} \\ &= 2cU(t,r) \end{aligned} \quad (36)$$

From equation (34), (32) and (35) we get :

$$\left.\begin{aligned} X^0 &= E^1_t(t,\theta)r - E^1_t(t,\theta)\int\frac{b^2}{U(t,r)+b^2}dr + \\ &\quad E^1_\theta(t,\theta)\int\frac{b}{U(t,r)+b^2}dr + E^3(t,\theta) \\ X^1 &= cr + E^1(t,\theta) \\ X^2 &= E^1_t(t,\theta)\int\frac{b}{U(t,r)+b^2}dr - \\ &\quad E^1_\theta(t,\theta)\int\frac{1}{U(t,r)+b^2}dr + E^2(t,\theta) \end{aligned}\right\} \quad (37)$$

Consider equation (33) and using (37), we get:

$$\left.\begin{aligned} X^0 &= E^3(t,\theta) \\ X^1 &= cr + d_1 \\ X^2 &= F^1(\theta) \end{aligned}\right\} \quad (38)$$

Consider equation (31) and using (38), we get $E^3(t,\theta) = ct + F^2(\theta)$.

Thus the system (3.1.121) takes the form

$$\left.\begin{aligned} X^0 &= ct + F^2(\theta) \\ X^1 &= cr + d_1 \\ X^2 &= F^1(\theta) \end{aligned}\right\} \quad (39)$$

Again consider (33) and using (39) we have:

$$F^2_\theta(\theta) + F^1_\theta(\theta) = bc$$

$$\Rightarrow F^2_\theta(\theta) = bc - bF^1_\theta(\theta)$$

$$\Rightarrow F^2(\theta) = bc\theta - bF^1(\theta) + d_2$$

System (39) becomes,

$$\left.\begin{aligned} X^0 &= ct + bc\theta - bF^1(\theta) + d_2 \\ X^1 &= cr + d_1 \\ X^2 &= F^1(\theta) \end{aligned}\right\} \quad (40)$$

Now consider (36) and using (40) we get:

$$\frac{\dot{U}(t,r)}{U(t,r)}(ct + bc\theta - bF^1(\theta) + d_2) - \frac{2b}{U(t,r)}\left(bc - bF^1_\theta(\theta)\right) + \frac{U_r(t,r)}{U(t,r)}(cr + d_1) + 2F_1^\theta(\theta) = 2c \quad (41)$$

Differentiating (41) with respect to 'θ', we get

$$\frac{\dot{U}(t,r)}{U(t,r)}\left(bc - bF^1_\theta(\theta)\right) + \left(\frac{2b}{U(t,r)} + 2\right)F^1_{\theta\theta}(\theta) = 0 \quad (42)$$

The solution of the equation (42) will be possible only if $F^1_{\ \theta}(\theta) = 0$

By backward substitution we have the solution:

$$\left.\begin{aligned} X^0 &= 0 \\ X^1 &= 0 \\ X^2 &= d_2 \\ X^3 &= f(z) \end{aligned}\right\} \quad (43)$$

Where $f(z)$ is an arbitrary function of z and $d_2 \in R$, thus the Killing vector field is of the form $(0,0,d_2,f(z))$.

After subtracting the Killing vectors the CCS can be written as

$$(0,0,0,f(z)) \quad (44)$$

clearly, in this case proper CCS form an infinite dimensional vector space.

Cases (B2) to (B4) are precisely same.

Case C1

In this the rank of the Riemann matrix is one, $U_r(t,r)=0$, $V_r(t,r)=0$, $U_t(t,r)\neq 0$, $V_t(t,r)\neq 0$, $U_{tt}(t,r)\neq 0$. These conditions imply that $U(t,r)=k(t)$, $V(t,r)=h(t)$ where $k(t)$ and $h(t)$ are nowhere zero functions of integration.

Here there exists two linearly independent solutions of $R_{abcd}k^d = 0$ namely z_a and r_a, which are both covariantly constant. This shows that $\dim S_p = 2$. The line element takes the form

$$ds^2 = -dt^2 + dr^2 + k(t)d\theta^2 + dz^2 - 2h(t)dtd\theta \quad (45)$$

Clearly the above space-time is seen to be 1+1+2 decomposable and CCS in this case takes the form[6]

$$X = \alpha(r,z)\frac{\partial}{\partial r} + \beta(r,z)\frac{\partial}{\partial z} + X' \quad (46)$$

Where $\alpha(r,z)$ and $\beta(r,z)$ are arbitrary functions of r and z, and X' is the curvature collineations in each of the two dimensional submanifolds of constants r and z. The procedure for calculating the CCS in the induced geometry on the submanifolds can be seen in[6]. The nonzero components of the induced metric on each of the two-dimensional submanifolds of constant r and z, are given by:

$$g_{00} = -1,\ g_{22} = k(t),\ g_{02} = g_{20} = h(t). \quad (47)$$

Calculations shows that nonzero components of Ricci tensor turn out to be

$$R_{00} = \frac{1}{4}\frac{2\ddot{k}(t)k(t) + 2\ddot{k}(t)(h(t))^2 - 2h(t)\dot{h}(t)\dot{k}(t) - (\dot{k}(t))^2}{\left[k(t) + (h(t))^2\right]^2}$$

$$R_{02} = R_{20} = -\frac{h(t)}{4\left[k(t) + (h(t))^2\right]^2}\left[\begin{aligned} &2\ddot{k}(t)k(t) + 2\ddot{k}(t)(h(t))^2 - \\ &2h(t)\dot{h}(t)\dot{k}(t) - (\dot{k}(t))^2 \end{aligned}\right]$$

$$R_{22} = -\frac{k(t)}{4\left[k(t) + (h(t))^2\right]^2}\left[\begin{aligned} &2\ddot{k}(t)k(t) + 2\ddot{k}(t)(h(t))^2 - \\ &2h(t)\dot{h}(t)\dot{k}(t) - (\dot{k}(t))^2 \end{aligned}\right]$$

and the Ricci scalar is given by

$$R = -\frac{1}{2\left[k(t) + (h(t))^2\right]^2}\left[\begin{aligned} &2\ddot{k}(t)k(t) + 2\ddot{k}(t)(h(t))^2 - \\ &2h(t)\dot{k}(t)\dot{h}(t) - (\dot{k}(t))^2 \end{aligned}\right]$$

Accordingly, the Einstein tensor $G_{\alpha\beta} = \frac{R}{2}g_{\alpha\beta}$, (where $\alpha,\beta = 0,2$) has non zero components

$$G_{00} = \frac{1}{4\left[k(t) + (h(t))^2\right]^2}\left[\begin{aligned} &2\ddot{k}(t)\,k(t) + 2\ddot{k}(t)\,(h(t))^2 - \\ &2h(t)\dot{k}(t)\dot{h}(t) - (\dot{k}(t))^2 \end{aligned}\right] = f(t) \quad (48)$$

$$G_{02} = \frac{1}{4\left[k(t) + (h(t))^2\right]^2}\left[\begin{aligned} &2\ddot{k}(t)k(t)h(t) + 2\ddot{k}(t)(h(t))^3 - \\ &2(h(t))^2\dot{k}(t)\dot{h}(t) - h(t)(\dot{k}(t))^2 \end{aligned}\right] = g(t) \quad (49)$$

$$G_{22} = -\frac{1}{4\left[k(t) + (h(t))^2\right]^2}\left[\begin{aligned} &2\ddot{k}(t)(k(t))^2 + 2\ddot{k}(t)(h(t))^2k(t) - \\ &2h(t)k(t)\dot{k}(t)\dot{h}(t) - k(t)(\dot{k}(t))^2 \end{aligned}\right] = p(t) \quad (50)$$

Expanding the equation $L_X G_{\alpha\beta} = 0$, by using (48) to (50)

$$\dot{f}(t)X^0 + 2f(t)X^0_{,0} + 2g(t)X^2_{,0} = 0 \quad (51)$$

$$\dot{g}(t)X^0 + g(t)X^0_{,0} + f(t)X^0_{,2} + p(t)X^2_{,0} + g(t)X^2_{,2} = 0 \quad (52)$$

$$\dot{p}(t)X^0 + 2g(t)X^0_{,2} = 0 \quad (53)$$

The non-trivial solution of the system will be possible only if we substitute $X^2_{,0} = 0$. Otherwise the solution will be trivial.

Now by solving (51) and (52), we get:

$$\left.\begin{aligned} X^0 &= \frac{C^1(\theta)}{\sqrt{f(t)}} \\ X^2 &= -\left[\frac{\dot{g}(t)}{g(t)\sqrt{f(t)}} - \frac{\dot{f}(t)}{2(f(t))^{3/2}}\right]\int C^1(\theta)d\theta - \frac{\sqrt{f(t)}}{g(t)}C^1(\theta) + C^2(t) \end{aligned}\right\} \quad (54)$$

Where $C^1(\theta)$ and $C^2(t)$ are functions of integration. Proceeding further with equation (54), we get the condition

$$-\frac{C^1_\theta(\theta)}{C^1(\theta)} = \frac{\dot{p}(t)}{2g(t)} = \mu \quad (55)$$

There exists following two possibilities:

(a) $\frac{\dot{p}(t)}{2g(t)} = \mu$, (b) $\frac{\dot{p}(t)}{2g(t)} \neq \mu$ where $\mu \in R$

First consider sub case (a), in this case there exists three possibilities which are

(i) $\mu > 0$ (ii) $\mu < 0$ (iii) $\mu = 0$

We discuss each possibility in turn

Case (a) i

In this case $\mu > 0$ and equation (55) implies that $C^1_\theta(\theta) + \mu C^1(\theta) = 0$. The solution of the equation is

$$C^1(\theta) = c_1 \cosh \mu\theta - c_2 \sinh \mu\theta \quad (56)$$

The CCS are of the form

$$\left.\begin{aligned} X^0 &= \frac{1}{\sqrt{f(t)}}\left(c_1 \cosh \mu\theta - c_2 \sinh \mu\theta\right) \\ X^2 &= -\left[\frac{\dot{g}(t)}{g(t)\sqrt{f(t)}} - \frac{\dot{f}(t)}{\left(f(t)\right)^{3/2}}\right]\frac{1}{\mu}\begin{pmatrix} c_1 \sinh \mu\theta \\ -c_2 \cosh \mu\theta \end{pmatrix} \\ &\quad -\frac{\sqrt{f(t)}}{g(t)}\left(c_1 \cosh \mu\theta - c_2 \sinh \mu\theta\right) + C^2(t) \end{aligned}\right\} \quad (57)$$

Where $c_1, c_2 \in R - \{0\}$ and provided that $\dot{p}(t) = 2\mu g(t)$. After subtracting the Killing vector fields, the CCS are

$$X = \left(0, \alpha(r,z), 0, \beta(r,z)\right) \quad (58)$$

Clearly, proper CCS form an infinite dimensional vector space.

Case (a) ii

In this sub case $\mu < 0$. Put $\mu = -m$ where $m \in R (m > 0)$. The CCS in this case are:

$$\left.\begin{aligned} X^0 &= \frac{1}{\sqrt{f(t)}}\left(c_1 \cosh m\theta + c_2 \sinh m\theta\right) \\ X^2 &= -\left[\frac{\dot{g}(t)}{g(t)\sqrt{f(t)}} - \frac{\dot{f}(t)}{\left(f(t)\right)^{3/2}}\right]\frac{1}{m}\begin{pmatrix} c_1 \sinh m\theta \\ +c_2 \cosh m\theta \end{pmatrix} \\ &\quad -\frac{\sqrt{f(t)}}{g(t)}\left(c_1 \cosh m\theta + c_2 \sinh m\theta\right) + C^2(t) \end{aligned}\right\} \quad (59)$$

Where $c_1, c_2 \in R - \{0\}$ and provided that $\dot{p}(t) = -2mg(t)$.

In this case the proper CCS are given in equation (58).

Case (a) iii

Here $\mu = 0$, CCS in this case are

$$\left.\begin{aligned} X^0 &= \frac{1}{\sqrt{f(t)}}c_1 \\ X^2 &= -\left[\frac{\dot{g}(t)}{g(t)\sqrt{f(t)}} - \frac{\dot{f}(t)}{\left(f(t)\right)^{3/2}}\right](c_1\theta + c_2) - \\ &\quad \frac{\sqrt{f(t)}}{g(t)}c_1 + C^2(t) \end{aligned}\right\} \quad (60)$$

provided that $\dot{p}(t) = 0$. After subtracting the Killing vector fields the proper CCS are given in equation (58).

Case (b)

In this case the CCS are

$$\left.\begin{aligned} X^0 &= 0 \\ X^1 &= \alpha(r,z) \\ X^2 &= 0 \\ X^3 &= \beta(r,z) \end{aligned}\right\} \quad (61)$$

Where $\alpha(r,z)$ and $\beta(r,z)$ are arbitrary functions of r and z. The proper CCS in this case are given in equation (58).

Cases (C2) and (C4) are precisely same.

5. Summary

In this paper a mathematical study of non-static spatially homogeneous rotating spacetimes according to their proper CCS is presented. An approach developed in[6] is adopted to study the proper CCS of above spacetimes by using the rank of 6×6 Riemann matrix. From this study we have the following results.

a) In the cases (A1) to (A24) rank of the 6×6 Riemann matrix is three and there exists a unique spacelike covariantly constant vector field which is the solution of equation $R_{abcd}k^d = 0$ .In these cases the spacetime (7) admits proper CCS which form an infinite dimensional vector space.

b) In the cases (B1) to (B4) the rank of the 6×6 Riemann matrix is two and there exists a unique spacelike covariantly constant vector field which is the solution of equation $R_{abcd}k^d = 0$. In these cases the spacetime (7) admits proper CCS which form an infinite dimensional vector space.

c) In cases (C1) to (C4) the rank of the 6×6 Riemann matrix is one and there exists two independent nowhere zero spacelike covariantly constant vector fields which are the solution of equation $R_{abcd}k^d = 0$ In these cases the spacetime (7) admits proper CCS which again form an infinite dimensional vector space.

When we say that CCS forms an infinite dimensional vector space, it actually means that CCS forms an infinite dimensional Lie algebra or Lie groups. Lie groups have fundamental importance in physical system like phase spaces and symmetry groups. When we study the dynamical system with infinite number of degrees of freedom such as PDEs and fields theories, then it is necessary to study the infinite dimensional Lie groups or Lie algebra

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