International Journal of Probability and Statistics 2021, 10(3): 74-93
DOI: 10.5923/j.ijps.20211003.02

New Compounded Family of Distributions, with Applications

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Abstract In this thesis, We propose and study new compounding families, the truncated natural discrete Lindley with three distributions severally (Exponential distribution, Weibull distribution and Lomax distribution) this approach generates new distributions that extend well-known families of distributions. At the same time, they offer more flexibility for modeling lifetime data. The structural properties of the new distributions are investigated. These include the compounding representation of the distribution, reliability analysis and statistical measures. Expressions for Lorenz and Bonferroni curves and Renyi entropy as a measure for uncertainty reduction are derived.. The maximum likelihood method is used to estimate the model parameters. Two real-life data applications are proposed to illustrate the importance of the new families.

Keywords Exponential distribution, Weibull distribution, Lomax distribution, Lindley distribution, Reliability analysis, Moment generating function, Order statistics and Quantile, Maximum likelihood estimation

1. Introduction

In statistical literature, we always assume that every real phenomenon can be modeled by some lifetime distributions. Modeling and analyzing lifetime data are important aspects of statistical research in many applied sciences such as engineering, medicine, economics and so on. Various recent probability distributions discussed modeling of such data by compounding the well-known lifetime distributions such as exponential, Weibull, generalized exponential, exponentiated Weibull and etc. with some discrete distributions such as zero-truncated binomial, geometric, Poisson, logarithmic and the power series distributions in general. The compounding approach gives new distributions that extend well-known families of distributions. At the same time, they offer more flexibility for modeling lifetime data. The flexibility of such compound distributions comes in terms of one or more failure rate shapes that may be decreasing, increasing, bathtub shaped or unimodal shaped. The relevance of mixtures in the analysis and improvements of coherent systems has been pinpointed in various investigations (see, for instance, Navarro et al. (2008) On the application and extension of system signatures in engineering reliability, where comparisons of coherent systems and mixed coherent systems are considered, Samaniego et al. (2007), Dynamic signatures and their use in comparing the reliability of new and used systems, where similar analysis are performed in a dynamic reliability setting, or Dugas et al. (2007), On optimal system designs in reliability-economics frameworks, who showed that solutions to certain optimization problems in reliability turn out to be mixed rather than coherent systems). We propose and study new compounding families will be named the complementary exponential truncated natural discrete Lindley distribution (CETNDL), the complementary Weibull truncated natural discrete Lindley distribution (CWTNDL) and the complementary Lomax truncated natural discrete Lindley distribution (CLTNDL), this approach generates new distributions that extend well-known families of distributions. At the same time, they offer more flexibility for modeling lifetime data.

Adamidis and Loukas (1998) introduced the basic concept of these models. The basic idea of introducing compound models or families is that a lifetime of a system with N (discrete random variable) components and the positive continuous random variable, say Xi (the lifetime of the ith component), can be denoted by the non-negative random variable $X = min(X_1, X_2, \dots X_N)$ (the minimum of an unknown number of any continuous random variables) or $X = max(X_1, X_2, \dots X_N)$ (the maximum of an unknown number of any continuous random variables), based on whether the components are in series or in parallel structure.

If we define $X = min(X_1, X_2, \dots X_N)$ with cdf $G(X; \theta)$; $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ $k \geq 1, X, \theta > 0$ and N is a truncated discrete random variable with probability mass function $P(N = n)$, the following steps explain how can we get the compounding of probability distribution.

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Received: Oct. 25, 2021; Accepted: Nov. 16, 2021; Published: Dec. 24, 2021
Published online at http://journal.sapub.org/ijps

Step (1) obtain the conditional cdf of $X_{(1)} \mid N = n$ as

$$G_{X_{(1)} \mid N=n}(x) = 1 - \{\bar{G}(x;\Theta)\}^n$$

where $\bar{G}(0;\Theta)$ is the survival function.

Step (2) obtain the joint cumulative distribution function as

$$P(X_{(1)} \le x, N = n) = P(N = n)[1 - \{\bar{G}(x;\Theta)\}^n],$$

Step (3) The marginal cumulative distribution of $X_{(1)}$ for the compounding distribution is given by

$$F(x) = \sum_{n=1}^{\infty} P(N = n)\left[1 - \{\bar{G}(x;\Theta)\}^n\right], x > 0$$

If we define $= \max(X_1, X_2, \dots \dots X_N)$, the following steps explain how can we get the Compounding of probability distribution

Step (1) Obtain the conditional cdf of $X_{(n)} \mid N = n$ as

$$G_{X_{(n)} \mid N=n}(x) = \{G(x;\Theta)\}^n.$$

Step (2) obtain the joint cumulative distribution function as

$$P(X_{(n)} \le x, N = n) = P(N = n)[\{G(x;\Theta)\}^n, x > 0.$$

The marginal cumulative distribution of $X_{(1)}$ for the compounding distribution is given by

$$F(x) = \sum_{n=1}^{\infty} P(N = n)[\{G(x;\Theta)\}^n], x > 0$$

The rest of paper is organized as follows: Section 2 introduces the definition of the probability density function (pdf) of the (CETNDL), (CWTNDL), (CLTNDL) distributions including its cumulative distribution function (cdf) and the sub-models of the new suggested model. The reliability analysis including the survival function, the hazard (or failure) rate function, the reversed hazard rate function, the cumulative hazard rate function, and the mean residual lifetime is explored in Section 3. The statistical properties of the new distribution such as the moments, the moment generating function, and the distribution of order statistics are investigated in Section 4, with a proposed algorithm for generating random data from the new distribution in this section. Section 5 introduces Lorenz and Bonferroni curves and Renyi entropy as measures of inequality and uncertainty, respectively. Section 6 discusses the estimation of parameters by using maximum likelihood estimation. Finally, Section 7 provides an application for modeling real data sets to illustrate the performance of the new distribution.

2. Generalization and Related Sub-Models

In this section, we introduce the pdf and the cdf of the three new general compounding families and then the special cases of all distributions.

2.1. Generalization

the series system

Let $X_1, X_2, \dots X_N$ be iid from $\boldsymbol{g(x)}$ $X = \min(X_1, X_2, \dots X_N)$ and x"s and N are independent.

1. The conditional cdf of X|N is given by

$$F_{X_{(1)}|N=n}(x) = F\left(x|n\right) = 1 - \bar{G}(x)^n$$

Now the joint cdf is

$$P\left(X_{(1)} \le x, N = n\right) = p(N = n)\,[1 - \bar{G}(x)^n]$$

This would allow us to reduce the $F_n\ (x)$, is given by

$$F_X\ (x) = \sum_{n=1}^{\infty} p(N = n)[1 - \bar{G}(x)^n]$$

If N follows a truncated natural discrete Lindley, then

$$P(N = n) = p(n) = \frac{\theta^2}{1 + 2\theta}\ (n + 2)(1 - \theta)^{n-1}, n = 1,2, \dots, \theta\epsilon\ (0,1)$$

The Cumulative distribution function (cdf) of X is given by

$$\bar{F}_X\ (x) = \sum_{n=1}^{\infty} \frac{\theta^2}{1 + 2\theta}\ (n + 2)(1 - \theta)^{n-1}[1 - \bar{G}(x)^n]$$

$$= \frac{\theta^2}{1 + 2\theta}\left\{\sum_{n=1}^{\infty} 2(1 - \theta)^{n-1}[1 - \bar{G}(x)^n] + \sum_{n=1}^{\infty} n(1 - \theta)^{n-1}[1 - \bar{G}(x)^n]\right\}$$

$$= \frac{\theta^2}{1 + 2\theta} 2\sum_{n=1}^{\infty}(1 - \theta)^{n-1} - 2\sum_{n=1}^{\infty}(1 - \theta)^{n-1}\,\bar{G}(x)^n + \sum_{n=1}^{\infty} n(1 - \theta)^{n-1} - \sum_{n=1}^{\infty} n(1 - \theta)^{n-1}\bar{G}(x)^n$$

We break the above sum into several components, as follows

To simplify, let

$$A = 2\sum_{n=1}^{\infty}(1 - \theta)^{n-1} = \frac{2}{\theta},$$

$$B = -2\bar{G}(x)\sum_{n=1}^{\infty}[(1-\theta)\bar{G}(x)]^{n-1}$$
$$= -2\bar{G}(x)\frac{1}{1-(1-\theta)\bar{G}(x)} = \frac{-2\bar{G}(x)}{1-(1-\theta)\bar{G}(x)},$$
$$C = \sum_{n=1}^{\infty} n(1-\theta)^{n-1} = \frac{-d}{d\theta}\sum_{n=1}^{\infty}(1-\theta)^{n} = \frac{-d}{d\theta}\frac{1-\theta}{\theta} = \frac{\theta+(1-\theta)}{\theta^2} = \frac{1}{\theta^2},$$

and

$$D = -\bar{G}(x)\sum_{n=1}^{\infty} n[(1-\theta)\bar{G}(x)]^{n-1}$$
$$= \frac{d}{d\theta}\sum_{n=1}^{\infty}[(1-\theta)\bar{G}(x)]^{n} = \frac{d}{d\theta}\frac{(1-\theta)\bar{G}(x)}{1-(1-\theta)\bar{G}(x)}$$
$$= \frac{-\bar{G}(x)[1-(1-\theta)\bar{G}(x)]-(1-\theta)\bar{G}(x)^2}{[1-(1-\theta)\bar{G}(x)]^2} = \frac{-\bar{G}(x)}{[1-(1-\theta)\bar{G}(x)]^2}.$$

Summing up, one gets

$$\bar{F}_X(x) = \frac{\theta^2}{1+2\theta}\left\{\frac{2}{\theta}+\frac{1}{\theta^2}-\frac{2\bar{G}(x)}{1-(1-\theta)\bar{G}(x)}-\frac{\bar{G}(x)}{[1-(1-\theta)\bar{G}(x)]^2}\right\}$$

Or $\bar{F}(x) = \frac{\theta^2}{1+2\theta}\left\{\frac{1+2\theta}{\theta^2}-\frac{3\bar{G}(x)-2(1-\theta)\bar{G}(x)^2}{[1-(1-\theta)\bar{G}(x)]^2}\right\}$ it follows that

$$F_X(x) = 1-\frac{\theta^2}{1+2\theta}\frac{\bar{G}(x)[3-2(1-\theta)\bar{G}(x)]}{[1-(1-\theta)\bar{G}(x)]^2}. \quad (1)$$

The associated pdf can be obtained as

$$f_X(x) = \frac{\theta^2}{1+2\theta}\frac{\{[[1-(1-\theta)\bar{G}(x)]^2][3g(x)-(1-\theta)g(x)\bar{G}(x)]+[3\bar{G}(x)-2(1-\theta)\bar{G}(x)^2]\{2(1-\theta)g(x)[1-(1-\theta)\bar{G}(x)]\}\}}{[1-(1-\theta)\bar{G}(x)]^4}$$

$$f_X(x) = \frac{\theta^2}{1+2\theta}\frac{g(x)[3-(1-\theta)\bar{G}(x)]}{[1-(1-\theta)\bar{G}(x)]^3} \quad (2)$$

Next, specific well known distributions further explored.

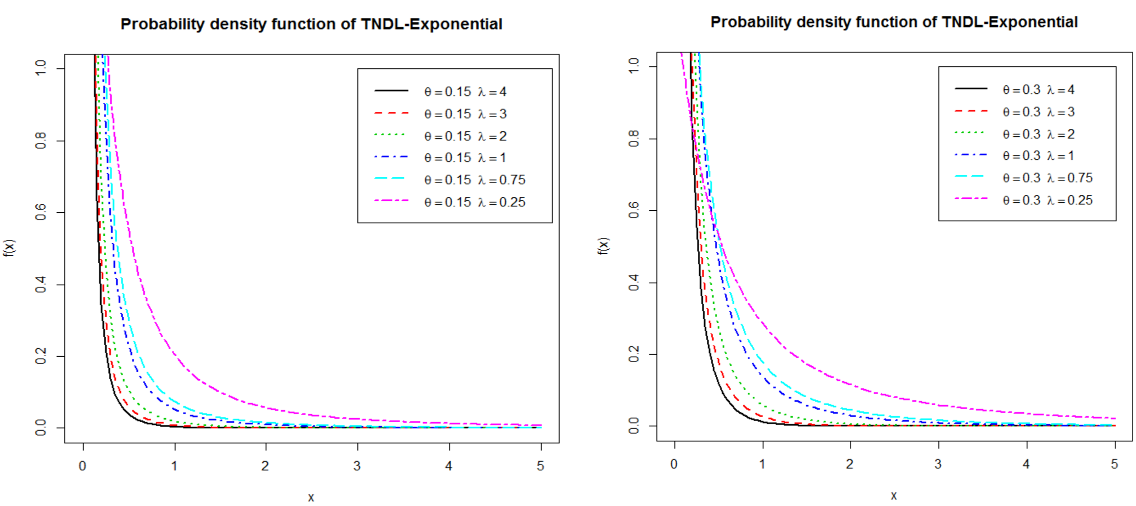
2.1.1. N~ Complementary Exponintial Truncated Natural Discrete Lindley Distribution

Where $\bar{G}(x) = e^{-\lambda x}, g(x) = \lambda e^{-\lambda x}$

Then the pdf of X for the compounding (CETNDL) distribution can be written as

$$f(x;\theta,\lambda) = \frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda x}(3-(1-\theta)e^{-\lambda x})}{(1-(1-\theta)e^{-\lambda x})^3}, x > 0, \theta \in (0,1) \quad (3)$$

The compounding (CETNDL) class of distributions is defined by the marginal cumulative distribution function of X is given by



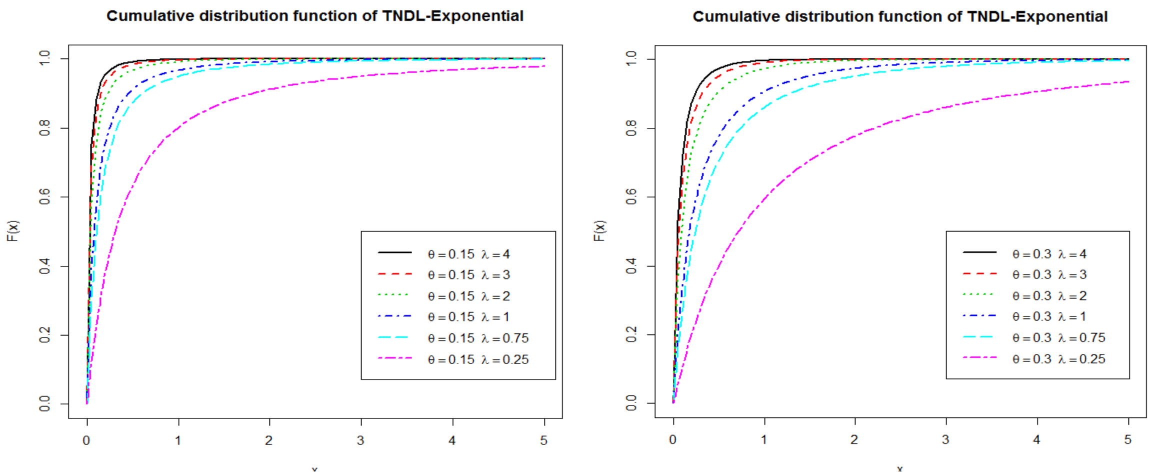


Figure 1. Increasing, Decreasing, Constant, Bathtub and Upside-Down Shapes for cdf, pdf of CETNDL distribution, and for different values

$$F(x;\theta,\lambda) = 1 - \left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda x}\left(3-2(1-\theta)e^{-\lambda x}\right)}{\left(1-(1-\theta)e^{-\lambda x}\right)^2}\right] \quad (4)$$

2.1.2. N~ Complementary Weibull Truncated Natural Discrete Lindley Distribution

Every (x) in exp. distribution can be replaced by $(\lambda x)^{\frac{1}{k}}$

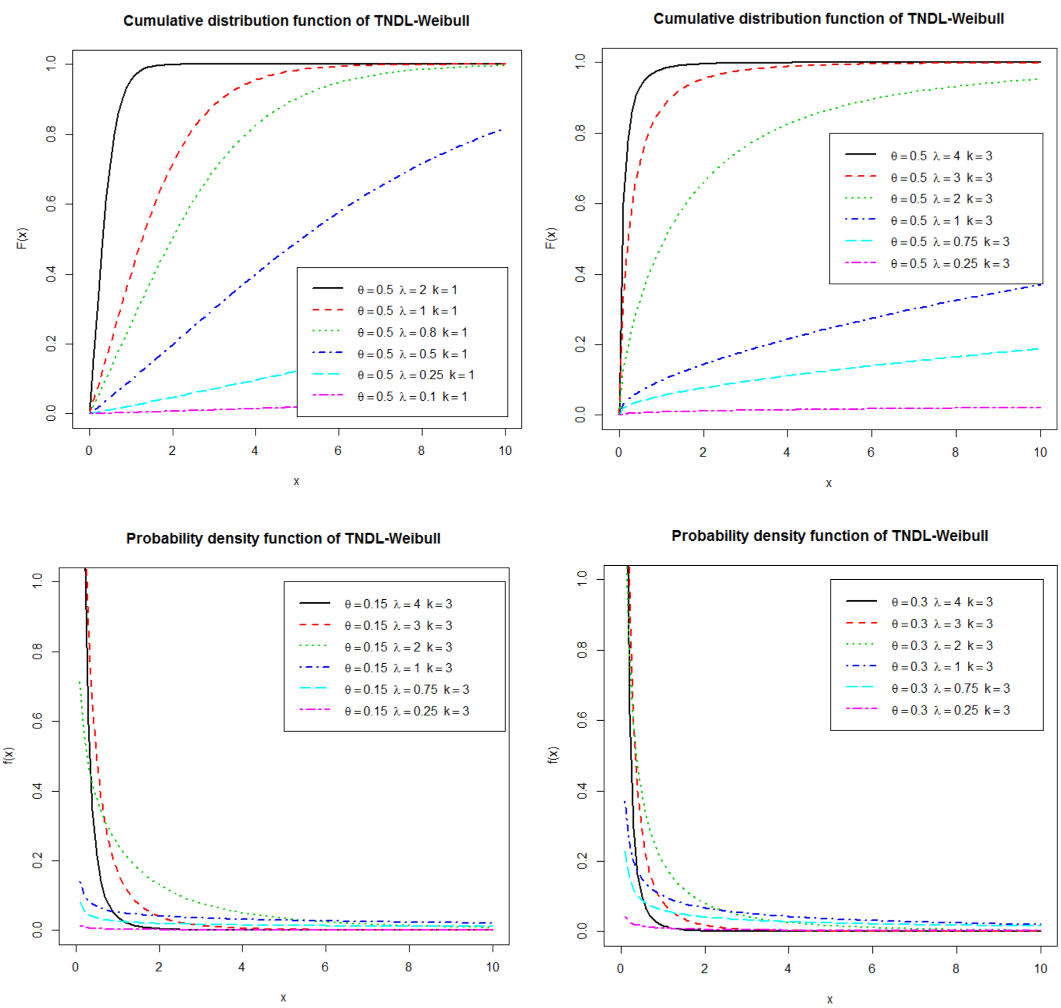


Figure 2. Increasing, Decreasing, Constant, Bathtub and Upside-Down Shapes for cdf, pdf of CWTNDL distribution,and for different values

Then the pdf of X for the compounding (CWTNDL) distribution can be written as

$$f(x;\theta,\lambda)=\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3} \tag{5}$$

the compounding (CWTNDL) class of distributions is defined by the marginal cumulative distribution function of X is given by

$$\mathrm{F}(x;\theta,\lambda)=1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^2}\right] \tag{6}$$

2.1.3. N~ Complementary Lomax Truncated Natural Discrete Lindley Distribution

where $g(x)=\frac{\lambda k}{1+\lambda x^{k+1}}$ and $\bar{G}(x)=-(1+\lambda x)^{-k}$

Then the pdf of X for the compounding (CLTNDL) distribution can be written as

$$f(x)=\frac{\theta^2}{1+2\theta}\frac{\frac{\lambda k}{1+\lambda x^{k+1}}(3-(1-\theta)-(1+\lambda x)^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3} \tag{7}$$

the compounding (CLTNDL) class of distributions is defined by the marginal cumulative distribution function of X is given by

$$F(x)=1-[\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda x)^{-k}(3-2(1-\theta)(-(1+\lambda x))^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2} \tag{8}$$

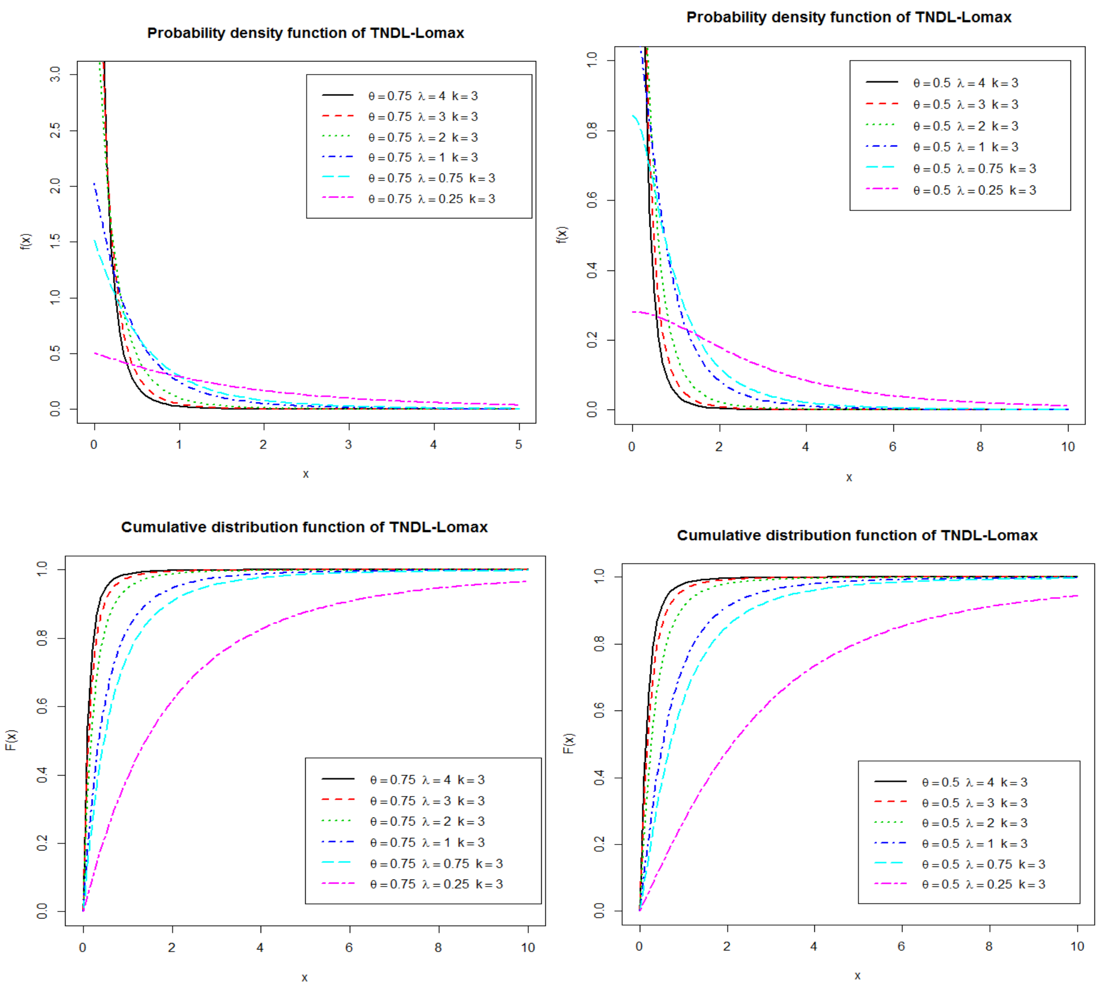


Figure 3. Increasing, Decreasing, Constant, Bathtub and Upside-Down Shapes for cdf, pdf of CLTNDL distribution,and for different values

3. Reliability Analysis

The time for an event to take place in an individual is called a survival time. Examples include the time that an individual survives after being diagnosed with a terminal illness or the time that an electronic component functions before failing. Two important functions for describing survival data are the survival function, the hazard function, The Reversed Hazard Rate Function, The Cumulative Hazard Rate Function, The Mean Residual Lifetime.

3.1. Survival Functions

The survival function is the probability that an observation survives longer than x, that is

$$\bar{F}(x) = P(X > x) = 1 - F(x)$$

Because $\bar{F}x$ is a probability, it is positive and ranges from 0 to 1. It is defined as $\bar{F}(-\infty) = 1$ and as t approaches to ∞, $\bar{F}(x)$ approaches to 0. (See Lee (1992), Therefore,

3.1.1. The Survival Probability for the CETNDL Distribution is Given by

$$s(x) = 1 - \mathrm{F}(X) = \frac{\theta^2}{1+2\theta}\left[\frac{e^{-\lambda x}\left(3-2(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^2}\right]. \quad (9)$$

3.1.2. The Survival Probability for the CWTNDL Distribution is Given by

$$s(x) = \frac{\theta^2}{1+2\theta}\left[\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^2}\right] \quad (10)$$

3.1.3. The Survival Probability for the CLTNDL Distribution is Given by

$$s(x) = \left[\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda x)^{-k}\left(3-2(1-\theta)(-(1+\lambda x))^{-k}\right)}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2}\right] \quad (11)$$

3.2. Hazard Rate Function

Nelson, (1982) defined the hazard rate is the rate of death/failure at an instant x, given that the individual survives up to time t. It measures how likely an observation is to fail as a function of the age of the observation. This function is also called the instantaneous failure rate or the force of mortality. It is defined as

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{\bar{F}(x)}$$

where $f(x)$ is the probability density function of X therefor

3.2.1. The Hazard Rate Function for the CETNDL Distribution is Given by

$$h(x) = \frac{f(x)}{s(x)} = \frac{\frac{\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{1-r}}{e^{-\lambda x}\left(3-2(1-\theta)e^{-\lambda x}\right)}, where\ r = (1-\theta)e^{-\lambda x} \quad (12)$$

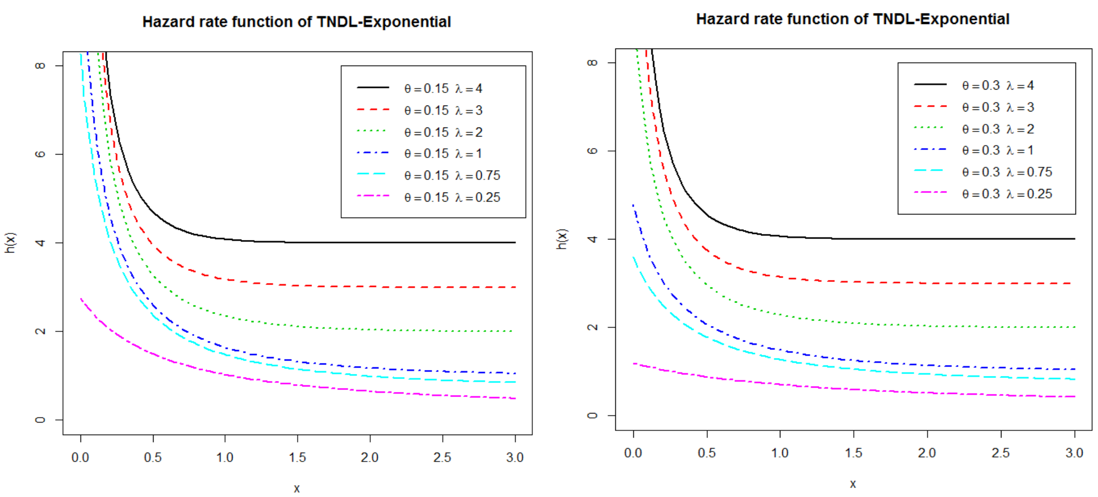


Figure 4. Increasing, Decreasing, Constant, Bathtub and Upside-Down Shapes for the Hazad Rate Function of the CETNDL

3.2.2. The Hazard Rate Function for the CWTNDL Distribution is Given by

$$h(x) = \frac{\lambda\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}}\left[3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right]^{-1} \tag{13}$$

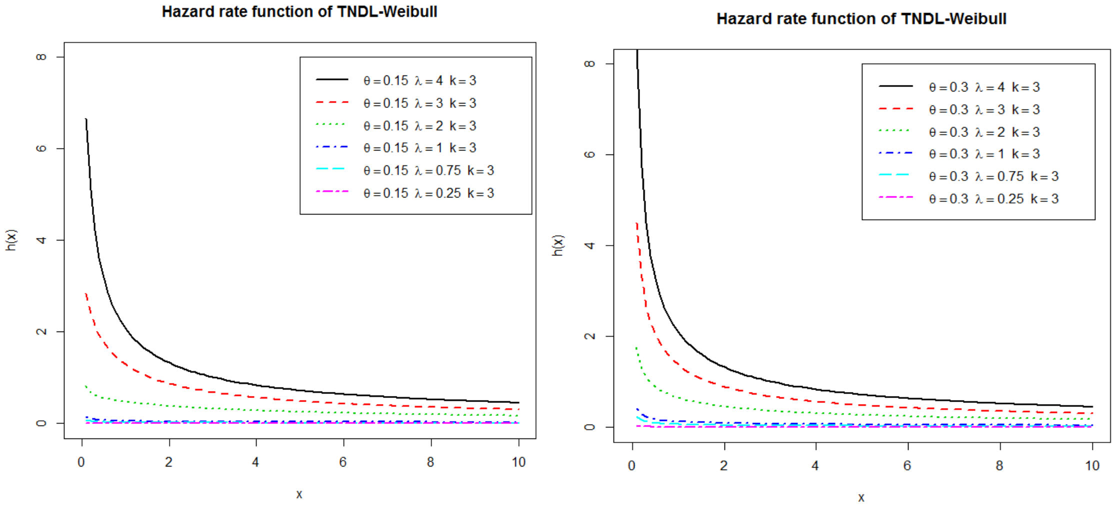


Figure 5. Increasing, Decreasing, Constnt, Bathtub and Upside-Down Shapes for The Hazad Rate Function of the CWTNDL

3.2.3. The Hazard Rate Function for the CLTNDL Distribution is Given by

$$h(x) = \frac{\frac{\lambda k}{1+\lambda x^{k+1}}(3-(1-\theta)-(1+\lambda x)^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2}[-(1+\lambda x)^{-k}(3-2(1-\theta)(-(1+\lambda x))^{-k})]^{-1} \tag{14}$$

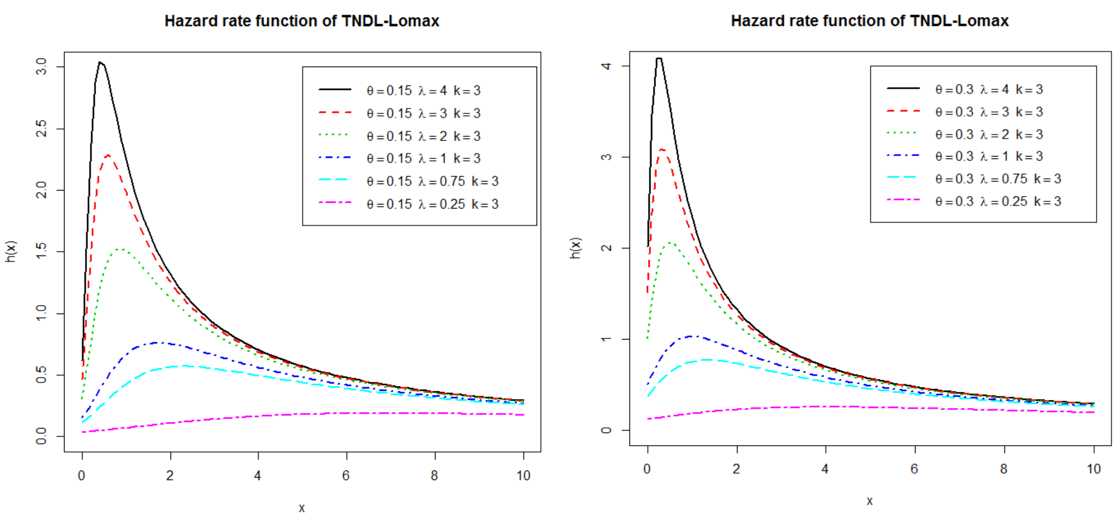


Figure 6. Increasing, Decreasing, Constant, Bathtub and Upside-Down Shapes for The Hazad Rate Function of the CLTNDL

3.3. The Reversed Hazard Rate Function

The reversed (reverse) hazard rate, also named retro hazard, was first mentioned by the name 'dual of the hazard rate' in BARLOW et al. (1963). The name 'reversed hazard rate' was first used by LAGAKOS et al. (1988). 6 It extends the concept of hazard rate to a reverse time direction and is defined as:

$$rh(x) = \frac{f(x)}{F(x)}, F(x) > 0$$

3.3.1. The Reversed Hazard Rate Function for the CETNDL Distribution is Given by

$$rh(x) = \frac{\frac{\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^3})}{1-[\frac{e^{-\lambda x}(3-2(1-\theta)e^{-\lambda x})}{(1-(1-\theta)e^{-\lambda x})^2}]}, F(x) > 0$$

$$= \frac{\theta^2\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{(1+2\theta)(1-(1-\theta)e^{-\lambda x})^5 - \theta^2 e^{-\lambda x}(3-2(1-\theta)e^{-\lambda x})(1-(1-\theta)e^{-\lambda x})}$$

$$= \frac{\theta^2\lambda e^{-\lambda x}(3-r)}{(1+2\theta)(1-r)^5 - \theta^2 e^{-\lambda x}(3-2r)(1-r)} \tag{15}$$

3.3.2. The Reversed Hazard Rate Function for the CWTNDL Distribution is Given by

$$rh(x) = \frac{\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}}{1-[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^2}]}$$

$$= \frac{\lambda\theta^2 e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1+2\theta)(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^3 - \theta^2 e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}$$

$$= \frac{\theta^2\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}(3-r)}{(1+2\theta)(1-r)^3 - \theta^2 e^{-\lambda(\lambda x)^{\frac{1}{k}}}(3-2r)(1-r)} \tag{16}$$

3.3.3. The Reversed Hazard Rate Function for the CLTNDL Distribution is Given by

$$rh(x) = \frac{\frac{\theta^2}{1+2\theta}\frac{\frac{\lambda k}{1+\lambda x^{k+1}}(3-(1-\theta)-(1+\lambda x)^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3}}{1-[\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda x)^{-k}(3-2(1-\theta)(-(1+\lambda x))^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2}}$$

$$r(x) = \frac{\lambda\theta^2 e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1+2\theta)(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^3 - \theta^2 e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}$$

$$= \frac{\theta^2\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}(3-r)}{(1+2\theta)(1-r)^3 - \theta^2 e^{-\lambda(\lambda x)^{\frac{1}{k}}}(3-2r)(1-r)} \tag{17}$$

3.4. The Cumulative Hazard Rate or Integrated Hazard Rate CHR is Defined as

$$H(x) = -\log(Sx)$$

3.4.1. The Cumulative Hazard Rate Function of the CETNDL Distribution is Given by

$$H(x) = -\log\left(\frac{\theta^2}{1+2\theta}\left[\frac{e^{-\lambda x}\left(3-2(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^2}\right]\right)$$

3.4.2. The Cumulative Hazard Rate Function of the CWTNDL Distribution is Given by

$$H(x) = -\log\left(\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^2}\right)$$

$$= -\left(\log\left(\frac{\theta^2}{1+2\theta}\right) + \log\left[\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^2}\right]\right)$$

$$= -\left(2\log(\theta) - \log(1+2\theta) - \lambda(\lambda x)^{\frac{1}{k}} + \log\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right) - 2\log\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)\right)$$

3.4.3. The Cumulative Hazard Rate Function of the CLTNDL Distribution is Given by

$$H(x) = -\log\left(\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda x)^{-k}(3-2(1-\theta)(-(1+\lambda x))^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2}\right)$$

$$= -\left(\log\left(\frac{\theta^2}{1+2\theta}\right) + \log\left[\frac{-(1+\lambda x)^{-k}(3-2(1-\theta)(-(1+\lambda x))^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2}\right]\right)$$

$$= -\big[2\log(\theta) - \log(1+2\theta) - k\log\big(-(1+\lambda x)\big) + \log(3-2(1-\theta)(-(1+\lambda x))^{-k}) - 2\log(1 - (1-\theta)(-(1+\lambda x))^{-k})\big]$$

where $H(x)$ is the total number of failure or deaths over an interval of time, and $H\ (x)$ is a non-decreasing function of satisfying. $H(0) = 0$, $(b) lim x\to\infty\ H(x) = \infty$.

4. Statistical Properties

This section investigates the statistical properties of the (CETNDL), (CWTNDL) and (CLTNDL) distributions

4.1. Distribution of Order Statistics

4.1.1. Distribution of Order Statistics for (CETNDL)

Let $X_1, X_2, \dots, X_n$ be a simple random sample of size n from CETNDL with cumulative distribution function $F(x)$ and probability density function $f(x)$ given by (1.1) and (1.2) respectively. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics obtained from this sample. The probability density function and the cumulative distribution function of the k^{th} order statistic, say $y = X_{(k)}$ are given by:

$$f_Y(y) = \frac{n!}{(k-1)!\,(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y)$$

thus

$$f_Y(y) = \frac{n!}{(k-1)!\,(n-k)!}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda y}\left(3-2(1-\theta)e^{-\lambda y}\right)}{(1-(1-\theta)e^{-\lambda y})^2}\right]\right]^{k-1}$$

$$\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda y}\left(3-2(1-\theta)e^{-\lambda y}\right)}{(1-(1-\theta)e^{-\lambda y})^2}\right]^{n-k}\left[\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^3}\right]$$

and

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j}[F(y)]^j\,[1-F(y)]^{n-j}$$

$$= \sum_{j=k}^{n}\binom{n}{j}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda y}\left(3-2(1-\theta)e^{-\lambda y}\right)}{(1-(1-\theta)e^{-\lambda y})^2}\right]\right]^j\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda y}\left(3-2(1-\theta)e^{-\lambda y}\right)}{(1-(1-\theta)e^{-\lambda y})^2}\right]^{n-j}$$

4.1.2. Distribution of Order Statistics for (CWTNDL)

Let $X_1, X_2, \dots, X_n$ be a simple random sample of size n from CWTNDL with cumulative distribution function $F(x)$ and probability density function $f(x)$ given by (1.1) and (1.2) respectively. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics obtained from this sample. The probability density function and the cumulative distribution function of the k^{th} order statistic, say $y = X_{(k)}$ are given by:

$$f_Y(y) = \frac{n!}{(k-1)!\,(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y)$$

thus

$$f_Y(y) = \frac{n!}{(k-1)!\,(n-k)!}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda y)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}})^2}\right]\right]^{k-1}$$

$$\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda y)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}})^2}\right]^{n-k}\left[\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda(\lambda y)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}}\right)^3}\right]$$

and

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j}[F(y)]^j\,[1-F(y)]^{n-j}$$

$$= \sum_{j=k}^{n}\binom{n}{j}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda y)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}})^2}\right]\right]^j\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda y)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda y)^{\frac{1}{k}}})^2}\right]^{n-j}$$

4.1.3. Distribution of Order Statistics for (CLTNDL)

Let $X_1, X_2, \dots, X_n$ be a simple random sample of size n from CLTNDL with cumulative distribution function $F(x)$ and probability density function $f(x)$ given by (1.1) and (1.2) respectively. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics obtained from this sample. The probability density function and the cumulative distribution function of the k^{th} order statistic, say $y = X_{(k)}$ are given by:

$$f_Y(y) = \frac{n!}{(k-1)!\,(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y)$$

thus

$$f_Y(y) = \frac{n!}{(k-1)!\,(n-k)!}\left[1-\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda y)^{-k}(3-2(1-\theta)(-(1+\lambda y))^{-k})}{(1-(1-\theta)(-(1+\lambda y))^{-k})^2}\right]^{k-1}$$

$$\left[\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda y)^{-k}(3-2(1-\theta)(-(1+\lambda y))^{-k})}{(1-(1-\theta)(-(1+\lambda y))^{-k})^2}\right]^{n-k}\left[\frac{\theta^2}{1+2\theta}\frac{\frac{\lambda k}{1+\lambda y^{k+1}}(3-(1-\theta)-(1+\lambda y)^{-k})}{(1-(1-\theta)(-(1+\lambda y))^{-k})^3}\right]$$

and

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j}[F(y)]^j\,[1-F(y)]^{n-j}$$

$$= \sum_{j=k}^{n}\binom{n}{j}\left[1-\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda y)^{-k}(3-2(1-\theta)(-(1+\lambda y))^{-k})}{(1-(1-\theta)(-(1+\lambda y))^{-k})^2}\right]^j$$

$$\left[\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda y)^{-k}(3-2(1-\theta)(-(1+\lambda y))^{-k})}{(1-(1-\theta)(-(1+\lambda y))^{-k})^2}\right]^{n-j}$$

4.2. The Quantile Function

4.2.1. The Quantile Function of (CETNDL) Distribution

The quantile function (x_q), is given by:

$$x_q = F(x_q)^{-1}, 0 < q < 1$$

Substitute from (1.2), the quantile function is

$$x_q = \frac{1}{\lambda}\left(\log(1-\theta) - \log\left[\frac{2(1-q) + \frac{3\theta^2}{(1+2\theta)(1-\theta)} - \sqrt{\frac{9\theta^4}{(1+2\theta)^2(1-\theta)^2} + \frac{4(1-q)\theta^2}{(1+2\theta)(1-\theta)}}}{2(1-q) + \frac{4\theta^2}{(1+2\theta)(1-\theta)}}\right]\right)$$

In particular when $q = 0.5$ the median can be defined as:

$$x_{0.5} = \frac{1}{\lambda}\left(\log(1-\theta) - \log\left[\frac{1 + \frac{3\theta^2}{(1+2\theta)(1-\theta)} - \sqrt{\frac{9\theta^4}{(1+2\theta)^2(1-\theta)^2} + \frac{2\theta^2}{(1+2\theta)(1-\theta)}}}{1 + \frac{4\theta^2}{(1+2\theta)(1-\theta)}}\right]\right)$$

4.2.2. The Quantile Function of (CWTNDL) Distribution

The quantile function (x_q), is given by:

$$x_q = F(x_q)^{-1}, 0 < q < 1$$

Substitute from (1.2), the quantile function is

$$x_q = \frac{1}{\lambda}\left(\frac{1}{\lambda}\left(\log(1-\theta) - \log\left[\frac{2(1-q) + \frac{3\theta^2}{(1+2\theta)(1-\theta)} - \sqrt{\frac{9\theta^4}{(1+2\theta)^2(1-\theta)^2} + \frac{4(1-q)\theta^2}{(1+2\theta)(1-\theta)}}}{2(1-q) + \frac{4\theta^2}{(1+2\theta)(1-\theta)}}\right]\right)\right)^k$$

In particular when $q = 0.5$ the median can be defined as:

$$x_{0.5} = \frac{1}{\lambda}\left(\frac{1}{\lambda}\left(\log(1-\theta) - \log\left[\frac{1 + \frac{3\theta^2}{(1+2\theta)(1-\theta)} - \sqrt{\frac{9\theta^4}{(1+2\theta)^2(1-\theta)^2} + \frac{2\theta^2}{(1+2\theta)(1-\theta)}}}{1 + \frac{4\theta^2}{(1+2\theta)(1-\theta)}}\right]\right)\right)^k$$

4.2.3. The Quantile Function of (CLTNDL) Distribution

The quantile function (x_q), is given by:

$$x_q = F(x_q)^{-1}, 0 < q < 1$$

the quantile function is

$$x_q = \frac{-1}{\lambda}\left(1 + \left(\frac{2(1-q) + \frac{3\theta^2}{(1+2\theta)(1-\theta)} - \sqrt{\frac{9\theta^4}{(1+2\theta)^2(1-\theta)^2} + \frac{4(1-q)\theta^2}{(1+2\theta)(1-\theta)}}}{2(1-q)(1-\theta) + \frac{4\theta^2}{(1+2\theta)}}\right)^{\frac{-1}{k}}\right)$$

In particular when $q = 0.5$ the median can be defined as:

$$x_{0.5} = x_q = \frac{-1}{\lambda}\left(1 + \left(\frac{1 + \frac{3\theta^2}{(1+2\theta)(1-\theta)} - \sqrt{\frac{9\theta^4}{(1+2\theta)^2(1-\theta)^2} + \frac{2\theta^2}{(1+2\theta)(1-\theta)}}}{(1-\theta) + \frac{4\theta^2}{(1+2\theta)}}\right)^{\frac{-1}{k}}\right)$$

4.3. The r^{th} Non-Central Moment

4.3.1. The $\mathbf{r^{th}}$ Non-Central Moment for (CETNDL) Distribution

The r^{th} non-central moment is given by:

$$\acute{\mu}_r = E(x^r) = \frac{\Gamma(r+1)}{\lambda^r}\frac{\theta^2}{(1+2\theta)}\sum_j^{\infty}\frac{(j+2)(1-\theta)^{j-1}}{j^r}$$

4.3.2. The $\mathbf{r^{th}}$ Non-Central Moment for (CWTNDL) Distribution

The r^{th} non-central moment is given by:

$$\acute{\mu}_r = E(x^r) = \frac{\Gamma(r+1)}{\lambda^{r(r+1)}}\frac{\theta^2}{(1+2\theta)}\sum_j^{\infty}\frac{(j+2)\theta^j}{j^{r+1}}$$

4.3.3. The $\mathbf{r^{th}}$ Non-Central Moment for (CLTNDL) Distribution

The r^{th} non-central moment is given by:

$$\acute{\mu}_r = E(x^r) = \frac{\Gamma(r-1)}{\lambda^{r-1}}\frac{\theta^2}{(1+2\theta)}\sum_j^{\infty}\frac{(j+2)\theta^{j-r}}{j^r}$$

4.4. The Moment Generating Function

4.4.1. The Following Theorem Gives the Moment Generating Function (mgf) of (CETNDL) Distribution $(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{\lambda})$

$$M_X(t) = E(e^{xt}) = \frac{\theta^2}{(1+2\theta)}\sum_{j=1}^{\infty} j(j+2)(1-\theta)^{j-1}\frac{\beta}{\beta j - t}$$

5. Measures of Inequality and Uncertainty

In this section Lorenz and Bonferroni curves are introduced as measures of inequality. Also, Renyi entropy will be mentioned as an important measure of uncertainty.

5.1. Lorenz and Bonferroni Curves

5.1.1. Lorenz and Bonferroni Curves for (CETNDL) can be Obtained as Follows

A. Lorenz curve can be obtained by

$$L(F(x)) = \frac{\int_0^x tf(t)dt}{E(X)}$$

$$= \frac{\left(\frac{\lambda\theta^2}{1+2\theta}\right)\int_0^x \frac{te^{-\lambda t}\left(3-(1-\theta)e^{-\lambda t}\right)}{(1-(1-\theta)e^{-\lambda t})^3}dt}{E(X)}$$

B. Bonferroni curve can be obtained by

$$L(F(x)) = \frac{\int_0^x tf(t)dt}{F(x)E(X)} = \frac{L(F(x))}{F(x)}$$

$$= \frac{\left(\frac{\lambda\theta^2}{1+2\theta}\right)\int_0^x \frac{te^{-\lambda t}\left(3-(1-\theta)e^{-\lambda t}\right)}{(1-(1-\theta)e^{-\lambda t})^3}dt}{E(X)}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda x}\left(3-2(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^2}\right]\right]^{-1}$$

5.1.2. Lorenz and Bonferroni Curves for (CWTNDL) can be Obtained as Follows

A. Lorenz curve can be obtained by

$$L(F(x)) = \frac{\int_0^x tf(t)dt}{E(X)} = \frac{\left(\frac{\lambda\theta^2}{1+2\theta}\right)\int_0^x \frac{e^{-\lambda(\lambda t)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda t)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda t)^{\frac{1}{k}}}\right)^3}dt}{E(X)}$$

B. Bonferroni curve can be obtained by

$$B(F(x)) = \frac{\int_0^x tf(t)dt}{F(x)E(X)} = \frac{L(F(x))}{F(x)}$$

$$= \frac{\left(\frac{\lambda\theta^2}{1+2\theta}\right)\int_0^x \frac{e^{-\lambda(\lambda t)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda t)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda t)^{\frac{1}{k}}}\right)^3}dt}{E(X)}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-2(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}})^2}\right]\right]^{-1}$$

5.1.3. Lorenz and Bonferroni Curves for (CLTNDL) can be Obtained as Follows

$$L(F(x)) = \frac{\int_0^x tf(t)dt}{E(X)} = \frac{\left(\frac{\lambda\theta^2}{1+2\theta}\right)\int_0^x \frac{\frac{tk}{1+\lambda t^{k+1}}(3-(1-\theta)-(1+\lambda t)^{-k})}{(1-(1-\theta)(-(1+\lambda t))^{-k})^3}dt}{E(X)}$$

Bonferroni curve can be obtained by

$$L(F(x)) = \frac{\int_0^x tf(t)dt}{F(x)E(X)} = \frac{L(F(x))}{F(x)}$$

$$= \frac{\left(\frac{\lambda\theta^2}{1+2\theta}\right)\int_0^x \frac{\frac{tk}{1+\lambda t^{k+1}}(3-(1-\theta)-(1+\lambda t)^{-k})}{(1-(1-\theta)(-(1+\lambda t))^{-k})^3}dt}{E(X)}\left[1-\left[\frac{\theta^2}{1+2\theta}\frac{-(1+\lambda x)^{-k}(3-2(1-\theta)(-(1+\lambda x))^{-k})}{(1-(1-\theta)(-(1+\lambda x))^{-k})^2}\right]\right]^{-1}$$

5.2. Renyi Entropy

According to Meeker and Escobar (1998) the entropy of random variable X with density function $f(x)$ is a measure of uncertainty or randomness of a system. One of the popular entropy measures is Rényi entropy which is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty f(x)^\gamma\, dx$$

5.2.1. Rényi Entropy for (CETNDL) is Obtained as Follows

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty f(x)^\gamma\, dx = \frac{1}{1-\gamma}\log\int_0^\infty\left(\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^3}\right)^\gamma dx$$

$$= \frac{1}{1-\gamma}\left(\gamma\log\left(\frac{\theta^2}{1+2\theta}\right)+\log\int_0^\infty\left(\frac{\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^3}\right)^\gamma dx\right)$$

5.2.2. Rényi Entropy for (CWTNDL) is Obtained as Follows

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty\left(\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}\right)^\gamma dx$$

$$= \frac{1}{1-\gamma}\left(\gamma\log\left(\frac{\theta^2}{1+2\theta}\right)+\log\int_0^\infty\left(\frac{\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}\right)^\gamma dx\right)$$

5.2.3. Rényi Entropy for (CLTNDL) is Obtained as Follows

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\int_0^\infty\left(\frac{\theta^2}{1+2\theta}\frac{\frac{\lambda k}{1+\lambda x^{k+1}}\left(3-(1-\theta)(-(1+\lambda x)^{-k})\right)}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3}\right)^\gamma dx$$

$$=\frac{1}{1-\gamma}\left(\gamma\log\left(\frac{\theta^2}{1+2\theta}\right)+\log\int_0^\infty\left(\frac{\frac{\lambda k}{1+\lambda x^{k+1}}\left(3-(1-\theta)(-(1+\lambda x)^{-k})\right)}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3}\right)^\gamma dx\right)$$

6. Maximum Likelihood Estimation

In this section, we will discuss the estimation of the unknown parameters of the (CETNDL), (CWTNDL), (CLTNDL) using the maximum likelihood estimation (MLE) method

6.1. Maximum Likelihood Estimation for (CETNDL) as Follows

The estimators of unknown parameters of the CETNDL model are obtained based on maximum likelihood (ML) method. Let $x_1,\dots,x_n$ be a random sample of size n from CETNDL, with set a parameters (θ,λ), the likelihood function of the density is given by,

$$L(\theta,\lambda;x)=\prod_{i=1}^n f(x_i;\theta,\lambda)$$

$$=\prod_{i=1}^n\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda x}\left(3-(1-\theta)e^{-\lambda x}\right)}{(1-(1-\theta)e^{-\lambda x})^3}$$

$$=\left(\frac{\lambda\theta^2}{1+2\theta}\right)^n\prod_{i=1}^n\frac{e^{-\lambda x_i}\left(3-(1-\theta)e^{-\lambda x_i}\right)}{(1-(1-\theta)e^{-\lambda x_i})^3}$$

Then, the log likelihood function is given by

$$l=\log L=n\log\lambda+2n\log\theta-n(\log(1+2\theta))+\sum_{i=1}^n\log\left(\frac{e^{-\lambda x_i}\left(3-(1-\theta)e^{-\lambda x_i}\right)}{(1-(1-\theta)e^{-\lambda x_i})^3}\right)$$

$$\sum_{i=1}^n\log\left(\frac{e^{-\lambda x_i}\left(3-(1-\theta)e^{-\lambda x_i}\right)}{(1-(1-\theta)e^{-\lambda x_i})^3}\right)=\sum_{i=1}^n-\lambda x_i+\log\left(3-(1-\theta)e^{-\lambda x_i}\right)-3\log(1-(1-\theta)e^{-\lambda x_i})$$

Thus

$$l=n\log\lambda+2n\log\theta-n(\log(1+2\theta))+\sum_{i=1}^n-\lambda x_i+\log\left(3-(1-\theta)e^{-\lambda x_i}\right)-3\log\left(1-(1-\theta)e^{-\lambda x_i}\right)$$

To maximize l, taken first order partial derivatives of l with respect to λ,θ and equating the results equations to zeros and solve them.

Taking derivatives to l with respect to λ,θ, respectively, one get:

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^n-x_i+\frac{(1-\theta)x_ie^{-\lambda x_i}}{3-(1-\theta)e^{-\lambda x_i}}-\frac{3(1-\theta)x_ie^{-\lambda x_i}}{1-(1-\theta)e^{-\lambda x_i}}$$

$$\frac{\partial l}{\partial\theta}=\frac{2n}{\theta}-\frac{2n}{1+2\theta}+\sum_{i=1}^n\frac{e^{-\lambda x_i}}{3-(1-\theta)e^{-\lambda x_i}}-\frac{e^{-\lambda x_i}}{1-(1-\theta)e^{-\lambda x_i}}$$

By equating $\frac{\partial l}{\partial\lambda}|_{\lambda=\hat{\lambda}}$ and $\frac{\partial l}{\partial\lambda}|_{\theta=\hat{\theta}}$ to zero and solving for $\hat{\lambda}$ and $\hat{\theta}$, the MLE of theses parameters are numerically solutions of the following two equations:

$$\frac{n}{\hat{\lambda}}-\sum_{i=1}^{n}x_i+\frac{\left(1-\hat{\theta}\right)x_ie^{-\hat{\lambda}x_i}}{3-\left(1-\hat{\theta}\right)e^{-\hat{\lambda}x_i}}-\frac{3\left(1-\hat{\theta}\right)x_ie^{-\lambda x_i}}{1-\left(1-\hat{\theta}\right)e^{-\lambda x_i}}=0$$

$$\frac{2n}{\hat{\theta}}-\frac{2n}{1+2\hat{\theta}}+\sum_{i=1}^{n}\frac{e^{-\hat{\lambda}x_i}}{3-(1-\theta)e^{-\hat{\lambda}x_i}}-\frac{e^{-\hat{\lambda}x_i}}{1-(1-\theta)e^{-\hat{\lambda}x_i}}=0$$

6.2. Maximum Likelihood Estimation for (CWTNDL) as Follows

The estimators of unknown parameters of the CWTNDL model are obtained based on maximum likelihood (ML) method. Let $x_1,\ldots,x_n$ be a random sample of size n from CWTNDL, with set a parameters (θ,λ,k), the likelihood function of the density is given by,

$$L(\theta,\lambda,k;x)=\prod_{i=1}^{n}f(x_i;\theta,\lambda,k)$$

$$=\prod_{i=1}^{n}\frac{\theta^2}{1+2\theta}\frac{\lambda e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}$$

$$=\left(\frac{\lambda\theta^2}{1+2\theta}\right)^n\prod_{i=1}^{n}\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}$$

Then, the log likelihood function is given by

$$l=\log L=n\log\lambda+2n\log\theta-n(\log(1+2\theta))+\sum_{i=1}^{n}\log\left(\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}\right)$$

$$\sum_{i=1}^{n}\log\left(\frac{e^{-\lambda(\lambda x)^{\frac{1}{k}}}\left(3-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)}{\left(1-(1-\theta)e^{-\lambda(\lambda x)^{\frac{1}{k}}}\right)^3}\right)=\sum_{i=1}^{n}-\lambda(\lambda x_i)^{\frac{1}{k}}+\log\left(3-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}\right)-3\log\left(1-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}\right)$$

Thus

$$l=n\log\lambda+2n\log\theta-n(\log(1+2\theta))+\sum_{i=1}^{n}-\lambda(\lambda x_i)^{\frac{1}{k}}+\log\left(3-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}\right)-3\log\left(1-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}\right)$$

To maximize l, taken first order partial derivatives of l with respect to λ,θ,k and equating the results equations to zeros and solve them.

Taking derivatives to l with respect to λ,θ, and k, respectively, one get:

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}-\left((\lambda x_i)^{\frac{1}{k}}+\frac{\lambda x_i}{k}(\lambda x_i)^{\frac{1}{k}-1}\right)+\frac{(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}\left((\lambda x_i)^{\frac{1}{k}}+\frac{\lambda x_i}{k}(\lambda x_i)^{\frac{1}{k}-1}\right)}{3-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}$$
$$-\frac{3(1-\theta)x_ie^{-\lambda(\lambda x_i)^{\frac{1}{k}}}\left((\lambda x_i)^{\frac{1}{k}}+\frac{\lambda x_i}{k}(\lambda x_i)^{\frac{1}{k}-1}\right)}{1-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}$$

$$\frac{\partial l}{\partial\theta}=\frac{2n}{\theta}-\frac{2n}{1+2\theta}+\sum_{i=1}^{n}\frac{e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}{3-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}-\frac{e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}{1-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}$$

$$\frac{\partial l}{\partial k}=\sum_{i=1}^{n}\frac{\lambda}{k^2}(\lambda x_i)^{\frac{1}{k}}\log(\lambda x_i)-\frac{(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}(\frac{\lambda}{k^2}(\lambda x_i)^{\frac{1}{k}}\log(\lambda x_i))}{3-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}+\frac{3(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}(\frac{\lambda}{k^2}(\lambda x_i)^{\frac{1}{k}}\log(\lambda x_i))}{1-(1-\theta)e^{-\lambda(\lambda x_i)^{\frac{1}{k}}}}$$

By equating $\frac{\partial l}{\partial \lambda}|_{\lambda=\hat{\lambda}}$, $\frac{\partial l}{\partial \theta}|_{\theta=\hat{\theta}}$, and $\frac{\partial l}{\partial k}|_{k=\hat{k}}$ to zero and solving for $\hat{\lambda}$, $\hat{\theta}$, and $\hat{k}$, the MLE of theses parameters are numerically solutions of the following three equations:

$$\frac{n}{\hat{\lambda}}-\sum_{i=1}^{n}\left((\hat{\lambda}x_i)^{\frac{1}{k}}+\frac{\hat{\lambda}x_i}{k}(\hat{\lambda}x_i)^{\frac{1}{k}-1}\right)+\frac{\left((\hat{\lambda}x_i)^{\frac{1}{k}}+\frac{\hat{\lambda}x_i}{k}(\hat{\lambda}x_i)^{\frac{1}{k}-1}\right)(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}{3-(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}$$
$$-\frac{3\left((\hat{\lambda}x_i)^{\frac{1}{k}}+\frac{\hat{\lambda}x_i}{k}(\hat{\lambda}x_i)^{\frac{1}{k}-1}\right)(1-\hat{\theta})x_i e^{-\lambda(\hat{\lambda}x_i)^{\frac{1}{k}}}}{1-(1-\hat{\theta})e^{-\lambda(\hat{\lambda}x_i)^{\frac{1}{k}}}}=0$$

$$\frac{2n}{\hat{\theta}}-\frac{2n}{1+2\hat{\theta}}+\sum_{i=1}^{n}\frac{e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}{3-(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}-\frac{e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}{1-(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}=0$$

$$\sum_{i=1}^{n}\frac{\hat{\lambda}}{\hat{k}^2}(\hat{\lambda}x_i)^{\frac{1}{k}}\log(\hat{\lambda}x_i)-\frac{(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}(\frac{\hat{\lambda}}{\hat{k}^2}(\hat{\lambda}x_i)^{\frac{1}{k}}\log(\hat{\lambda}x_i))}{3-(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}+\frac{3(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}(\frac{\hat{\lambda}}{\hat{k}^2}(\hat{\lambda}x_i)^{\frac{1}{k}}\log(\hat{\lambda}x_i))}{1-(1-\hat{\theta})e^{-\hat{\lambda}(\hat{\lambda}x_i)^{\frac{1}{k}}}}=0$$

6.3. Maximum Likelihood Estimation for (CLTNDL) as Follows

The estimators of unknown parameters of the CLTNDL model are obtained based on maximum likelihood (ML) method. Let $x_1,\dots,x_n$ be a random sample of size n from CETNDL, with set a parameters (θ,λ,k), the likelihood function of the density is given by,

$$L(\theta,\lambda,k;x)=\prod_{i=1}^{n}f(x_i;\theta,\lambda,k)$$
$$=\prod_{i=1}^{n}\frac{\theta^2}{1+2\theta}\frac{\frac{\lambda k}{1+\lambda x^{k+1}}\left(3-(1-\theta)(-(1+\lambda x)^{-k})\right)}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3}$$
$$=\left(\frac{\lambda\theta^2}{1+2\theta}\right)^n\prod_{i=1}^{n}\frac{\frac{\lambda k}{1+\lambda x^{k+1}}\left(3-(1-\theta)(-(1+\lambda x)^{-k})\right)}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3}$$

Then, the log likelihood function is given by

$$l=\log L=n\log\lambda+2n\log\theta-n(\log(1+2\theta))+\sum_{i=1}^{n}\log\left(\frac{\frac{\lambda k}{1+\lambda x^{k+1}}\left(3-(1-\theta)(-(1+\lambda x)^{-k})\right)}{(1-(1-\theta)(-(1+\lambda x))^{-k})^3}\right)$$
$$\sum_{i=1}^{n}\log\left(\frac{\frac{\lambda k}{1+\lambda x^{k+1}}\left(3-(1-\theta)(-(1+\lambda x_i)^{-k})\right)}{(1-(1-\theta)(-(1+\lambda x_i))^{-k})^3}\right)$$
$$=n\log(\lambda k)+\sum_{i=1}^{n}-(k+1)\log(1+\lambda x_i)+\log\left(3-(1-\theta)(-(1+\lambda x_i)^{-k})\right)-3\log(1-(1-\theta)(-(1+\lambda x_i))^{-k})$$

Thus

$$l=2n\log\lambda+2n\log\theta-n(\log(1+2\theta))+\log(k)$$
$$+\sum_{i=1}^{n}-(k+1)\log(1+\lambda x_i)+\log\left(3-(1-\theta)(-(1+\lambda x_i)^{-k})\right)-3\log\left(1-(1-\theta)(-(1+\lambda x_i)^{-k})\right)$$

To maximize l, taken first order partial derivatives of l with respect to λ,θ, and k and equating the results equations to zeros and solve them.

Taking derivatives to l with respect to λ,θ, respectively, one get:

$$\frac{\partial l}{\partial \lambda}=\frac{2n}{\lambda}+\sum_{i=1}^{n}\frac{-(k+1)x_i}{1+\lambda x_i}+\frac{(kx_i)(1-\theta)(-(1+\lambda x_i))^{-k-1}}{3-(1-\theta)(-(1+\lambda x_i))^{-k}}-\frac{3(1-\theta)(kx_i)(-(1+\lambda x_i))^{-k-1}}{1-(1-\theta)(-(1+\lambda x_i))^{-k}}$$

$$\frac{\partial l}{\partial \theta}=\frac{2n}{\theta}-\frac{2n}{1+2\theta}+\sum_{i=1}^{n}\frac{(-(1+\lambda x_i))^{-k}}{3-(1-\theta)(-(1+\lambda x_i))^{-k}}-\frac{3(-(1+\lambda x_i))^{-k}}{1-(1-\theta)(-(1+\lambda x_i))^{-k}}$$

$$\frac{\partial l}{\partial k}=\frac{1}{k}+\sum_{i=1}^{n}-\log(1+\lambda x_i)+\frac{(1-\theta)(-(1+\lambda x_i)^{-k})\log(1+\lambda x_i)}{3-(1-\theta)(-(1+\lambda x_i))^{-k}}-\frac{3(1-\theta)(-(1+\lambda x_i)^{-k})\log(1+\lambda x_i)}{1-(1-\theta)(-(1+\lambda x_i))^{-k}}$$

By equating $\frac{\partial l}{\partial \lambda}|_{\lambda=\hat{\lambda}}$, $\frac{\partial l}{\partial \theta}|_{\theta=\hat{\theta}}$, and $\frac{\partial l}{\partial k}|_{k=\hat{k}}$ to zero and solving for $\hat{\lambda}$, $\hat{\theta}$, and $\hat{k}$, the MLE of theses parameters are numerically solutions of the following three equations:

$$\frac{2n}{\hat{\lambda}}+\sum_{i=1}^{n}\frac{-(\hat{k}+1)x_i}{1+\hat{\lambda}x_i}+\frac{(\hat{k}x_i)\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}-1}}{3-\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}}}-\frac{3\left(1-\hat{\theta}\right)\left(\hat{k}x_i\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}-1}}{1-\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}}}=0$$

$$\frac{2n}{\hat{\theta}}-\frac{2n}{1+2\hat{\theta}}+\sum_{i=1}^{n}\frac{(-(1+\hat{\lambda}x_i))^{-\hat{k}}}{3-\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}}}-\frac{3(-(1+\hat{\lambda}x_i))^{-\hat{k}}}{1-\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}}}=0$$

$$\frac{1}{\hat{k}}+\sum_{i=1}^{n}-\log\left(1+\hat{\lambda}x_i\right)+\frac{\left(1-\hat{\theta}\right)\left(-(1+\hat{\lambda}x_i)^{-\hat{k}}\right)\log(1+\hat{\lambda}x_i)}{3-\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}}}-\frac{3\left(1-\hat{\theta}\right)\left(-(1+\hat{\lambda}x_i)^{-k}\right)\log(1+\hat{\lambda}x_i)}{1-\left(1-\hat{\theta}\right)(-(1+\hat{\lambda}x_i))^{-\hat{k}}}=0$$

Table (1). Maximum likelihood, standard errors and associated asymptotic confidence interval estimates for CETNDL, CLTNDL and CWTNDL distributions based in series and systems given data set I

Distribution	System	Parameter	MLE	St. Error	95% Asymptotic CI		
					Lower	Upper	AIL
CETNDL	Series	θ	1.0443	0.2461	0.5620	1.5267	0.9647
		λ	0.1100	0.0202	0.0702	0.1498	0.0796
CLTNDL	Series	θ	9.9940	11.6713	0.0001	32.869	32.8689
		λ	0.3673	0.4235	0.0001	1.1975	1.1974
		k	2.3206	0.6236	1.0982	3.5430	2.4448
CWTNDL	Series	θ	0.1268	0.1126	0.0940	0.3476	0.2536
		λ	0.1109	0.0452	0.0223	0.1996	0.1773
		k	0.6697	0.0487	0.5742	0.7652	0.191

St.Error: Standard error, AIL: Average interval length

Table (2). Maximum likelihood, standard errors and associated asymptotic confidence interval estimates for CETNDL, CLTNDL and CWTNDL distributions based in series and systems given data set II

Distribution	System	Parameter	MLE	St. Error	95% Asymptotic CI		
					Lower	Upper	AIL
CETNDL	Series	θ	0.6517	0.1186	0.4192	0.8842	0.4650
		λ	0.0078	0.0012	0.0053	0.0104	0.0051
CLTNDL	Series	θ	2.0300	0.7898	0.4820	3.5781	3.0961
		λ	0.0122	0.0066	0.0009	0.0253	0.0244
		k	2.4026	0.4902	1.4416	3.3635	1.9219
CWTNDL	Series	θ	0.3743	0.1374	0.1050	0.6437	0.5387
		λ	0.0588	0.0137	0.0319	0.0858	0.0539
		k	0.8565	0.0659	0.7273	0.9857	0.2584

St. Error: Standard error, AIL: Average interval length

7. Application

In this section we present an application of the Complementary Exponential Truncated Natural Discrete Lindley Distribution, Complementary Weibull Truncated Natural Discrete Lindley Distribution and Complementary Lomax Truncated Natural Discrete Lindley Distribution to two real data sets and fit the Exponential Distribution, Weibull Distribution and Lomax Distribution.

The first data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang (2003). The data are given as **follows**: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 4.3 The MLEs of the parameters, the values of K-S statistic, BIC, AIC, HQIC, NLC and P-Value are summarized in Table (1, 2 and 3). From this tables, we note that the TNL-Exponential distribution is better than the Exponential distribution in terms of fitting to this data, that the TNL-Weibull distribution is better than the Weibull distribution in terms of fitting to this data and the TNL-Lomax distribution is better than the Lomax distribution in terms of fitting to this data.

The AIC (Akaike Information Criterion) is given by

$$AIC = -2\log(liklihood) + 2k$$

where k is the number of estimated parameters.
where is the number of the sample size in the model. The Bayesian information criterion (BIC) is given by

$$BIC = Kln(n) - 2\log(likelihood)$$

where n is the sample size of the training set. The lower BIC score signals a better mode.

The second data set is obtained from: Leukemia 33, the data are:

c(194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 102,15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 18, 209, 41,58, 60, 48, 56, 87, 11, 102, 12, 5, 100, 14, 29, 37, 186,29, 104, 7, 4, 72, 270, 283, 7, 57, 33, 100, 61, 502, 220,120, 141, 22, 603, 35, 98, 54, 181, 65, 49, 12, 239, 14, 18,39, 3, 12, 5, 32, 9, 14, 70, 47, 62, 142, 3, 104, 85, 67,169, 24, 21, 246, 47, 68, 15, 2, 91, 59, 447, 56, 29, 176,225, 77, 197, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188,230, 152, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27,21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 59, 153,104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 5, 61, 31, 118,326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 156, 11, 216,139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 26, 71,39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 62, 11,191, 14, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52,208, 95).

The MLEs of the parameters the values of K-S statistic, BIC, AIC, HQIC, NLC and P-Value are summarized in Table (1, 2 and 3). From this tables, we note that the TNL-Exponential distribution is better than the Exponential distribution in terms of fitting to this data, that the TNL-Weibull distribution is better than the Weibull distribution in terms of fitting to this data and the TNL-Lomax distribution is better than the Lomax distribution in terms of fitting to this data.

Table (1). Comparison between the Exponential Distribution and Complementary Exponential Truncated Natural Discrete Lindley Distribution

Distribution	Real Data	Parameter	MLE (St. Error)	Goodness of fit test					
				AIC	BIC	HQIC	NLC	KS	p-value
Exponential	I	λ	0.1068 (0.0094)	830.7	833.5	831.8	414.3	0.0846	0.3183
	II	λ	0.0107 (0.00073)	2360	2363	2361	1179	0.0726	0.2113
TNL-Exponential	I	θ	1.0444 (0.2461)	832.7	838.4	835	414.3	0.0809	0.3721
		λ	0.1101 (0.0203)						
	II	θ	0.6517 (0.1186)	2356	2362	2358	1176	0.0492	0.6807
		λ	0.0078 (0.00128)						

The MLEs of the parameters, the values of K-S statistic, BIC, AIC, HQIC, NLC and P-Value are summarized in Table (1). From this table, we note that the TNL-Exponential distribution is better than the Exponential distribution in terms of fitting to this data.

Table (2). Comparison between the Weibull Distribution and Complementary Weibull Truncated Natural Discrete Lindley Distribution

Distribution	Real Data	Parameter	MLE (St. Error)	Goodness of fit test					
				AIC	BIC	HQIC	NLC	KS	p-value
Weibull	I	λ	0.3150 (0.0211)	832.2	837.9	834.5	414.1	0.0700	0.5569
		k	0.9544 (0.0615)						
	II	λ	0.1154 (0.0091)	2359	2366	2362	1178	0.0519	0.6136
		k	1.0816 (0.0563)						
TNL-Weibull	I	θ	0.1268 (0.1127)	825.2	833.8	828.7	409.6	0.0318	0.9995
		λ	0.1110 (0.0452)						
		k	0.6697 (0.0487)						
	II	θ	0.3744 (0.1374)	2355	2365	2359	1174	0.0393	0.8972
		λ	0.0588 (0.0137)						
		k	0.8565 (0.0659)						

The MLEs of the parameters, the values of K-S statistic, BIC, AIC, HQIC, NLC and P-Value are summarized in Table (2). From this table, we note that the TNL-Weibull distribution is better than the Weibull distribution in terms of fitting to this data.

Table (3). Comparison between the Lomax Distribution and Complementary Lomax Truncated Natural Discrete Lindley Distribution

Distribution	Real Data	Parameter	MLE (St. Error)	Goodness of fit test					
				AIC	BIC	HQIC	NLC	KS	p-value
Lomax	I	λ	0.00825 (0.00387)	831.7	837.4	834	413.8	0.0969	0.1804
		k	13.9467 (6.2253)						
	II	λ	0.0021 (0.00012)	2357	2364	2360	1176	0.0407	0.8715
		k	6.0650 (0.5068)						
TNL-Lomax	I	θ	9.9941 (11.6713)	825.1	833.6	828.5	409.5	0.0301	0.9998
		λ	0.3674 (0.4236)						
		k	2.3206 (0.6237)						
	II	θ	2.0300 (0.7898)	2361	2371	2365	1178	0.0416	0.8956
		λ	0.0122 (0.0066)						
		k	2.4026 (0.4902)						

The MLEs of the parameters, the values of K-S statistic, BIC, AIC, HQIC, NLC and P-Value are summarized in Table (3). From this table, we note that the TNL-Lomax distribution is better than the Lomax distribution in terms of fitting to this data.

8. Conclusions

In this chapter a three new families of lifetime distributions called the complementary exponential truncated natural discrete Lindley distribution (CETNDL), the complementary Weibull truncated natural discrete Lindley distribution (CWTNDL) and the complementary Lomax truncated natural discrete Lindley distribution (CLTNDL) is introduced, this family is obtained by compounding the exponential distribution with truncated natural discrete Lindley distribution, compounding the Weibull distribution with truncated natural discrete Lindley distribution and compounding the Lomax distribution with truncated natural discrete Lindley distribution.

It is observed the (CETNDL), (CWTNDL) and (CLTNDL) can have increasing, decreasing, constant, bathtub and upside-down hazard rate functions which are eligible for data analysis purposes. Several properties of the (CETNDL), (CWTNDL) and (CLTNDL) distributions such as quantiles, median, mean deviations, moments, Lorenz and Bonferroni curves,. This family contains several new distributions such as complementary exponentiated truncated natural discrete Lindley distribution, complementary Weibull logarithmic truncated natural discrete Lindley distribution, complementary weibull geometric truncated natural discrete Lindley distribution and, complementary Weibull Poisson truncated natural discrete Lindley distribution, cdf, pdf, hazard function of these special sub-models are presented. The method of maximum likelihood is used for estimating the model parameters. Finally the complementary exponential truncated natural discrete Lindley (CETNDL), the complementary Weibull truncated natural discrete Lindley (CWTNDL) and the complementary Lomax truncated natural discrete Lindley (CLTNDL) series models are fitted to real data sets to show the flexibility and potentiality of the proposed of distributions.

ACKNOWLEDGMENTS

The authorswould like to thank the Editor and the anonymous referees for very careful reading and valuable comments which greatly improved the paper.

Disclosure Statement

No potential conflict of interest was reported by the authors.

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