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A New Two-Parameter Compound G Family: Copulas, Properties and Applications

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Abstract In the work, we propose and study a new two-parameter compound G family of continuous distributions. Relevant statistical properties are mathematically derived. Many new G families of bivariate distributions are presented using Renyi's copula, Clayton copula, Ali-Mikhail-Haq copula, Farlie-Gumbel-Morgenstern copula, and modified Farlie-Gumbel-Morgenstern copula. Based on a special case we presented a new Lomax extension and studied its relevant statistical properties. The maximum likelihood method is used and employed for estimating the model parameters. Two applications to real-life data sets are presented for illustrating the superiority of the new family.

Keywords Poisson Family, Order Statistics, Farlie-Gumbel-Morgenstern, Topp Leone Family, Maximum Likelihood Estimation, Clayton copula, Generating Function, Moments, Ali-Mikhail-Haq copula

1. Introduction

The statistical literature contains many new G families of continuous distributions which have been generated either by merging (compounding) common G families of continuous distributions or by adding one or more parameters to the G family. These novel G families have been employed for modeling real-life datasets in many applied studies such as insurance, engineering, econometrics, biology, medicine, statistical forecasting, and environmental sciences see Gupta et al. (1998) for the exponentiated-G family, Marshall and Olkin (1997) for the Marshall-Olkin-G family, Eugene et al. (2002) for beta generalized-G family, Yousof et al. (2015) for the transmuted exponentiated generalized-G family, Nofal et al. (2017) for the generalized transmuted-G family, Rezaei et al. (2017) for the Topp Leone generated family, Merovci et al. (2017) for the exponentiated transmuted-G family, Brito et al. (2017) for the Topp-Leone odd log-logistic-G family, Yousof et al. (2017a) for Burr type X G family, Aryal and Yousof (2017) for exponentiated generalized-G Poisson family, Hamedani et al. (2017) for type I general exponential class of distributions, Cordeiro et al. (2018) for Burr type XII G family, Korkmaz et al. (2018a) for the exponential Lindley odd log-logistic-G family, Korkmaz et al. (2018b) for the Marshall-Olkin generalized-G Poisson family, Yousof et al. (2018) for Burr-Hatke family of distributions, Hamedani et al. (2018) for the extended G family, Hamedani et al. (2019) for the type II general exponential G family, Nascimento et al. (2019) for the odd Nadarajah-Haghighi family of distributions, Yousof et al. (2020) for the Weibull G Poisson family, Karamikabir et al. (2020) for the Weibull Topp-Leone generated family, Merovci et al. (2020) for the Poisson Topp Leone G family, Korkmaz et al. (2020) for the Hjorth's IDB generator of distributions, Alizadeh et al. (2020a) for flexible Weibull generated family of distributions, Alizadeh et al. (2020b) for the transmuted odd log-logistic-G family, Altun et al. (2021) for the Gudermannian generated family of distributions and El-Morshedy et al. (2021) for the Poisson generated exponential G family among others.

Due to Rezaei et al. (2017), the cumulative distribution function (CDF) of the Topp Leone generated G (TLG-G) family of distributions can be expressed as

$$H_{\boldsymbol{\mathcal{P}}}(x) = G_{\underline{\mho}}(x)^{ab}\left[2 - G_{\underline{\mho}}(x)^{b}\right]^{a}, \tag{1}$$

The corresponding probability density function (PDF) to (1) is

$$h_{\boldsymbol{\mathcal{P}}}(x) = 2ab g_{\underline{\mho}}(x) G_{\underline{\mho}}(x)^{ab-1}\left[1 - G_{\underline{\mho}}(x)^{b}\right]\left[2 - G_{\underline{\mho}}(x)^{b}\right]^{a-1}, \tag{2}$$

where $\underline{\boldsymbol{\mathcal{P}}} = a, b, \underline{\mho}$ refers to the parameters vector of the new family and $\underline{\mho}$ refers to the parameters vector of any base-line model. For $b = 1$, the TLG-G family reduces to the one-parameter Topp Leone G (TL-G) family. Suppose that $Z_1, Z_2, \ldots, Z_N$ be an independent and identically random variables (iid RVs) with common CDF follows the TLG-G family and N be RV with probability mass function

$$P(N = \boldsymbol{n}) = \frac{1}{\boldsymbol{n}!\,(e^1 - 1)}|_{\boldsymbol{n} \in \mathbb{N}},$$

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defining

$M_N = \min\{Z_1, Z_2, \ldots, Z_N\}|_{\boldsymbol{n}\in\mathbb{N}}$,

then

$$F(x) = \sum_{\boldsymbol{n}=0}^{\infty} \Pr(M_N \leq x | N = \boldsymbol{n}) \times \Pr(N = \boldsymbol{n})|_{\boldsymbol{n}\in\mathbb{N}}. \quad (3)$$

Using equations (2) and the last equation, we can write

$$F_{\underline{\boldsymbol{\mathcal{P}}}}(x) = \frac{1}{1-exp\,(-1)}\left[1 - exp\left(-\left\{G_{\underline{\mho}}(x)^b\left[2 - G_{\underline{\mho}}(x)^b\right]\right\}^a\right)\right]|_{a>0,b>0}. \quad (4)$$

Equation (4) can be called as quasi Poisson Topp Leone generated-G (QPTLG-G) family of distributions. The corresponding PDF of the QPTLG-G family can be expressed as

$$f_{\underline{\boldsymbol{\mathcal{P}}}}(x) = 2ab\frac{g_{\underline{\mho}}(x)G_{\underline{\mho}}(x)^{ab-1}\left[1-G_{\underline{\mho}}(x)^b\right]\left[2-G_{\underline{\mho}}(x)^b\right]^{a-1}}{[1-exp\,(-1)]\,exp\left(1\left\{G_{\underline{\mho}}(x)^b\left[2-G_{\underline{\mho}}(x)^b\right]\right\}^a\right)}. \quad (5)$$

The QPTLG-G family may be useful in modeling:

I. The "monotonically increasing hazard rate" real-life datasets as illustrated in Section 6 (Figures 1 and 2 (top left plots)).

II. The real-life datasets which do not have extreme values as shown in Section 6 (Figures 1 and 2 (bottom right plots) and (bottom left plots)).

III. The real-life datasets which their nonparametric Kernel density estimations are left skewed bimodal and right skewed bimodal as given in Section 6 (Figures 1 and 2 (top right plots)).

The QPTLG-G family proved adequate superiority against many other well-known G families as illustrated below:

I. In modeling the times failure of the aircraft windshield items, the QPTLG-G family is better than the odd log-logistic-G family, the generalized mixture-G family, the transmuted Topp-Leone-G family, the Gamma-G family, the Burr-Hatke-G family, the McDonald-G family, the exponentiated-G family, the Kumaraswamy-G family, and the proportional reversed hazard rate-G family under the consistent- information criteria, Akaike information criteria, Hannan-Quinn information criteria and Bayesian information criteria.

II. In modeling the times of service of the aircraft windshield items, the QPTLG-G family is better than the odd log-logistic-G family, the generalized mixture-G family, the transmuted Topp-Leone-G family, the Gamma-G family, the Burr-Hatke-G family, the McDonald-G family, the exponentiated-G family, the Kumaraswamy-G family, and the proportional reversed hazard rate-G family under the consistent- information criteria, Akaike information criteria, Hannan-Quinn information criteria and Bayesian information criteria.

2. Copula

For modeling of the bivariate real data sets, we shall derive some new bivariate QPTLG-G (Bv-QPTLG-G) type distributions using "Farlie-Gumbel-Morgenstern copula" (FGMC for short) copula (see Morgenstern (1956), Farlie (1960). Gumbel (1960 and 1961), Johnson and Kotz (1975 and 1977)), modified FGMC (see Balakrishnan and Lai (2009)) which contains for internal types, " Clayton copula " (see Nelsen (2007)), "Renyi's entropy copula (REC) (Pougaza and Djafari (2010))" and "Ali-Mikhail-Haq copula (AMHC)" (see Ali et al. (1978)). The multivariate QPTLG-G (Mv-QPTLG-G) type can be easily derived based on the Clayton copula. However, future works may be allocated to study these new models.

2.1. BQPTLG-G type via Clayton Copula

Let $X_1 \sim \mathrm{QPTLE} - \mathrm{G}(\underline{\boldsymbol{\mathcal{P}}}_1)$ and $X_2 \sim \mathrm{QPTLE} - \mathrm{G}(\underline{\boldsymbol{\mathcal{P}}}_2)$. Then depending on the continuous marginals $\overline{u} = 1 - u$ and $\overline{d} = 1 - d$, the Clayton copula can be expressed as

$$\mathcal{W}_{\mathrm{V}}(\overline{u}, \overline{d}) = \left[\max\left(\overline{u}^{-\mathrm{V}} + \overline{d}^{-\mathrm{V}} - 1\right); 0\right]^{-\frac{1}{\mathrm{V}}},$$

where

$$\mathrm{V} \in [-1, \infty) - \{0\}, \overline{u} \in (0,1), \overline{d} \in (0,1).$$

Let $\overline{u} = 1 - F_{\underline{\boldsymbol{\mathcal{P}}}_1}(x_1)|_{\underline{\boldsymbol{\mathcal{P}}}_1}$, $\overline{d} = 1 - F_{\underline{\boldsymbol{\mathcal{P}}}_2}(x_2)|_{\underline{\boldsymbol{\mathcal{P}}}_2}$ and

$$F_{\underline{\boldsymbol{\mathcal{P}}}_i}(x_i)|_{i=1,2} = \frac{1}{1-exp\,(-1)}\left\{1 - exp\left(-\left\{G_{\underline{\mho}}(x)^{b_i}\left[2 - G_{\underline{\mho}}(x)^{b_i}\right]\right\}^{a_i}\right)\right\}.$$

Then, the BQPTLG-G type distribution can be obtained from $\mathcal{W}_{\mathrm{V}}(\overline{u}, \overline{d})$. A straightforward multivariate extension via Clayton copula can be derived.

2.2. BQPTLG-G Type via REC

The REC can be derived using the continuous marginal functions $u = 1 - \overline{u} = F_{\underline{\boldsymbol{\mathcal{P}}}_1}(x_1) \in (0,1)$ and $d = 1 - \overline{d} = F_{\underline{\boldsymbol{\mathcal{P}}}_1}(x_2) \in (0,1)$ as follows

$$F(x_1, x_2) = C\left(F_{\underline{\boldsymbol{\mathcal{P}}}_1}(x_1), F_{\underline{\boldsymbol{\mathcal{P}}}_1}(x_2)\right) = x_2 u + x_1 d - x_1 x_2.$$

2.3. BQPTLG-G Type via FGMC

Considering the FGMC, the joint CDF can be written as

$$\mathcal{W}_{\mathrm{V}}(u, d) = ud + ud\mathrm{V}\overline{u}\,\overline{d},$$

where the continuous marginal function $u \in (0,1)$, $d \in (0,1)$ and $\mathrm{V} \in [-1,1]$ where $\mathcal{W}_{\mathrm{V}}(u, 0) = \mathcal{W}_{\mathrm{V}}(0, d) = 0|_{(u,d\in(0,1))}$, which under the grounded minimum condition and $u = \mathcal{W}_{\mathrm{V}}(u, 1)$ and $d = \mathcal{W}_{\mathrm{V}}(1, d)$ under the grounded maximum condition. Clearly, the grounded minimum (maximum) conditions are valid for any copula. Setting $\overline{u} = \overline{u}_{\underline{\boldsymbol{\mathcal{P}}}_1}|_{\underline{\boldsymbol{\mathcal{P}}}_1>0}$ and $\overline{d} = \overline{d}_{\underline{\boldsymbol{\mathcal{P}}}_2}|_{\underline{\boldsymbol{\mathcal{P}}}_2>0}$. Then, we have

$$F(x_1, x_2) = ud\left(1 + \mathrm{V}\overline{u}\,\overline{d}\right).$$

Then, the joint PDF can be expressed as

$$C_{\mathrm{V}}(u, d) = 1 + \mathrm{V}u^* d^*,$$

where

$$u^* = 1 - 2u \text{ and } d^* = 1 - 2d,$$

or

$$f_{\mathrm{V}}(x_1, x_2) = f_{\underline{\mathcal{P}}_1}(x_1) f_{\underline{\mathcal{P}}_2}(x_2) \theta \left(F_{\underline{\mathcal{P}}_1}(x_1), F_{\underline{\mathcal{P}}_2}(x_2)\right),$$

where the two function $f_{\mathrm{V}}(x_1, x_2)$ and $\theta_{\mathrm{V}}(u, d)$ are PDFs corresponding to the joint CDFs $F_{\mathrm{V}}(x_1, x_2)$ and $\mathcal{W}_{\mathrm{V}}(u, d)$.

2.4. BQPTLG-G Type via Modified FGMC

The modified formula of the modified FGMC can expressed as

$$\mathcal{W}_{\mathrm{V}}(u, d) = \mathrm{V}\boldsymbol{\mathcal{O}}(u)^* \boldsymbol{\mathcal{K}}(d)^* + ud,$$

with $\boldsymbol{\mathcal{O}}(u)^* = u\boldsymbol{\mathcal{O}}(u)$ and $\boldsymbol{\mathcal{K}}(d)^* = d\boldsymbol{\mathcal{K}}(d)$ where $\boldsymbol{\mathcal{O}}(u) \in (0,1)$ and $\boldsymbol{\mathcal{K}}(d) \in (0,1)$ are two continuous functions where $\boldsymbol{\mathcal{O}}(u = 0) = \boldsymbol{\mathcal{O}}(u = 1) = \boldsymbol{\mathcal{K}}(d = 0) = \boldsymbol{\mathcal{K}}(d = 1) = 0$. Then, let

$$a(\boldsymbol{\mathcal{O}}(u)^*) = \inf\left\{\boldsymbol{\mathcal{O}}(u)^* \colon \partial\boldsymbol{\mathcal{O}}(u)^*/\partial u \mid_{l_1(u)}\right\} < 0,$$

$$w(\boldsymbol{\mathcal{O}}(u)^*) = \sup\left\{\boldsymbol{\mathcal{O}}(u)^* \colon \partial\boldsymbol{\mathcal{O}}(u)^*/\partial u \mid_{l_1(u)}\right\} < 0,$$

$$b(\boldsymbol{\mathcal{K}}(d)^*) = \inf\left\{\boldsymbol{\mathcal{K}}(d)^* \colon \partial\boldsymbol{\mathcal{K}}(d)^*/\partial d \mid_{l_2(d)}\right\} > 0,$$

and

$$\eta(\boldsymbol{\mathcal{K}}(d)^*) = \sup\left\{\boldsymbol{\mathcal{K}}(d)^* \colon \partial\boldsymbol{\mathcal{K}}(d)^*/\partial d \mid_{l_2(d)}\right\} > 0.$$

Then, for $1 \leq min\left(a(\boldsymbol{\mathcal{O}}(u)^*) w(\boldsymbol{\mathcal{O}}(u)^*), b(\boldsymbol{\mathcal{K}}(d)^*) \eta(\boldsymbol{\mathcal{K}}(d)^*)\right)$ we have

$$\frac{\partial\boldsymbol{\mathcal{O}}(u)^*}{\partial u} = -\frac{u}{\partial u}\partial\boldsymbol{\mathcal{O}}(u) + \boldsymbol{\mathcal{O}}(u),$$

where

$$l_1(u) = \left\{\frac{\partial}{\partial u}\boldsymbol{\mathcal{O}}(u)^* \text{ exists}\right\},$$

and

$$l_2(d) = \left\{\frac{\partial}{\partial d}\boldsymbol{\mathcal{K}}(d)^* \text{ exists}\right\}.$$

The following four types can be derived and considered:

Type I:

The new bivariate version via modified FGMC type I can written as

$$\mathcal{W}_{\mathrm{V}}(u, d) = \mathrm{V}\boldsymbol{\mathcal{O}}(u)^* \boldsymbol{\mathcal{K}}(d)^* + ud.$$

Type II:

Consider $\mathbf{A}(u; \mathrm{V}_1)$ and $\mathbf{B}(d; \mathrm{V}_2)$ which satisfy the above conditions where

$$\mathbf{A}(u; \mathrm{V}_1)|_{(\mathrm{V}_1 > 0)} = u^{\mathrm{V}_1}(1 - u)^{1 - \mathrm{V}_1}$$

and

$$\mathbf{B}(d; \mathrm{V}_2)|_{(\mathrm{V}_2 > 0)} = d^{\mathrm{V}_2}(1 - d)^{1 - \mathrm{V}_2}.$$

Then, the corresponding bivariate version (modified FGMC Type II) can be derived from

$$\mathcal{W}_{\mathrm{V}_0, \mathrm{V}_1, \mathrm{V}_2}(u, d) = ud + \mathrm{V}_0 ud\mathbf{A}(u; \mathrm{V}_1)\mathbf{B}(d; \mathrm{V}_2).$$

Type III:

Let $\widetilde{\mathbf{A}(u)} = u[log(1 + \overline{u})]|_{(\overline{u} = 1 - u)}$ and $\widetilde{\mathbf{B}(d)} = d\left[log\left(1 + \overline{d}\right)\right]|_{(\overline{d} = 1 - d)}$. Then, the associated CDF of the BQPTLG-G-FGM (modified FGMC type III) as

$$\mathcal{W}_{\mathrm{V}}(u, d) = ud + ud\mathrm{V}\widetilde{\mathbf{A}(u)}\widetilde{\mathbf{B}(d)}.$$

Type IV:

Using the quantile concept, the CDF of the BQPTLG-G-FGM (modified FGMC type IV) model can be obtained using

$$\mathcal{W}(u, d) = uF^{-1}(u) + dF^{-1}(d) - F^{-1}(u)F^{-1}(d)$$

where $F^{-1}(u) = Q(u)$ and $F^{-1}(d) = Q(d)$.

2.5. BQPTLG-G type via AMHC

Under the "stronger Lipschitz condition" and following Ali et al. (1978), the joint CDF of the Archimedean Ali-Mikhail-Haq copula can written as

$$\mathcal{W}_{\mathrm{V}}(v, d) = \frac{1}{1 - \mathrm{V}\overline{v}\overline{d}} vd|_{\mathrm{V} \in (-1,1)},$$

the corresponding joint PDF of the Archimedean Ali-Mikhail-Haq copula can be express as

$$\theta_{\mathrm{V}}(v, d) = \frac{1}{\left[1 - \mathrm{V}\overline{v}\overline{d}\right]^2}\left(1 - \mathrm{V} + 2\mathrm{V}\frac{vd}{1 - \mathrm{V}\overline{v}\overline{d}}\right)|_{\mathrm{V} \in (-1,1)},$$

then for any $\overline{v} = 1 - F_{\underline{\mathcal{P}}_1}(x_1) = |_{[\overline{v} = (1 - v) \in (0,1)]}$ and $\overline{d} = 1 - F_{\underline{\mathcal{P}}_2}(x_2)|_{[\overline{d} = (1 - d) \in (0,1)]}$ we have

$$\mathcal{W}_{\mathrm{V}}(x_1, x_2) = \frac{1}{1 - \mathrm{V}\left[1 - F_{\underline{\mathcal{P}}_1}(x_1)\right]\left[1 - F_{\underline{\mathcal{P}}_2}(x_2)\right]}\left[F_{\underline{\mathcal{P}}_1}(x_1) F_{\underline{\mathcal{P}}_2}(x_2)\right]|_{\mathrm{V} \in (-1,1)},$$

and

$$\theta_{\mathrm{V}}(x_1, x_2) = \frac{1}{\left\{1 - \mathrm{V}\left[1 - F_{\underline{\mathcal{P}}_1}(x_1)\right]\left[1 - F_{\underline{\mathcal{P}}_2}(x_2)\right]\right\}^2}$$
$$\times \left(1 - \mathrm{V} + 2\mathrm{V}\left\{\frac{F_{\underline{\mathcal{P}}_1}(x_1) F_{\underline{\mathcal{P}}_2}(x_2)}{1 - \mathrm{V}\left[1 - F_{\underline{\mathcal{P}}_1}(x_1)\right]\left[1 - F_{\underline{\mathcal{P}}_2}(x_2)\right]}\right\}\right)|_{\mathrm{V} \in (-1,1)}.$$

3. Mathematical Properties

3.1. Linear Representation

In this section, useful representation for the QPTLG-G PDF (5) is presented. First, expanding the quantity $A_{a,b,\underline{\mho}}(x)$ where

$$A_{\underline{\mathcal{P}}}(x) = exp\left(-\left\{G_{\underline{\mho}}(x)^b\left[2 - G_{\underline{\mho}}(x)^b\right]\right\}^a\right).$$

Then,

$$A_{\underline{\mathcal{P}}}(x) = \sum_{h=0}^{+\infty} \frac{(-1)^h}{h!}\left\{G_{\underline{\mho}}(x)^b\left[2 - G_{\underline{\mho}}(x)^b\right]\right\}^{ah}$$

Compiling the expansion of $A_{1,a,b,\underline{\mho}}(x)$ in to (5), we have

$$f_{\underline{\mathcal{P}}}(x) = \sum_{h=0}^{+\infty} \frac{a2^{a(h+1)}(-1)^h}{h!(1-e^{-1})} g_{\underline{\mho}}(x)G_{\underline{\mho}}(x)^{b(a+ah)-1}\left[1 - G_{\underline{\mho}}(x)^b\right]\left[1 - \frac{1}{2}G_{\underline{\mho}}(x)^b\right]^{a(h+1)-1}. \quad (6)$$

Consider the power series

$$\left(1 - \frac{\Delta_1}{\Delta_2}\right)^{\Delta_3} = \sum_{l=0}^{\infty}(-1)^l\binom{\boldsymbol{\Delta}_3}{l}\left(\frac{\Delta_1}{\Delta_2}\right)^l, \quad (7)$$

which holds for $\left|\frac{\Delta_1}{\Delta_2}\right| < 1$ and $\boldsymbol{\Delta}_3 > 0$ real non-integer. Using (7), the QPTLG-G class in (6) can be written as

$$f_{\underline{\mathcal{P}}}(x) = \sum_{h,l=0}^{+\infty} \frac{a2^{a(h+1)-l}(-1)^{h+l}}{h!\,(1-e^{-1})}\binom{a(h+1)-1}{l}\begin{bmatrix} g_{\underline{\mho}}(x)G_{\underline{\mho}}(x)^{b(a+ah+l)-1} \\ -g_{\underline{\mho}}(x)G_{\underline{\mho}}(x)^{b(a+ah+l+1)-1}\end{bmatrix}$$

Which can be summarized as

$$f_{\underline{\mathcal{P}}}(x) = \sum_{h,l=0}^{+\infty}\left\{\mathcal{W}_{h,l}\,\pi_{b(a+ah+l)}(x;\underline{\mho}) - \mathcal{W}^*_{h,l}\,\pi_{b(a+ah+l+1)}(x;\underline{\mho})\right\}, \quad (8)$$

where

$$\mathcal{W}_{h,l} = \frac{1}{b(a+ah+l)}C_{h,l},$$

$$\mathcal{W}^*_{h,l} = \frac{1}{b(a+ah+l+1)}C_{h,l},$$

and

$$C_{h,l} = \frac{a(-1)^{h+l}2^{a(h+1)-l}}{h!\,(1-e^{-1})}\binom{a(h+1)-1}{l}$$

and $\pi_{\boldsymbol{\xi}}(x) = \boldsymbol{\xi}g(x)G(x)^{\boldsymbol{\xi}-1}$. Equation (8) reveals that the density of the QPTLG-G family can then be expressed as a linear representation of exp-G PDFs. Also, the CDF of the QPTLG-G family can also be expressed as a mixture of exp-G CDFs. By integrating (8), we have

$$F_{\underline{\mathcal{P}}}(x) = \sum_{h,l=0}^{+\infty}\left\{\mathcal{W}_{h,l}\,\mathbf{H}_{b(a+ah+l)}(x;\underline{\mho}) - \mathcal{W}^*_{h,l}\,\mathbf{H}_{b(a+ah+l+1)}(x;\underline{\mho})\right\}, \quad (9)$$

where $\mathbf{H}_{\boldsymbol{\xi}}(x)$ is the CDF of the exp-G family with power parameter $\boldsymbol{\xi}$.

3.2. Ordinary Moments

The $\boldsymbol{r}^{th}$ ordinary moment of X where X follows QPTLG-G family with parameters $(a, b, \underline{\mho})$ is given by $\mu'_{\boldsymbol{r}} = \mathbb{E}(X^{\boldsymbol{r}}) = \int_{-\infty}^{\infty} x^{\boldsymbol{r}} f_{\underline{\mathcal{P}}}(x)dx.$ Then we obtain

$$\mu'_{\boldsymbol{r},X} = \sum_{h,l=0}^{+\infty}\left\{\mathcal{W}_{h,l}\,\mathbb{E}(Y^{\boldsymbol{r}}_{b(a+ah+l)}) - \mathcal{W}^*_{h,l}\,\mathbb{E}(Y^{\boldsymbol{r}}_{b(a+ah+l+1)})\right\}. \quad (10)$$

where $Y_{\boldsymbol{\xi}}$ denotes the density of the exp-G model with power parameter $\boldsymbol{\xi}$. The expected value $\mathbb{E}(X)$ can be derived from when $\boldsymbol{r} = 1$ in (10). The integrations in $\mathbb{E}(Y^{\boldsymbol{r}}_{b(a+ah+l)})$ and $(Y^{\boldsymbol{r}}_{b(a+ah+l+1)})$ can be computed numerically for most parent distributions. The $\boldsymbol{n}^{th}$ central moment of X, variance $(\mathrm{V}(X))$, skewness $(\mathrm{S}(X))$, kurtosis $(\mathrm{K}(X))$ and dispersion index $(\mathrm{DI}(X))$ measures can be derived using well-known relationships. The s^{th} incomplete moment, say $\mathbf{I}_{s,X}(t)$, of X can be expressed from (9) as $\mathbf{I}_{s,X}(t) = \int_{-\infty}^{t} x^s f_{\underline{\mathcal{P}}}(t)dx$. Then

$$\mathbf{I}_{s,X}(t) = \sum_{h,l=0}^{+\infty}\begin{bmatrix} \mathcal{W}_{h,l}\int_{-\infty}^{t} x^s\,\pi_{b(a+ah+l)}(x;\underline{\mho})dx \\ -\mathcal{W}^*_{h,l}\int_{-\infty}^{t} x^s\pi_{b(a+ah+l+1)}(x;\underline{\mho})dx\end{bmatrix}. \quad (11)$$

The mean deviation about the mean and mean deviation about the median of X are given by

$$A_{1,X} = E(|X - \mu_1'|) = 2\mu_1' F(\mu_1') - 2\mathbf{I}_1(\mu_{1,X}')$$

and

$$A_{2,X} = \mathbb{E}(|X - M|) = \mu_1' - 2\mathbf{I}_{1,X}(M),$$

respectively, where $\mu_{1,X}' = E(X)$, $M = Median(X) = Q\left(\frac{1}{2}\right)$ is the median, $F(\mu_{1,X}')$ is obtained from (4) and $\mathbf{I}_{1,X}(t)$ is the first incomplete moment given by (11) with $s = 1$ as

$$\mathbf{I}_{1,X}(t) = \sum_{h,l=0}^{+\infty} \left\{ \mathcal{W}_{h,l} \mathbf{J}_{b(a+ah+l)}(x; \underline{\mho}) dx - \mathcal{W}_{h,l}^{*} \mathbf{J}_{b(a+ah+l+1)}(x; \underline{\mho}) dx \right\},$$

where $\mathbf{J}_{\boldsymbol{\xi}}(x) = \int_{-\infty}^{t} x \pi_{\boldsymbol{\xi}}(x)\, dx$ is the first incomplete moment of the exp-G distribution. The moment generating function $M_X(t) = \mathbb{E}(e^{tX})$ of X can be derived as

$$M_X(t) = \sum_{h,l=0}^{+\infty} \left\{ \mathcal{W}_{h,l}\, M_{b(a+ah+l)}(t; \underline{\mho}) - \mathcal{W}_{h,l}^{*}\, M_{b(a+ah+l+1)}(t; \underline{\mho}) \right\},$$

where $M_{\boldsymbol{\xi}}(t)$ is the moment generating function of $Y_{\boldsymbol{\xi}}$.

3.3. Moment of the Residual Life

The $\boldsymbol{n}^{th}$ moment of the residual life, say

$$\boldsymbol{v}_{\boldsymbol{n},X}(t) = \mathbb{E}[(X - t)^{\boldsymbol{n}} \mid X > t, \boldsymbol{n} \in \mathbb{N}].$$

Then, the $\boldsymbol{n}^{th}$ moment of the residual life of X can be given as

$$\boldsymbol{v}_{\boldsymbol{n},X}(t) = \frac{1}{1 - F_{a,b,\underline{\mho}}(t)} \int_t^{\infty} (x - t)^{\boldsymbol{n}} f_{\underline{\boldsymbol{\mathcal{P}}}}(t) dx.$$

Therefore, using (8) we have

$$\boldsymbol{v}_{\boldsymbol{n},X}(t) = \frac{1}{1 - F_{\underline{\boldsymbol{\mathcal{P}}}}(t)} \sum_{\boldsymbol{r}=0}^{\boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{r}} (-t)^{\boldsymbol{n}-\boldsymbol{r}} \sum_{h,l=0}^{+\infty} \begin{bmatrix} \mathcal{W}_{h,l} \int_t^{\infty} x^{\boldsymbol{r}}\, \pi_{b(a+ah+l)}(x; \underline{\mho}) dx \\ -\mathcal{W}_{h,l}^{*} \int_t^{\infty} x^{\boldsymbol{r}}\, \pi_{b(a+ah+l+1)}(x; \underline{\mho}) dx \end{bmatrix}.$$

The life expectation can then be defined by

$$\boldsymbol{v}_{1,X}(t) = \mathbb{E}[(X - t) \mid X > t, \boldsymbol{n} = 1],$$

which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $\boldsymbol{n} = 1$ in the last equation.

3.4. Moment of the Reversed Residual Life

The $\boldsymbol{n}^{th}$ moment of the reversed residual life, say

$$\boldsymbol{V}_{\boldsymbol{n},X}(t) = \mathbb{E}[(t - X)^{\boldsymbol{n}} \mid X \le t, t > 0, \boldsymbol{n} \in \mathbb{N}].$$

The $\boldsymbol{n}^{th}$ moment of the reversed residual life of X can be given as

$$\boldsymbol{V}_{\boldsymbol{n},X}(t) = \frac{1}{F_{\underline{\boldsymbol{\mathcal{P}}}}(t)} \int_0^{t} (t - x)^{\boldsymbol{n}} f_{\underline{\boldsymbol{\mathcal{P}}}}(t) dx.$$

Then, the $\boldsymbol{n}^{th}$ moment of the reversed residual life of X becomes

$$\boldsymbol{V}_{\boldsymbol{n},X}(t) = \frac{1}{F_{\underline{\boldsymbol{\mathcal{P}}}}(t)} \sum_{\boldsymbol{r}=0}^{\boldsymbol{n}} (-1)^{\boldsymbol{r}} \binom{\boldsymbol{n}}{\boldsymbol{r}} t^{\boldsymbol{n}-\boldsymbol{r}} \sum_{h,l=0}^{+\infty} \begin{bmatrix} \mathcal{W}_{h,l} \int_0^{t} x^{\boldsymbol{r}}\, \pi_{b(a+ah+l)}(x; \underline{\mho}) dx \\ -\mathcal{W}_{h,l}^{*} \int_0^{t} x^{\boldsymbol{r}}\, \pi_{b(a+ah+l+1)}(x; \underline{\mho}) dx \end{bmatrix}.$$

The mean inactivity time (MIT) is given by

$$\boldsymbol{V}_{1,X}(t) = \mathbb{E}[(t - X) \mid X \le t, t > 0, \boldsymbol{n} = 1],$$

and it refers to the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

3.5. Probability Weighted Moments

The $(s,\boldsymbol{r})^{th}$ PWM of X following the QPTLG-G family, say $R_{s,\boldsymbol{r}}$, is formally defined by

$$R_{s,\boldsymbol{r}} = E\{X^s F(X)^{\boldsymbol{r}}\} = \int_{-\infty}^{\infty} x^s F_{\underline{\mathcal{P}}}(x)^{\boldsymbol{r}} f_{\underline{\mathcal{P}}}(x)dx.$$

Using equations (4) and (5), we can write

$$F_{\underline{\mathcal{P}}}(x)^{\boldsymbol{r}} f_{\underline{\mathcal{P}}}(x) = \sum_{h,l=0}^{+\infty} \begin{Bmatrix} \mathrm{V}_{h,l}(\boldsymbol{r})\, \pi_{b(a+ah+l)}(x;\underline{\mho}) \\ -\mathrm{V}^*_{h,l}(\boldsymbol{r})\, \pi_{b(a+ah+l+1)}(x;\underline{\mho}) \end{Bmatrix}$$

where

$$\mathrm{V}_{h,l}(\boldsymbol{r}) = \frac{1}{b(a+ah+l)}\,\omega_{h,l}^{(\boldsymbol{r})},$$

$$\mathrm{V}^*_{h,l}(\boldsymbol{r}) = \frac{1}{b(a+ah+l+1)}\,\omega_{h,l}^{(\boldsymbol{r})},$$

$$\omega_{h,l}^{(\boldsymbol{r})} = \sum_{p=0}^{+\infty} \frac{a(1+p)^h(-1)^{p+h+l}2^{a(h+1)-l}}{h!\,[1-exp(-1)]^{1+\boldsymbol{r}}} \binom{\boldsymbol{r}}{p} \binom{a(h+1)-1}{l}$$

and $\pi_{\xi}(x)$ is defined above. Then, the $(s,\boldsymbol{r})$ th PWM of X can be expressed as

$$R_{s,\boldsymbol{r}} = \sum_{h,l=0}^{+\infty} \left\{\mathrm{V}_{h,l}(\boldsymbol{r})\mathbb{E}\left(Y^s_{b(a+ah+l)}\right) - \mathrm{V}^*_{h,l}(\boldsymbol{r})\,\mathbb{E}\left(Y^s_{b(a+ah+l+1)}\right)\right\} dx.$$

3.6. Order Statistics

Let $X_1, X_2, \ldots, X_{\boldsymbol{n}}$ be a random sample from the QPTLG-G family of distributions and let $X_{1:\boldsymbol{n}}, X_{2:\boldsymbol{n}}, \ldots, X_{\boldsymbol{n}:\boldsymbol{n}}$ be the corresponding order statistics. The PDF of i^{th} order statistic, say $X_{i:\boldsymbol{n}}$, can be written as

$$f_{i:\boldsymbol{n}}(x) = \tfrac{1}{B(i,\boldsymbol{n}+1-i)} f_{\underline{\mathcal{P}}}(x) \textstyle\sum_{j=0}^{\boldsymbol{n}-i}(-1)^j \binom{\boldsymbol{n}-i}{j} F_{\underline{\mathcal{P}}}(x)^{i-1+j}, \tag{12}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (4) and (5) in equation (12) and using a power series expansion, we get

$$f(x)F(x)^{i-1+j} = \sum_{h,l=0}^{+\infty} \begin{Bmatrix} \mathrm{V}_{h,l}(i-1+j)\, \pi_{b(a+ah+l)}(x;\underline{\mho}) \\ -\mathrm{V}^*_{h,l}(i-1+j)\, \pi_{b(a+ah+l+1)}(x;\underline{\mho}) \end{Bmatrix},$$

where

$$\mathrm{V}_{h,l}(i-1+j) = \frac{1}{b(a+ah+l)}\,\omega_{h,l}^{(i-1+j)},$$

$$\mathrm{V}^*_{h,l}(i-1+j) = \frac{1}{b(a+ah+l+1)}\,\omega_{h,l}^{(i-1+j)},$$

and

$$\omega_{h,l}^{(i-1+j)} = \sum_{p=0}^{+\infty} \frac{a(1+p)^h(-1)^{p+h+l}2^{a(h+1)-l}}{h!\,[1-exp(-1)]^{j+i}} \binom{\boldsymbol{r}}{p} \binom{a(h+1)-1}{l}.$$

Then, the PDF of $X_{i:\boldsymbol{n}}$ can be written as

$$f_{i:\boldsymbol{n}}(x) = \sum_{j=0}^{\boldsymbol{n}-i} \frac{(-1)^j}{B(i,\boldsymbol{n}+1-i)} \binom{\boldsymbol{n}-i}{j} \sum_{h,l=0}^{+\infty} \begin{Bmatrix} \mathrm{V}_{h,l}(i-1+j)\, \pi_{b(a+ah+l)}(x;\underline{\mho}) \\ -\mathrm{V}^*_{h,l}(i-1+j)\, \pi_{b(a+ah+l+1)}(x;\underline{\mho}) \end{Bmatrix}.$$

Then, the density function of the QPTLG-G order statistics is a mixture of exp-G PDFs. Based on the last result, we note that the main properties of $X_{i:\boldsymbol{n}}$ follow from those properties of $Y_{b(a+ah+l)}$ and $Y_{b(a+ah+l+1)}$. For example, the s^{th} moments of $X_{i:\boldsymbol{n}}$ can be expressed as

$$\mathbb{E}(X^s_{i:\boldsymbol{n}}) = \textstyle\sum_{j=0}^{\boldsymbol{n}-i} \frac{(-1)^j}{B(i,\boldsymbol{n}+1-i)} \binom{\boldsymbol{n}-i}{j} \sum_{h,l=0}^{+\infty} \begin{Bmatrix} V_{h,l}(i-1+j)\, \mathbb{E}\left(Y^s_{b(a+ah+l)}\right) \\ -V^*_{h,l}(i-1+j)\, \mathbb{E}\left(Y^s_{b(a+ah+l+1)}\right) \end{Bmatrix}. \tag{13}$$

Analogously to the ordinary moments we can derive the L-moments. However, the L-moments can also be estimated via a linear combination of the order statistics. Whenever the mean of the distribution exists, the L-moments are also existing.

Based on the moments in equation (13), explicit expressions for the L-moments as infinite weighted linear combinations of the means of suitable QPTLG-G order statistics can be derived. The L-moments can be expressed as a linear function of the expected order statistics and can be defined by

$$\mathbf{L}_{\varsigma}(x;t,\varsigma)=\frac{1}{\varsigma}\sum_{t=0}^{\varsigma-1}\boldsymbol{\xi}(t,\varsigma)\,\mathbb{E}\left(X_{\varsigma-t:\varsigma}\right),\ \varsigma\geq 1,$$

where

$$\boldsymbol{\xi}(t,\varsigma)=(-1)^{t}\binom{\varsigma-1}{t}.$$

4. Studying a Special Model

In this section, we shall focus on a new special QPTLG-G model based on the Lomax distribution called the QPTLG-G Lomax (PTLGL) distribution. Below, we present two theorems related to the exp-L distribution. The two theorems are employed in deriving the mathematical properties in Table 1.

Table 1. Theoretical results of the PTLGL model

Property	Result	Support
$\mathbb{E}(X^{\boldsymbol{r}})$	$\sum_{h,l=0}^{+\infty}\sum_{d=0}^{\boldsymbol{r}}\theta^{\boldsymbol{r}}(-1)^{d}\binom{\boldsymbol{r}}{d}\left\{\begin{array}{c}\mathcal{W}_{h,l}[b(a+ah+l)]B\left([b(a+ah+l)],\frac{d-\boldsymbol{r}}{\beta}+1\right)\\-\mathcal{W}_{h,l}^{*}[b(a+ah+l+1)]B\left([b(a+ah+l+1)],\frac{d-\boldsymbol{r}}{\beta}+1\right)\end{array}\right\}.$	$\beta>\boldsymbol{r}$
$M_X(t)$	$\sum_{h,l,\boldsymbol{r}=0}^{+\infty}\sum_{d=0}^{\boldsymbol{r}}\frac{t^{\boldsymbol{r}}}{\boldsymbol{r}!}\theta^{\boldsymbol{r}}(-1)^{d}\binom{\boldsymbol{r}}{d}$ $\times\left\{\begin{array}{c}\mathcal{W}_{h,l}[b(a+ah+l)]B\left([b(a+ah+l)],\frac{d-\boldsymbol{r}}{\beta}+1\right)\\-\mathcal{W}_{h,l}^{*}[b(a+ah+l+1)]B\left([b(a+ah+l+1)],\frac{d-\boldsymbol{r}}{\beta}+1\right)\end{array}\right\}.$	$\beta>\boldsymbol{r}$
$\mathbf{I}_{s,X}(t)$	$\sum_{h,l=0}^{+\infty}\sum_{d=0}^{s}\theta^{s}(-1)^{d}\binom{s}{d}$ $\times\left\{\begin{array}{c}\mathcal{W}_{h,l}[b(a+ah+l)]B_t\left([b(a+ah+l)],\frac{d-s}{\beta}+1\right)\\-\mathcal{W}_{h,l}^{*}[b(a+ah+l+1)]B_t\left([b(a+ah+l+1)],\frac{d-s}{\beta}+1\right)\end{array}\right\}.$	$\beta>s$
$\mathbf{I}_{1,X}(t)$	$\sum_{h,l=0}^{+\infty}\sum_{d=0}^{1}\theta(-1)^{d}\binom{1}{d}\left\{\begin{array}{c}\mathcal{W}_{h,l}[b(a+ah+l)]B_t\left([b(a+ah+l)],\frac{d-1}{\beta}+1\right)\\-\mathcal{W}_{h,l}^{*}[b(a+ah+l+1)]B_t\left([b(a+ah+l+1)],\frac{d-1}{\beta}+1\right)\end{array}\right\}.$	$\beta>1$
$\boldsymbol{v}_{\boldsymbol{n},X}(t)$	$\frac{1}{1-F_{\underline{V}}(t)}\sum_{h,l=0}^{+\infty}\sum_{d=0}^{\boldsymbol{n}}\theta^{\boldsymbol{n}}(-1)^{d}\binom{\boldsymbol{n}}{d}$ $\times\left\{\begin{array}{c}\mathcal{W}_{h,l,u}(\boldsymbol{v},\boldsymbol{n})[b(a+ah+l)]B_t\left([b(a+ah+l)],\frac{d-\boldsymbol{n}}{\beta}+1\right)\\-\mathcal{W}_{h,l,u}^{*}(\boldsymbol{v},\boldsymbol{n})[b(a+ah+l+1)]B_t\left([b(a+ah+l+1)],\frac{d-\boldsymbol{n}}{\beta}+1\right)\end{array}\right\},$ where $\mathcal{W}_{h,l,u}(\boldsymbol{v},\boldsymbol{n})=\mathcal{W}_{h,l}\sum_{h=0}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{h}(-t)^{\boldsymbol{n}-h},$ and $\mathcal{W}_{h,l,u}^{*}(\boldsymbol{v},\boldsymbol{n})=\mathcal{W}_{h,l}^{*}\sum_{h=0}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{h}(-t)^{\boldsymbol{n}-h}$	$t>0$, $\boldsymbol{n}\in\mathbb{N}$, $\beta>\boldsymbol{n}$
$\boldsymbol{v}_{1,X}(t)$	$\frac{1}{1-F_{\underline{V}}(t)}\sum_{h,l=0}^{+\infty}\sum_{d=0}^{1}\theta(-1)^{d}\binom{1}{d}$ $\times\left\{\begin{array}{c}\mathcal{W}_{h,l,u}(\boldsymbol{v},1)[b(a+ah+l)]B_t\left([b(a+ah+l)],\frac{d-1}{\beta}+1\right)\\-\mathcal{W}_{h,l,u}^{*}(\boldsymbol{v},1)[b(a+ah+l+1)]B_t\left([b(a+ah+l+1)],\frac{d-1}{\beta}+1\right)\end{array}\right\},$	$t>0$, $\boldsymbol{n}=1$, $\beta>1$

Property	Result	Support
	where $$\mathcal{W}_{h,l,u}(\boldsymbol{v},1)=\mathcal{W}_{h,l}\sum\nolimits_{h=0}^{1}\binom{1}{h}(-t)^{1-h},$$ and $$\mathcal{W}^*_{h,l,u}(\boldsymbol{v},1)=\mathcal{W}^*_{h,l}\sum\nolimits_{h=0}^{1}\binom{1}{h}(-t)^{1-h}.$$	
$\boldsymbol{V}_{\boldsymbol{n},X}(t)$	$$\frac{1}{F_{\underline{V}}(t)}\sum_{h,l=0}^{+\infty}\sum_{d=0}^{\boldsymbol{n}}\theta^{\boldsymbol{n}}(-1)^d\binom{\boldsymbol{n}}{d} \times\left\{\begin{matrix}\mathcal{W}_{h,l,u}(\boldsymbol{V},\boldsymbol{n})[b(a+ah+l)]B_t\left([b(a+ah+l)],\frac{d-\boldsymbol{n}}{\beta}+1\right)\\ \mathcal{W}^*_{h,l,u}(\boldsymbol{V},\boldsymbol{n})[b(a+ah+l+1)]B_t\left([b(a+ah+l+1)],\frac{d-\boldsymbol{n}}{\beta}+1\right)\end{matrix}\right\},$$ where $$\mathcal{W}_{h,l,u}(\boldsymbol{V},\boldsymbol{n})=\mathcal{W}_{h,l}\sum\nolimits_{h=0}^{\boldsymbol{n}}(-1)^h\binom{\boldsymbol{n}}{\boldsymbol{r}}t^{\boldsymbol{n}-h},$$ and $$\mathcal{W}^*_{h,l,u}(\boldsymbol{V},\boldsymbol{n})=\mathcal{W}^*_{h,l}\sum\nolimits_{h=0}^{\boldsymbol{n}}(-1)^h\binom{\boldsymbol{n}}{\boldsymbol{r}}t^{\boldsymbol{n}-h}.$$	$t>0$, $\boldsymbol{n}\in\mathbb{N}$, $\beta>\boldsymbol{n}$
$\boldsymbol{V}_{1,X}(t)$	$$\frac{1}{F_{\underline{V}}(t)}\sum_{h,l=0}^{+\infty}\sum_{d=0}^{1}\theta(-1)^d\binom{1}{d} \times\left\{\begin{matrix}\mathcal{W}_{h,l,u}(\boldsymbol{V},1)[b(a+ah+l)]B_t\left([b(a+ah+l)],\frac{d-1}{\beta}+1\right)\\ \mathcal{W}^*_{h,l,u}(\boldsymbol{V},1)[b(a+ah+l+1)]B_t\left([b(a+ah+l+1)],\frac{d-1}{\beta}+1\right)\end{matrix}\right\},$$ Where $$\mathcal{W}_{h,l,u}(\boldsymbol{V},1)=\mathcal{W}_{h,l}\sum\nolimits_{h=0}^{1}(-1)^h\binom{1}{\boldsymbol{r}}t^{1-h},$$ and $$\mathcal{W}^*_{h,l,u}(\boldsymbol{V},1)=\mathcal{W}^*_{h,l}\sum\nolimits_{h=0}^{1}(-1)^h\binom{1}{\boldsymbol{r}}t^{1-h}.$$	$t>0$, $\boldsymbol{n}=1$, $\beta>\boldsymbol{n}$
$R_{s,\boldsymbol{r}}$	$$\sum_{h,l=0}^{+\infty}\sum_{d=0}^{S}\theta^s(-1)^d\binom{s}{d} \times\left\{\begin{matrix}\mathrm{V}_{h,l}(\boldsymbol{r})\,[b(a+ah+l)]B\left([b(a+ah+l)],\frac{d-s}{\beta}+1\right)\\ -\mathrm{V}^*_{h,l}(\boldsymbol{r})\,[b(a+ah+l+1)]B\left([b(a+ah+l+1)],\frac{d-S}{\beta}+1\right)\end{matrix}\right\}.$$	$\beta>s$
$\mathbb{E}(X^s_{i:\boldsymbol{n}})$	$$\sum_{j=0}^{\boldsymbol{n}-i}\frac{(-1)^j}{B(i,\boldsymbol{n}-i+1)}\binom{\boldsymbol{n}-i}{j}\sum_{h,l=0}^{+\infty}\sum_{d=0}^{s}\theta^s(-1)^d\binom{s}{d} \times\left\{\begin{matrix}\mathrm{V}_{h,l}(i-1+j)\,[b(a+ah+l)]B\left([b(a+ah+l)],\frac{d-s}{\beta}+1\right)\\ -\mathrm{V}^*_{h,l}(i-1+j)\,[b(a+ah+l+1)]B\left([b(a+ah+l+1)],\frac{d-s}{\beta}+1\right)\end{matrix}\right\}.$$	$\beta>s$

Theorem I:

Let $Y_{\boldsymbol{\xi}+1}$ *be a random variable having the exp-L distribution with power parameter* $\boldsymbol{\xi}+1$*. Then the CDF of the exp-L model can be expressed as*

$$G_{\boldsymbol{\xi}+1,\beta,\theta}(x)=\left[1-\left(\frac{1}{\theta}x+1\right)^{-\beta}\right]^{\boldsymbol{\xi}+1}$$

Then, the $\boldsymbol{r}^{th}$ *ordinary moment of* $Y_{\boldsymbol{\Delta}+1}$ *is given by*

$$\mathbb{E}(X^{\boldsymbol{r}})=\sum_{d=0}^{\boldsymbol{r}}(\boldsymbol{\xi}+1)\,\theta^{\boldsymbol{r}}(-1)^d\binom{\boldsymbol{r}}{d}B\left(\boldsymbol{\xi}+1,\frac{d-\boldsymbol{r}}{\beta}+1\right)|_{\beta>\boldsymbol{r}},$$

where

$$B(\varsigma_1,\varsigma_2)=\int_0^1 u^{\varsigma_1-1}(1-u)^{\varsigma_2-1}\,du$$

is the complete beta function.

Theorem 2:

Let $Y_{\boldsymbol{\xi}+1}$ *be a random variable having the exp-L distribution with power parameter* $\boldsymbol{\xi}+1$. *Then, the* $\boldsymbol{r}^{th}$ *incomplete moment of* $Y_{\boldsymbol{\xi}+1}$ *is given by*

$$\mathbf{I}_{\boldsymbol{r},Y}(t)=\sum_{d=0}^{\boldsymbol{r}}(\boldsymbol{\xi}+1)\theta^{\boldsymbol{r}}(-1)^{d}\binom{\boldsymbol{r}}{d}B_t\left(\boldsymbol{\xi}+1,\frac{d-\boldsymbol{r}}{\beta}+1\right)|_{\beta>\boldsymbol{r}},$$

where

$$B_t(\varsigma_1,\varsigma_2)=\int_0^t u^{\varsigma_1-1}(1-u)^{\varsigma_2-1}\,du$$

is the incomplete beta function.

5. Estimation

Let $x_1,\ldots,x_{\boldsymbol{n}}$ be a random sample from the QPTLG-G distribution with parameters $a,b,\underline{\mho}$. For determining the maximum likelihood estimators (MLEs) of $a,b,\underline{\mho}$, we have the log-likelihood function

$$\ell(a,b,\underline{\mho})=\boldsymbol{n}\log 2-\boldsymbol{n}\log(1-e^{-1})+\boldsymbol{n}\log a+\boldsymbol{n}\log b$$
$$+\sum_{i=1}^{\boldsymbol{n}}\log g_{\underline{\mho}}(x_i)+(ab-1)\sum_{i=1}^{\boldsymbol{n}}\log G_{\underline{\mho}}(x)+\sum_{i=1}^{\boldsymbol{n}}\log\left[1-G_{\underline{\mho}}(x)^b\right]$$
$$+(a-1)\sum_{i=1}^{\boldsymbol{n}}\log\left[2-G_{\underline{\mho}}(x)^b\right]-\sum_{i=1}^{\boldsymbol{n}}\left\{G_{\underline{\mho}}(x)^b\left[2-G_{\underline{\mho}}(x)^b\right]\right\}^a.$$

The components of the score vector are

$$U_a=\frac{\partial}{\partial a}\ell(a,b,\underline{\mho})=\frac{n}{a}+b\sum_{i=1}^{\boldsymbol{n}}\log G_{\underline{\mho}}(x)+\sum_{i=1}^{\boldsymbol{n}}\log\left[2-G_{\underline{\mho}}(x)^b\right]-\sum_{i=1}^{\boldsymbol{n}}\frac{log\left\{G_{\underline{\mho}}(x)^b\left[2-G_{\underline{\mho}}(x)^b\right]\right\}}{\left\{G_{\underline{\mho}}(x)^b\left[2-G_{\underline{\mho}}(x)^b\right]\right\}^{-a}},$$

$$U_b=\frac{\partial}{\partial b}\ell(a,b,\underline{\mho})=\frac{n}{b}+a\sum_{i=1}^{\boldsymbol{n}}\log G_{\underline{\mho}}(x)-\sum_{i=1}^{\boldsymbol{n}}\frac{G_{\underline{\mho}}(x)^b\log\left[G_{\underline{\mho}}(x)\right]}{\left[1-G_{\underline{\mho}}(x)^b\right]}-(a-1)\sum_{i=1}^{\boldsymbol{n}}\frac{G_{\underline{\mho}}(x)^b\log\left[G_{\underline{\mho}}(x)\right]}{\left[2-G_{\underline{\mho}}(x)^b\right]}$$
$$-a\sum_{i=1}^{\boldsymbol{n}}\frac{\left\{2G_{\underline{\mho}}(x)^b\,logG_{\underline{\mho}}(x)\left[1-G_{\underline{\mho}}(x)^b\right]\right\}}{\left\{G_{\underline{\mho}}(x)^b\left[2-G_{\underline{\mho}}(x)^b\right]\right\}^{-(a-1)}},$$

and

$$U_{\underline{\mho}}=\frac{\partial}{\partial\underline{\mho}}\ell(a,b,\underline{\mho})=\sum_{i=1}^{\boldsymbol{n}}\frac{g'_{\underline{\mho}}(x_i)}{g_{\underline{\mho}}(x_i)}+(ab-1)\sum_{i=1}^{\boldsymbol{n}}\frac{g_{\underline{\mho}}(x_i)}{G_{\underline{\mho}}(x)}-b\sum_{i=1}^{\boldsymbol{n}}\frac{g_{\underline{\mho}}(x_i)G_{\underline{\mho}}(x)^{b-1}}{\left[1-G_{\underline{\mho}}(x)^b\right]}$$
$$-(a-1)b\sum_{i=1}^{\boldsymbol{n}}\frac{g_{\underline{\mho}}(x_i)G_{\underline{\mho}}(x)^{b-1}}{\left[2-G_{\underline{\mho}}(x)^b\right]}-2ab\sum_{i=1}^{\boldsymbol{n}}g_{\underline{\mho}}(x)G_{\underline{\mho}}(x)^{ab-1}\left[1-G_{\underline{\mho}}(x)^b\right]\left[2-G_{\underline{\mho}}(x)^b\right]^{a-1}.$$

Setting the nonlinear system of equations $U_a=U_b=0$ and $U_{\underline{\mho}}=0$ and solving them simultaneously yields the MLE. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ.

6. Modeling Real-Life Data of Aircraft Windshields

Two real-life data applications to illustrate the importance and flexibility of the family is presented under the L case. The fits of the PTLGL are compared with other Lomax extensions shown in Table 2. However, many other Lomax extensions can be used in comparison such as the five-parameter Lomax distribution (Mead (2016)), new generalized of the Lomax distribution (Oguntunde et al. (2017)), the generalized odd Lomax Generated Family (Marzouk et al. (2019)) and the Zubair-inverse lomax distribution (Falgore (2020)). Table 3 below gives the goodness-of-fit (G-O-F) statistic tests which are used for comparing competitive models.

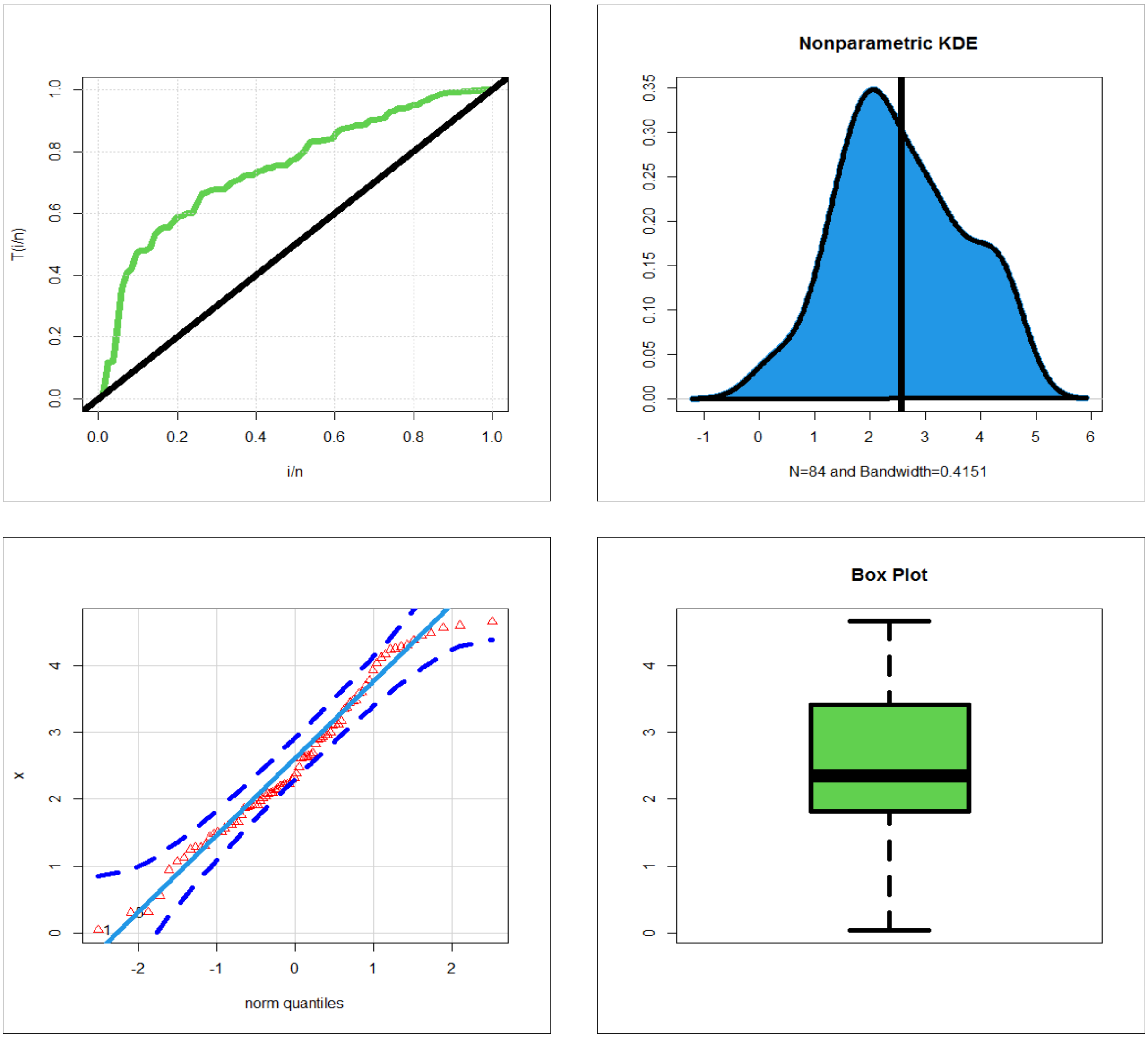

Figure 1. TTT, NKDE, Q-Q and box plot for the 1st data

Table 2. The competitive models

N.	Model	Abbreviation	Author
1	Special generalized mixture-L	SGML	Chesneau and Yousof (2021)
2	Odd log-logistic-L	OLLL	Altun et al. (2018)
3	Reduced OLL-L	ROLLL	Altun et al. (2018)
4	Reduced Burr-Hatke-L	RBHL	Yousof et al. (2018)
5	Transmuted Topp-Leone-L	TTLL	Yousof et al. (2017b)
6	Reduced TTL-L	RTTLL	Yousof et al. (2017b)
7	Gamma-L	GamL	Cordeiro et al. (2015)
8	Kumaraswamy-L	KL	Lemonte and Cordeiro (2013)
9	Beta-L	BL	Lemonte and Cordeiro (2013)
10	Exponentiated-L	exp-L	Gupta et al. (1998)
11	L	L	Lomax (1954)
12	Proportional reversed hazard rate-L	PRHRL	-

Table 3. The goodness-of-fit (G-O-F) statistic tests

N	G-O-F	Abbreviation
I	Akaike Information Criteria	AICr
II	Consistent- Akaike Information Criteria	CAICr
II	Bayesian- Information Criteria	BICr
IV	Hannan-Quinn- Information Criteria	HQICr

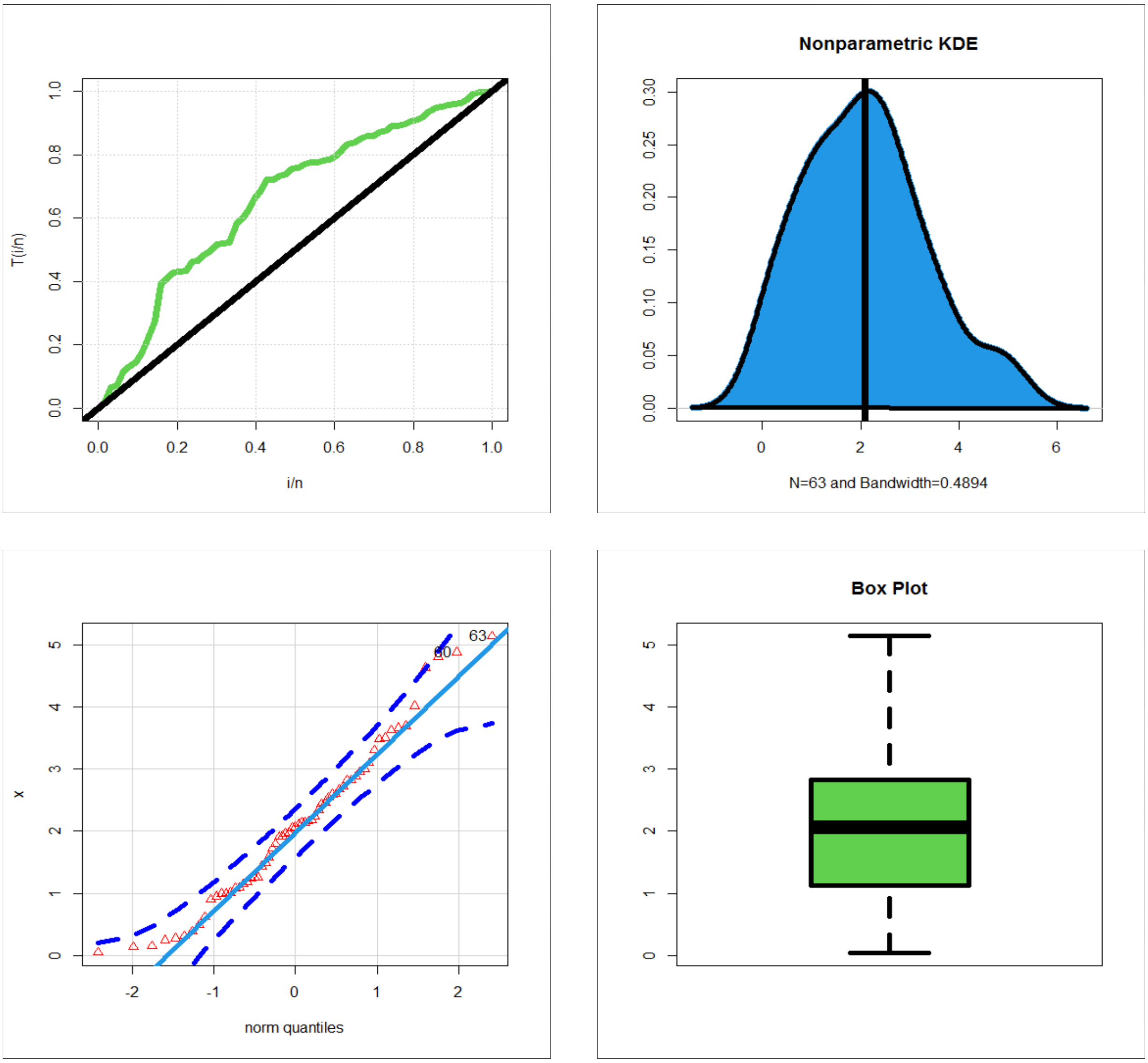

Figure 2. TTT, NKDE, Q-Q and box plot for the 2nd data

The 1st real-life data set (aircraft windshield consists of 84 aircraft windshield item) represents the data of failure times of 84 aircraft windshield. The 2nd real-life Data set (aircraft windshield consists of 63 aircraft windshield item) represents the data of service times of 63 aircraft windshield. The two real data were reported by Murthy et al. (2004). The “nonparametric Kernel density estimation (KDE)” tool is employed for exploring the initial PDF shape. The “normality” is also checked by the plot of the “Quantile-Quantile” (Q-Q). The initial HRF shapes explored via the “total time in test (TTT)” plot. The extremes are explored by the “box plot”. Based on Figures 1 and 2 (top left plots), it is shown that the HRFs are "monotonically increasing HRF" for the two data sets. Based on Figures 1 and 2 (top right plots), it is noted that the PDFs are asymmetric functions for the two data sets. Based on Figures 1 and 2 (bottom left plots), it is noted that the “normality” is exists. Based on Figures 1 and 2 (bottom right plots), we proved that no extremes are spotted.

Tables 4 and 6 gives the MLEs and the corresponding standard errors (SEs) for the two real-life datasets. Tables 5 and 7 list the four G-O-F statistic tests for the two real-life datasets. Figures 3 and 4 give the Probability- Probability (P-P) plots, Kaplan-Meier Survival (KMS) plot, estimated PDF (E-PDF), estimated CDF (E-CDF) and estimated HRF (E-HRF) plot for the two data sets respectively. Based on Tables 5 and 7, it is noted that the PTLGL model gives the lowest values for all G-O-F statistics with AICr = 269.8712, CAICr = 270.6404, BICr = 282.0253 and HQICr = 274.7570 for the 1st data, and AICr = 208.582, CAICr = 209.6347, BICr = 218.2977 and HQICr = 212.7966 for the 2nd data among all fitted competitive models. So, it could be selected as the best model under these G-O-F criteria.

Table 4. MLEs and SEs for 1st data

Model	Estimates			
PTLGL(a, b, β, θ)	**0.48687**	**6.98711**	**196.35669**	**391.79935**
	(0.43517)	**(6.4228)**	**(475.067)**	**(939.6757)**
TTLL(a, b, β, θ)	-0.8075	2.47663	15608	38628
	(0.1396)	(0.5418)	(1602.4)	(123.94)
BL(a, b, β, θ)	3.60360	33.6387	4.83070	118.837
	(0.6187)	(63.715)	(9.2382)	(428.93)
PRHRL(b, β, θ)	3.73×10^{6}	4.71×10^{-1}	4.5×10^{6}	
	(1.01×10^{6})	(0.00001)	(37.1468)	
SGML(b, β, θ)	-1.04×10^{-1}	9.83×10^{6}	1.18×10^{7}	
	(0.1223)	(4843.3)	(501.04)	
RTTLL(a, b, β)	-0.84732	5.52057	1.15678	
	(0.1001)	(1.1848)	(0.0959)	
OLLL(b, β, θ)	2.32636	$7.17\times e^{5}$	2.3×10^{6}	
	(2.14×10^{-1})	($1.19\times e^{4}$)	(2.6×10^{1})	
exp-L(b, β, θ)	3.62610	20074.5	26257.7	
	(0.6236)	(2041.8)	(99.743)	
GamL(b, β, θ)	3.58760	52001.4	37029.7	
	(0.5133)	(7955.0)	(81.16)	
ROLLL(b,β)	3.89056	0.57316		
	(0.3652)	(0.0195)		
RBHL(β, θ)	1080175	5136722		
	(983309)	(232313)		
L(β, θ)	51425.44	1317902		
	(5933.52)	(296.120)		

Table 5. G-O-F statistics for 1st data

Model	AICr	BICr	CAICr	HQICr
PTLGL	**269.8712**	**282.0253**	**270.6404**	**274.757**
OLLL	274.847	282.139	275.147	277.779
TTLL	279.140	288.863	279.646	283.049
GamL	282.808	290.136	283.105	285.756
BL	285.435	295.206	285.935	289.365
exp-L	288.799	296.127	289.096	291.747
ROLLL	289.690	294.552	289.839	291.645
SGML	292.175	299.467	292.475	295.106
RTTLL	313.962	321.254	314.262	316.893
PRHRL	331.754	339.046	332.054	334.686
L	333.977	338.862	334.123	335.942
RBHL	341.208	346.070	341.356	343.162

Table 6. MLEs and SEs for 2nd data

Model	Estimates			
PTLGL(a,b,β,θ)	**0.10082**	**21.76866**	**24.38403**	**40.54141**
	(<0.0001)	**(<0.0001)**	**(24.88439)**	**(41.9256)**
BL(a,b,β,θ)	1.9218	31.2594	4.9684	169.572
	(0.318)	(316.84)	(50.528)	(339.21)
KL$(\mathrm{a},b,\beta,\theta)$	1.6691	60.5673	2.56490	65.0640
	(0.257)	(86.013)	(4.7589)	(177.59)
TTLL(a,b,β,θ)	-0.607	1.78578	2123.39	4822.79
	(0.2137)	(0.4152)	(163.92)	(200.01)
RTTLL(a,b,β)	-0.6715	2.74496	1.01238	
	(0.18746)	(0.6696)	(0.1141)	
PRHRL(b,β,θ)	1.59×10^{6}	3.93×10^{-1}	1.30×10^{6}	
	(2.01×10^{3})	(0.001×10^{-1})	(0.95×10^{6})	
SGML(b,β,θ)	-1.04×10^{-1}	6.45×10^{6}	6.33×10^{6}	
	(4.1×10^{-10})	(3.21×10^{6})	(3.8573)	
GamL(b,β,θ)	1.9073	35842.433	39197.57	
	(0.3213)	(6945.074)	(151.653)	
OLLL(b,β,θ)	1.66419	6.340×10^{5}	2.01×10^{6}	
	(1.8×10^{-1})	(1.68×10^{4})	(7.22×10^{6})	
exp-L(b,β,θ)	1.9145	22971.15	32882.0	
	(0.348)	(3209.53)	(162.22)	
RBHL(β,θ)	14055522	53203423		
	(422.01)	(28.5232)		
ROLLL(b,β)	2.37233	0.69109		
	(0.2683)	(0.0449)		
L(β,θ)	99269.8	207019.4		
	(11864)	(301.237)		

Table 7. G-O-F statistics for 2nd data

Model	AICr	BICr	CAICr	HQICr
PTLGL	**208.582**	**218.2977**	**209.6347**	**212.7966**
KL	209.735	218.308	210.425	213.107
TTLL	212.900	221.472	213.589	216.271
GamL	211.666	218.096	212.073	214.195
SGML	211.788	218.218	212.195	214.317
BL	213.922	222.495	214.612	217.294
exp-L	213.099	219.529	213.506	215.628
OLLL	215.808	222.238	216.215	218.337
PRHRL	224.597	231.027	225.004	227.126
L	222.598	226.884	222.798	224.283
ROLLL	225.457	229.744	225.657	227.143
RTTLL	230.371	236.800	230.778	232.900
RBHL	229.201	233.487	229.401	230.887

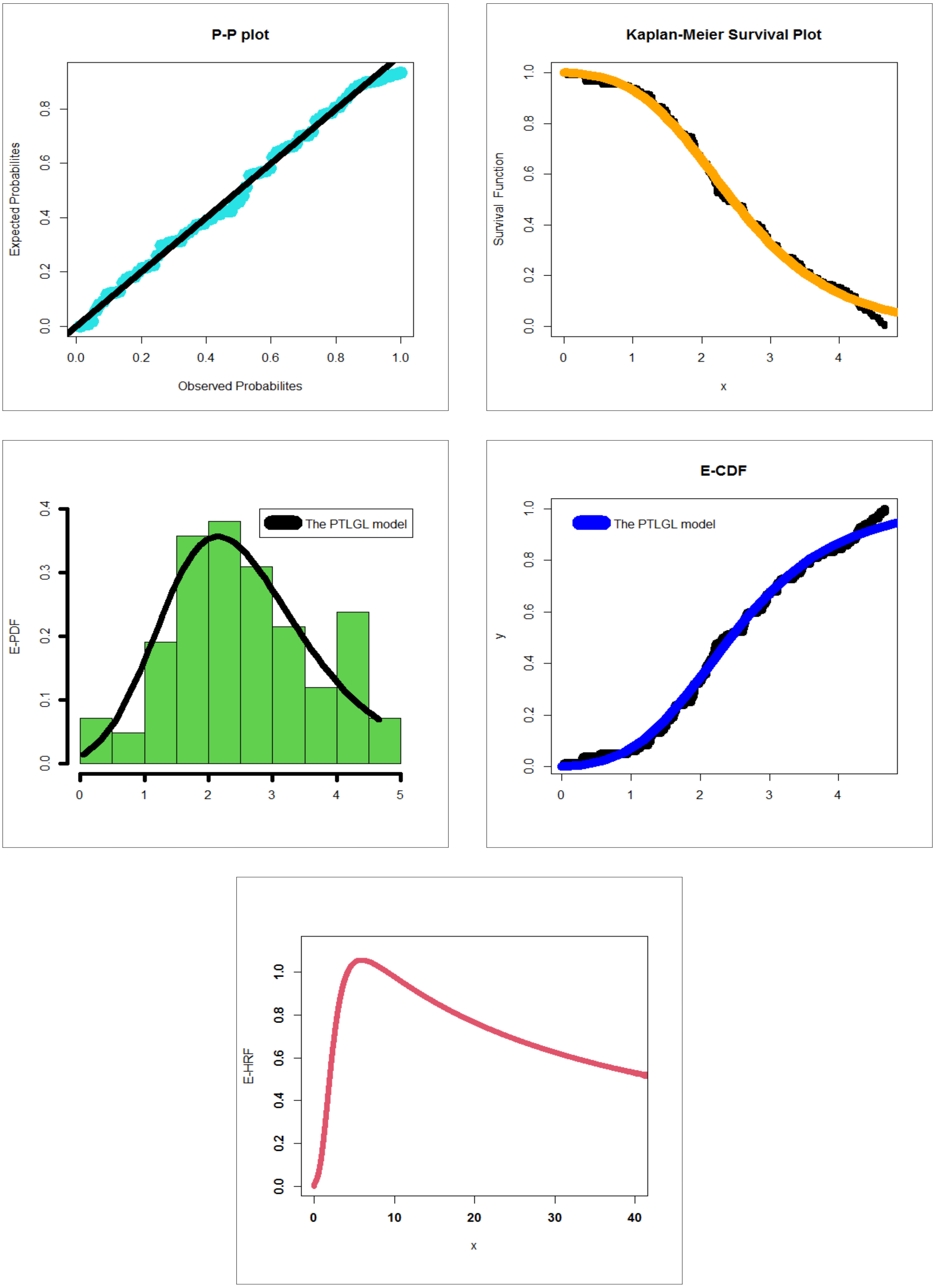

Figure 3. EPDF, EHRF, P-P, KMS plots for the 1st data set

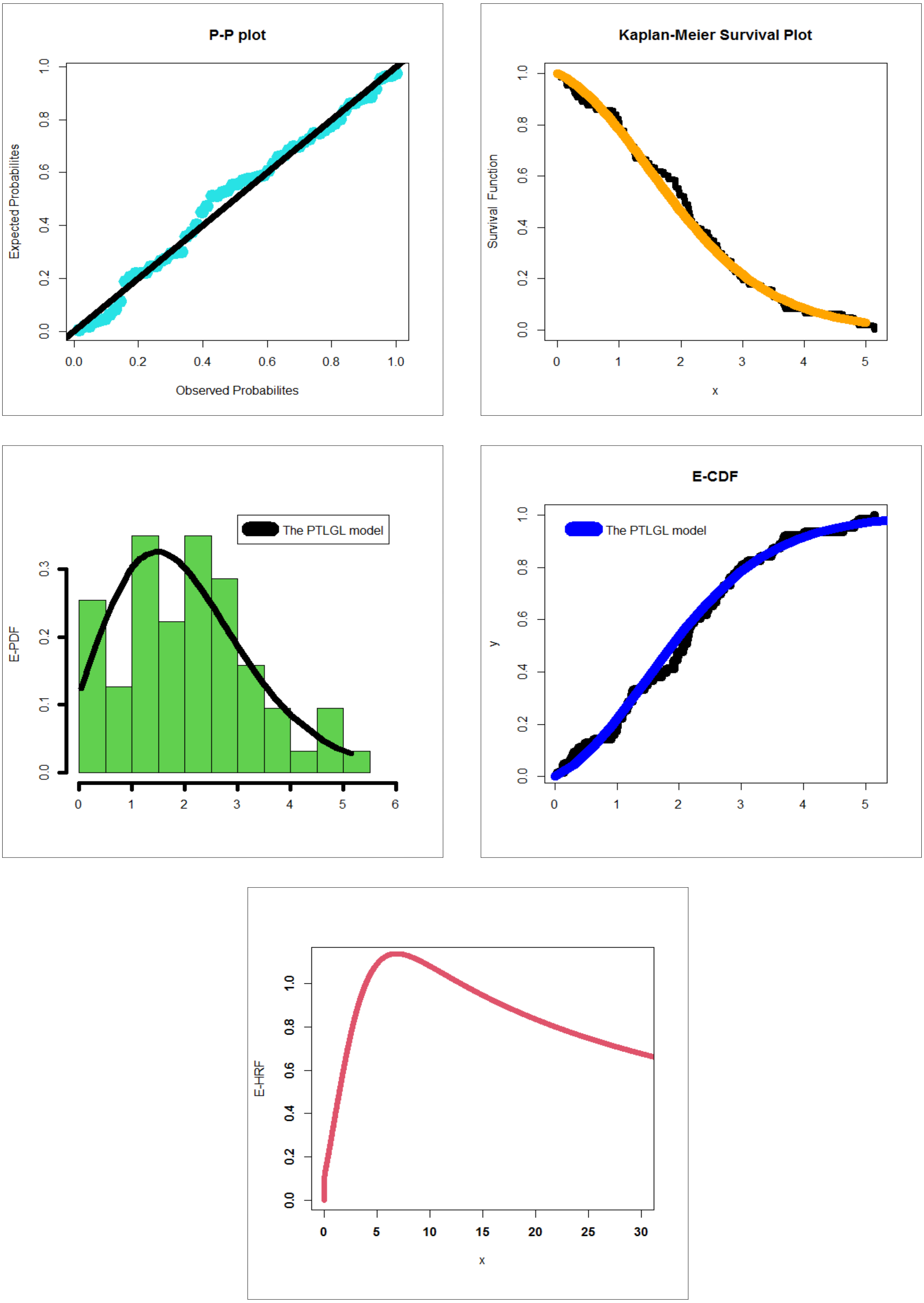

Figure 4. EPDF, EHRF, P-P, KMS plots for the 2nd data set

7. Conclusions

A new compound G family of distributions called the Poisson Topp Leone generated-G (QPTLG-G) family is defined and studied. The QPTLG-G family is constructed by compounding the Poisson and the Topp Leone generated G families. Special case based on the Lomax model called the Poisson Topp Leone generated Lomax (PTLGL) model is studied and analyzed. Relevant properties of the PTLGL model including moment of the residual life, ordinary moments, Moment of the reversed residual life, incomplete moments, probability weighted moments, order statistics and mean deviation are derived and numerically analyzed. Several new bivariate QPTLG-G families using the “Clayton copula”, “Farlie-Gumbel-Morgenstern copula”, “modified Farlie-Gumbel-Morgenstern copula”, “Ali-Mikhail-Haq copula” and “Renyi’s entropy copula” are investigated. Two different applications to real-life datasets are presented to illustrate the applicability and importance of the QPTLG-G family. For the two real datasets: The “initial density shapes” are explored by the nonparametric Kernel density function estimated, the “normality condition” is checked by the “Quantile-Quantile plot”, the shape of the hazard rates are discovered by the “total time in test” graphical tool, the extremes are explored by the “box plots”. Based on the two applications, the PTLGL distribution gives the lowest values for all statistic tests where AICr = 269.8712, CAICr = 270.6404, BICr = 282.0253 and HQICr = 274.7570 for the failure times data; AICr = 208.582, CAICr = 209.6347, BICr = 218.2977 and HQICr = 212.7966 for the service times data.

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