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# Berry-Esseen Type Bound in Partially Linear Regression Model under Mixing Sequences

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**Abstract** It is well-known that the confidence intervals of $\beta$ and $g(\cdot)$ in partially linear regression model lie in the limit distributions of their estimators. However the accuracy of the confidence intervals depends on how fast the theoretical distributions of the estimators converge to their limits. As a results, Berry-Esseen type bounds can be used to assess the accuracy. The aim of this paper is to study the Barry-Esseen type bounds for the estimators of $\beta$ and $g(\cdot)$ in the partially linear regression model with $\varepsilon_i$ satisfying $\varepsilon_i = \sum_{j=-\infty}^{\infty} \psi_j e_{i-j}$ with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\{e_i\}$ being stationary $(\alpha, \beta)$-mixing random variables. By choosing suitable weighted functions, the Berry-Esseen type bounds for the estimators $\beta$ and $g(\cdot)$ can achieve $O(n^{-1/4})$ and $O(n^{-1/8})$ respectively. Simulation studies are conducted to demonstrate the performance of the proposed procedure.

**Keywords** Partially linear model, Berry-esseen bound, $(\alpha, \beta)$-mixing sequence

## 1. Introduction

Partially linear regression model is a combination of linear and nonparametric parts in which the relationship between the response and some explanatory variables are linear whereas the other predictors are emerged in the model in unspecified association form. Opsomer and Ruppert (1999) argued for the advantage of partially linear regression model, including that there is less worry of overfitting, that they are more easily interpretable, and that the estimator is more efficient for the parametric components. Also, various estimation and variable selection methods for the partially linear regression model have been developed which we refer to, Horowitz (2009), Liu et al. (2011), Roozbeh et al. (2012), Amini and Roozbeh (2016), Roozbeh and Arashi (2016), Roozbeh (2018) and Amini and Roozbeh (2019) to mention a few.

Since its introduction by Engle et al. (1986), partially linear regression models have been studied by many authors. For example, Heckman (1986), Rice (1986), Chen (1988) and Speckman (1988) studied the consistency properties of the estimator of $\beta$ under different assumptions. Schick (1996) and Liang and Härdle (1997) extended the root $n$ consistency and asymptotic results for the case of heteroscedasticity. Härdle et al. (2000) provided a good comprehensive reference of the partially linear model. Chen et al. (1998) and Gao et al. (1994) established the strong consistency and asymptotic normality, respectively, for the least squares estimators and weighted least-squares estimator (WLSE, for short) of $\beta$ based on nonparametric estimates $g(\cdot)$ and $f(\cdot)$. You et al. (2007) further studied the model and developed an inferential procedure which includes a test of heteroscedasticity, a two-step estimator of $f(\cdot)$, mean square errors of $\beta$ and $g(\cdot)$ and a bootstrap goodness of fit test. If $g(t)=0, \sigma_i^2 = f(u_i)$, then the model boils down to the heteroscedastic linear model, whose asymptotic properties of the WLSE of $\beta$ were studied by Carroll (1982), Robinson (1987) and Carroll and Härdle (1989), respectively.

In this paper, we will further study the limit behaviors of the estimators in the partially linear regression model under $(\alpha, \beta)$-mixing random variables, the concept of which was first introduced by Bradley (1985) as follows.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Denote $S_n = \sum_{i=1}^{n} X_i, n \geq 1$, and $S_0 = 0$. Let $n$ and $m$ be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$. Given $\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{F}$, let

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$$\lambda(\mathcal{A},\mathcal{B})=\sup_{X\in L_{1/\alpha}(\mathcal{A}),Y\in L_{1/\beta}(\mathcal{B})}\frac{|EXY-EXEY|}{\|X\|_{1/\alpha}\|Y\|_{1/\beta}},$$

where $0<\alpha,\beta<1,\alpha+\beta=1,$ and $\|X\|_p=(E|X|^p)^{1/p}$. Define the $(\alpha,\beta)$-mixing coefficients by

$$\lambda(n)=\sup_{k\geq1}\lambda(\mathcal{F}_1^k,\mathcal{F}_{k+n}^{\infty}),\ \ n\geq0.$$

**Definition 1.1.** A sequence $\{X_n,n\geq1\}$ of random variable is said to be $(\alpha,\beta)$-mixing if $\lambda(n)\downarrow0$ as $n\to\infty.$

Since the concept of $(\alpha,\beta)$-mixing was introduced by Bradley (1985), many limit theorems were established. Bradley (1985) discussed central limit theorems under absolute regularity for $(\alpha,\beta)$-mixing sequences. Shao (1993) established limit theorems of $(\alpha,\beta)$-mixing sequences; Cai (1991) obtained strong consistency and rates for recursive nonparametric conditional probability density estimators under $(\alpha,\beta)$-mixing conditions; Lu and Lin (1997) gave the bounds of covariance of $(\alpha,\beta)$-mixing sequences; Shen and Zhang (2011) studied some convergence theorems for $(\alpha,\beta)$-mixing random variables, and obtained some new strong laws of large numbers for weighted sums of $(\alpha,\beta)$-mixing random variables; Gao (2016) investigated the $(\alpha,\beta)$-mixing sequences which are stochastically dominated, and presented some strong stability; Yu (2016) showed the Resenthal-type inequality of the $(\alpha,\beta)$-mixing sequences, and investigated the strong convergence theorems.

The aim of this paper is to further study the Barry-Esseen type bounds for the estimators of $\beta$ and $g(\cdot)$ in the partially linear regression model (2.1) with $\varepsilon_i$ satisfying $\varepsilon_i=\sum_{j=-\infty}^{\infty}\psi_j e_{i-j}$ with $\sum_{j=-\infty}^{\infty}|\psi_j|<\infty$ and $\{e_i\}$ being stationary $(\alpha,\beta)$-mixing random variables. By choosing suitable weighted functions, the Berry-Esseen type bounds for the estimators $\beta$ and $g(\cdot)$ can achieve $O(n^{-1/4})$ and $O(n^{-1/8})$ respectively.

This work is organised as follows: In Section 2, we recall the partially linear regression model and construct the partial least squares estimator for both the parametric and non-parametric components. The main results and numerical analysis (simulations and real data) are presented in Section 3. The proofs of the main results are provided in Section 4.

Throughout this paper, the symbols $C$, $c_1$, $c_2$, ... denote positive constants whose values may be different in different places. Let $\log x=\ln\max(x,e)$ and $I(A)$ be the indicator function of the set $A$. Let $a\wedge b=\min\{a,b\}$.

## 2. Model and Estimation

Consider the following partially linear regression model:

$$y_i=x_i\beta+g(t_i)+\varepsilon_i,\ 1\leq i\leq n, \tag{2.1}$$

where $\beta$ is an unknown parameter of interest, $(x_i,t_i)$ are nonrandom design points, $y_i$ are the response variables, $\varepsilon_i$ are random errors, $g(\cdot)$ is an unknown functions defined on closed interval [0,1].

If $\beta$ is the true parameter, then model (2.1) is reduced to a nonparametric regression model $y_i=x_i=g(t_i)+\varepsilon_i$. Since $Ee_i=0,$ we have $g_n^*(t,\beta)=\sum_{i=1}^{n}W_{ni}(x)(y_i-x_i\beta)$. Using the least squares method, we obtain $\hat{\beta}_L$ of $\beta$ by minimizing

$$SS(\beta)=\sum_{i=1}^{n}W_{ni}(x)(y_i-x_i\beta-g_n^*(t,\beta))^2$$

The minimizer is found to be

$$\hat{\beta}_n=\sum_{i=1}^{n}\sum_{i=1}^{n}\tilde{x}_i y_i/S_n^2, \tag{2.2}$$

where $S_n^2=\sum_{i=1}^{n}\tilde{x}_i^2$, $\tilde{x}_i=x-\sum_{i=1}^{n}W_{nj}(t_i)x_j$ and $y_i=y_i-\sum_{i=1}^{n}W_{nj}(t_i)y_j,1\leq i\leq n.$ Then based on $\hat{\beta}_L$, we defined the nonparametric function $g(\cdot)$ by

$$\hat{g}_n(t)=\sum_{j=1}^{n}\sum_{i=1}^{n}W_{ni}(t)(y_i-x_i\hat{\beta}_L), \tag{2.3}$$

To derive the Berry-Esseen bounds for the estimators, we make the following assumptions:

**A1.** There exist a functions $h(\cdot)$ on [0,1] such that $x_i=h(t_i)+v_i$, and

(i) $\lim_{n\to\infty}n^{-1}\sum_{i=1}^{n}v_i^2=\Sigma_0(0<\Sigma_0<\infty);$

(ii) $\lim_{n\to\infty}\sup_n(\sqrt{n}\log n)^{-1}\cdot\max_{1\leq m\leq n}|\sum_{i=1}^{n}v_{j_i}|<\infty.$

**A2.** $g(\cdot)$ and $h(\cdot)$ are defined on [0,1] and satisfy Lipschitz condition of order 1.

**A3.** The probability weight function $W_{nj}$ are defined on [0,1] and satisfy

(i) $\max_{1\leq j\leq n}\sum_{i=1}^{n}W_{ni}(t_i)=O(1);$

(ii) $\sup_{0\leq t\leq1}\max_{1\leq j\leq n}\sum_{i=1}^{n}W_{ni}(t_i)=O(a_n);$

(iii) $\max_{1\le i\le n}\sum_{j=1}^{n} W_{nj}(t_i)I(|t_i-t_j|=O(1).$

**A4.** $\max_{1\le i\le n}|v_i|=O(n^{\theta})$ for some $0\le\theta 1/2$

**A5.** There exist positive integers $p=p(n)$ and $q=q(n)$ such that $p+q\le n,\quad p\to\infty,\quad q\to\infty,$ $q/p\to 0, np^{-1/2}w_n^{1/2}\lambda(q)^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}\to 0.$

**A6.** The spectral density $f_1(.)$ of $e_i$ satisfies that $0<C_1\le f_1(\omega)\le C_2<\infty$ for $\omega\in(-\pi,\pi]$.

**Remark 2.1** Conditions (A1)-(A3) have been used frequently by many authors. For example, Gao et al. [1996], Sun et al. [2002], You et al. [2005], Liang et al. [2006], You and Chen [2007] and so on. (A4) is adopted in Sun et al. [2002], You et al. [2005], Liang et al. [2006], Liang and Fan [2009] and so forth. Moreover, if functions $g(\cdot)$ and $h(\cdot)$ satisfy a Lipschitz condition of order 1 on [0,1], then (A3) (iii) implies that $\max_{1\le i\le n}|g_i|=O(b_n)$ and $\max_{1\le i\le n}|h_i|=O(b_n)$.

# 3. Main Results and Numerical Analysis

## 3.1. Main Results

In this subsection, we present the Berry-Esseen type bounds for the estimators $\beta$, $g(\cdot)$. We first introduce some notations which will be used in the theorem below.

$$\Gamma_n^2=Var\left(\sum_{i=1}^{n}u_i\varepsilon_i\right),\ u(q)=\sum_{i=q}^{\infty}\lambda^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}(i),\ \Theta_n^2(t)=Var\left[\sum_{i=1}^{n}W_{ni}(t)\varepsilon_i\right],$$

$$\phi_{1n}=q/p,\ \phi_{2n}=p/n,\ \phi_{3n}=n\left(\sum_{|j|>n}|\psi_j|\right)^2,$$

$$\gamma_{1n}=nqp^{-1}a_n,\ \gamma_{2n}=pa_n,\ \gamma_{3n}=\Theta_n^{-2}(t)\left(\sum_{|j|>n}|\psi_j|\right)^2,$$

$$\mu_{1n}=(n^{\epsilon-\tau}\phi_{1n}+\phi_{1n}^{1+\tau})^{1/(3+2\tau)}+\phi_{1n}^{1/2}+(n^{\epsilon-\tau}\phi_{2n}+\phi_{2n}^{1+\tau})^{1/(3+2\tau)}+\phi_{2n}^{1/2}+\phi_{2n}^{\tau}+\phi_{3n}^{1/3}+u(q)+(np^{-1}\lambda(q))^{1/4},$$

$$\mu_{2n}=b_n^{(2+2\tau)/(3+2\tau)}+(n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2}n+a_n^{\tau+1}\log^{2\tau+2}n)^{1/(3+2\tau)}+b_n\log n+n^{1/2}b_n^2,$$

$$\nu_{1n}=\gamma_{1n}^{1/2}+(n^{\epsilon}\gamma_{1n}a_n^{\tau}+\gamma_{1n}^{1+\tau})^{1/(3+2\tau)}+\gamma_{2n}^{\tau}+\gamma_{2n}^{1/2}+(n^{\epsilon}\gamma_{2n}a_n^{\tau}+\gamma_{2n}^{1+\tau})^{1/(3+2\tau)}+\gamma_{3n}^{1/3}+(n^2p^{-1}a_n\lambda(q))^{1/4},$$

$$\nu_{2n}=(n^{1/2}\Theta_n^{-1}(t)+a_n^{1/2}\log n)^{(2+\delta)/(3+\delta)}.$$

**Theorem 3.1.** *Let* $\{\varepsilon_i, 1\le i\le n,\}$ *be a mean zero* $(\alpha,\beta)$ *-mixing sequence, with* $\sum_{n=1}^{\infty}(\lambda(n))^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}<\infty$, *where* $0<\alpha,\beta<1$ *and* $\alpha+\beta=1$. *Suppose that conditions* (**A1**) - (**A3**) *are satisfied,* $\Gamma_n^2\ge Cn$ *and* $E|e_0|^{2+\delta}<\infty$ *for some* $\delta>0$ . *Let* $\lambda(n)=O(n^{-\phi})$ *for some* $\phi>(2+\delta)/\delta$. *Assume that* $\mu_{1n}\to 0,\quad \nu_{1n}\to 0$ . *Then, for* $0<\tau\le(\phi\delta-\delta-2)/(2\phi+\delta+2)$ *and any* $\epsilon\in(0,\tau]$, *we have*

$$\sup_u\left|P\left(\frac{S_n^2(\beta_n-\beta)}{\Gamma_n}\le u\right)-\Phi(u)\right|=:O(\mu_{1n}+\mu_{2n});\tag{3.1}$$

**Corollary 3.1** *Suppose that (A1)-(A4) hold with* $a_n=n^{\theta-1/2}\log^{-2}n$ *and* $b_n=n^{2\theta-3)/8}$, *where* $0<\theta<1/2$, *and* $E|e_0|^{2+\delta}<\infty$ *for some* $\delta>\sqrt{3}-1$. *Let* $\sum_{|j|>n}|\psi_j|=O(n^{-1/2+3(\theta-1/2)(1+\tau)/(6+4\tau)})$, $u(n)=O(n^{(\theta-1/2)(1+\tau)/[2\theta(3+2\tau)]})$ *for some* $(\sqrt{3}-1)/2\le\tau<\delta/2$, *and* $\lambda(n)=O(n^{-\phi})$ *for some* $\phi\ge\max\{(\delta+2)(\tau+1)/(\delta-2\tau)$ , $(1/2-\theta)(7+6\tau)/[2\theta(3+\tau)]\}$. *Then*

$$\sup_u \left| P\left( \frac{S_n^2(\beta_n - \beta)}{\Gamma_n} \le u \right) - \Phi(u) \right| =: O(n^{(\theta-1/2)(1+\tau)/(3+2\tau)})$$

**Theorem 3.2.** *Suppose that the conditions in theorem 3.1 hold. Let* $w_n(t) = \max_{1\le j\le n} W_{nj}(t)$, $\Gamma_n^2 \le cn$ *and* $w_n(t) = O(\Theta_n^2(t))$. *If* $\nu_{1n}$ *and* $\nu_{2n}$ *converge to zero, then*

$$\sup_{t\in[0,1]} \sup_u \left| P\left( \frac{(\hat{g}(t) - E\hat{g}(t))}{\Theta_n(t)} \le u \right) - \Phi(u) \right| =: O(\nu_{1n} + \nu_{2n}). \tag{3.2}$$

**Corollary 3.2** *Set* $a_n = n^{-(3+2\theta)/4}$ *for some* $0 < \theta < 1/2$ *and* $b_n = o(n^{-1/4})$. *Suppose that* (**A1**) - (**A4**) *hold with* $a_n^{-1}\sum_{i=1}^n W_{ni}^2 \ge \varrho_0 > 0$ *for each* $t \in [0,1]$, *and* $E|e_0|^{2+\delta} < \infty$ *for some* $\delta > \sqrt[n]{3} - 1$. *Let* $\sum_{|j|>n} |\psi_j| = O(n^{-1/2+(\theta-1/2)\tau/(12+8\tau)})$, $u(n) = O(n^{(\theta-1/2)(1+r)/[4\theta(3+2r)]})$ *for some* $(\sqrt{3}-1)/2 \le \tau < \delta/2$, *and* $\lambda(n) = O(n^{-\phi})$ *for some* $\phi \ge \max\{(\delta+2)(\tau+1)/(\delta-2\tau), (1/2-\theta)(13+10\tau)/[4\theta(3+2\tau)]\}$, *then*

$$\sup_{t\in[0,1]} \sup_u \left| P\left( \frac{(\hat{g}(t) - E\hat{g}(t))}{\Theta_n(t)} \le u \right) - \Phi(u) \right| =: O(n^{(\theta-1/2)(1+\tau)/(6+4\tau)}).$$

### 3.2. Numerical Simulation

In this subsection, We carry out a simulation to study the asymptotic normality of the estimators $\hat{\beta}_n$ and $\hat{g}_n$ of $\beta$ and $g$, respectively. The observations are generated for the following model:

$$y_i = 2x_i + sin(x_{ni}) + \varepsilon_i, \quad i = 1, 2, \cdots, n,$$

where $\varepsilon_i$ is an AR(1) type process $\varepsilon_i = 0.2\varepsilon_{i-1} + e_i$ with $\{e_i\}$ be an MA(1) process specified by $e_i = 0.5\xi_i - 1.3\xi_{i-1}$ and $\xi_i \overset{i.i.d.}{\sim} N(0,1)$, for $1 \le i \le n$. Let $x_i = 0.5t^2 + u_i$, where $t_i = (i-0.5)/n$ and $u_i \overset{i.i.d.}{\sim} N(0.5, 0.5)$ for $1 \le i \le n$. Here, we choose the nearest neighbor weights to be the weight functions $w_{ni}(\cdot)$. For any $x \in A = [0,1]$, we rewrite $|t_{ni} - t|, |x_{n2} - x|, \cdots, |t_{nn} - t|$ as follows

$$|t_{nR_1(x)} - t| \le |t_{nR_2(t)} - t| \le \cdots \le |t_{nR_n(t)} - t|,$$

if $|t_{in} - t| = |t_{jn} - t|$, then $|t_{in} - t|$ is permutated before $|t_{jn} - t|$ when $t_{in} \le t_{jn}$. Take $k_n = \lfloor n^{0.8} \rfloor$ and defined the nearest neighbor weight function as follows:

$$W_{ni}(t) = \begin{cases} 1/k_n, & if\ |t_{in} - t| \le |t_{R_{k_n}(x),n} - t|, \\ \\ 0, & otherwise. \end{cases}$$

We generate the observed data with sample size $n$ as $n = 50, 150, 250$ and $t = 0.1, 0.2, \cdots, 1$, respectively. We used R software to compute $U_{n1} = \dfrac{s_n^2(\hat{\beta}_n - \beta)}{\Gamma_n}$ and $U_{n2} = \dfrac{\hat{g}_n(t) - E\hat{g}_n(t)}{\Theta_n(t)}$, and obtained the Q-Q plots of $\hat{\beta}_n$ and $\hat{g}_n$ respectively, based on 500 replications.

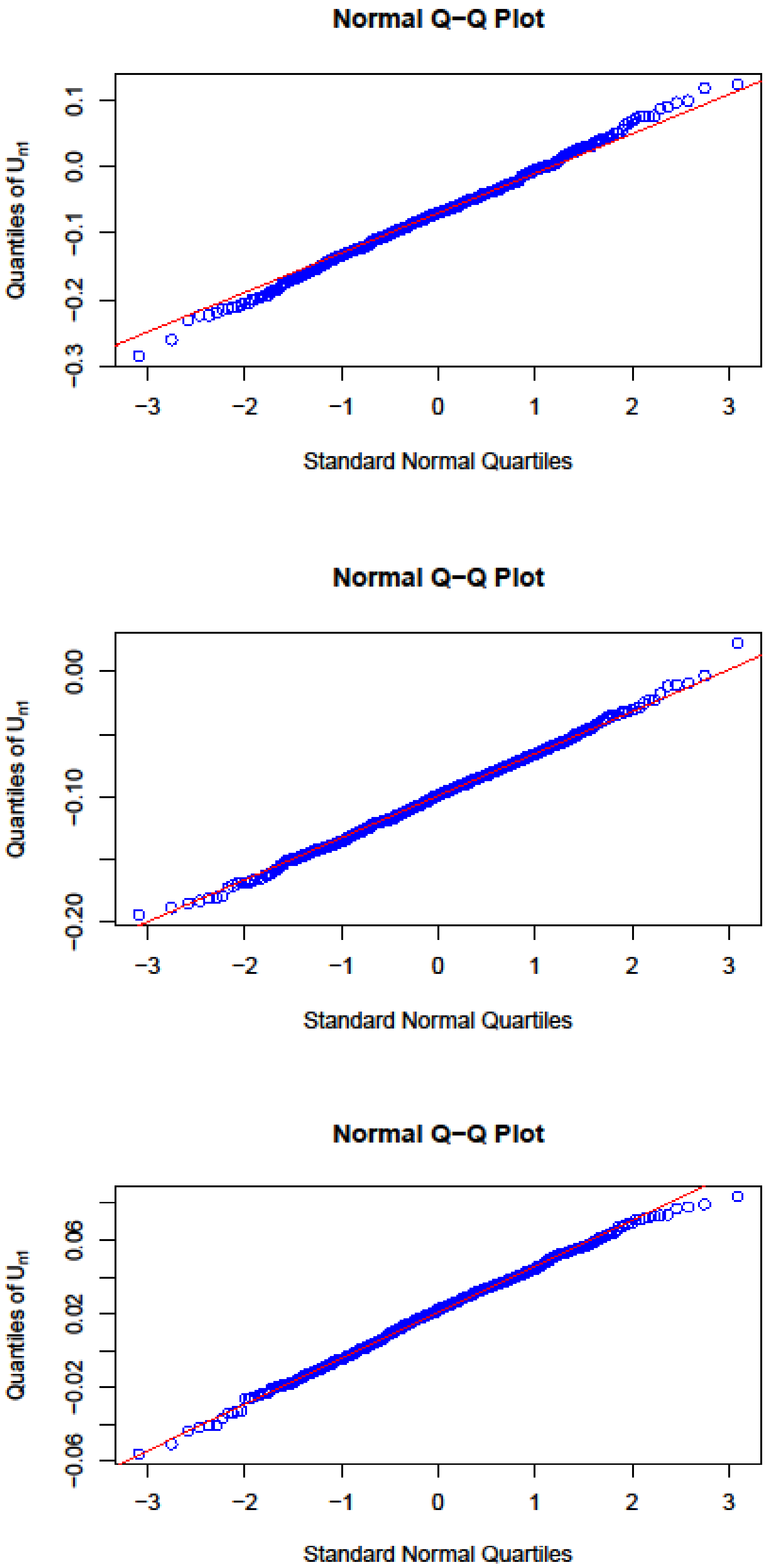


**Figure 1.** Q-Q plot of $U_{n1}$ with $n$ =50, 100 and 150, respectively

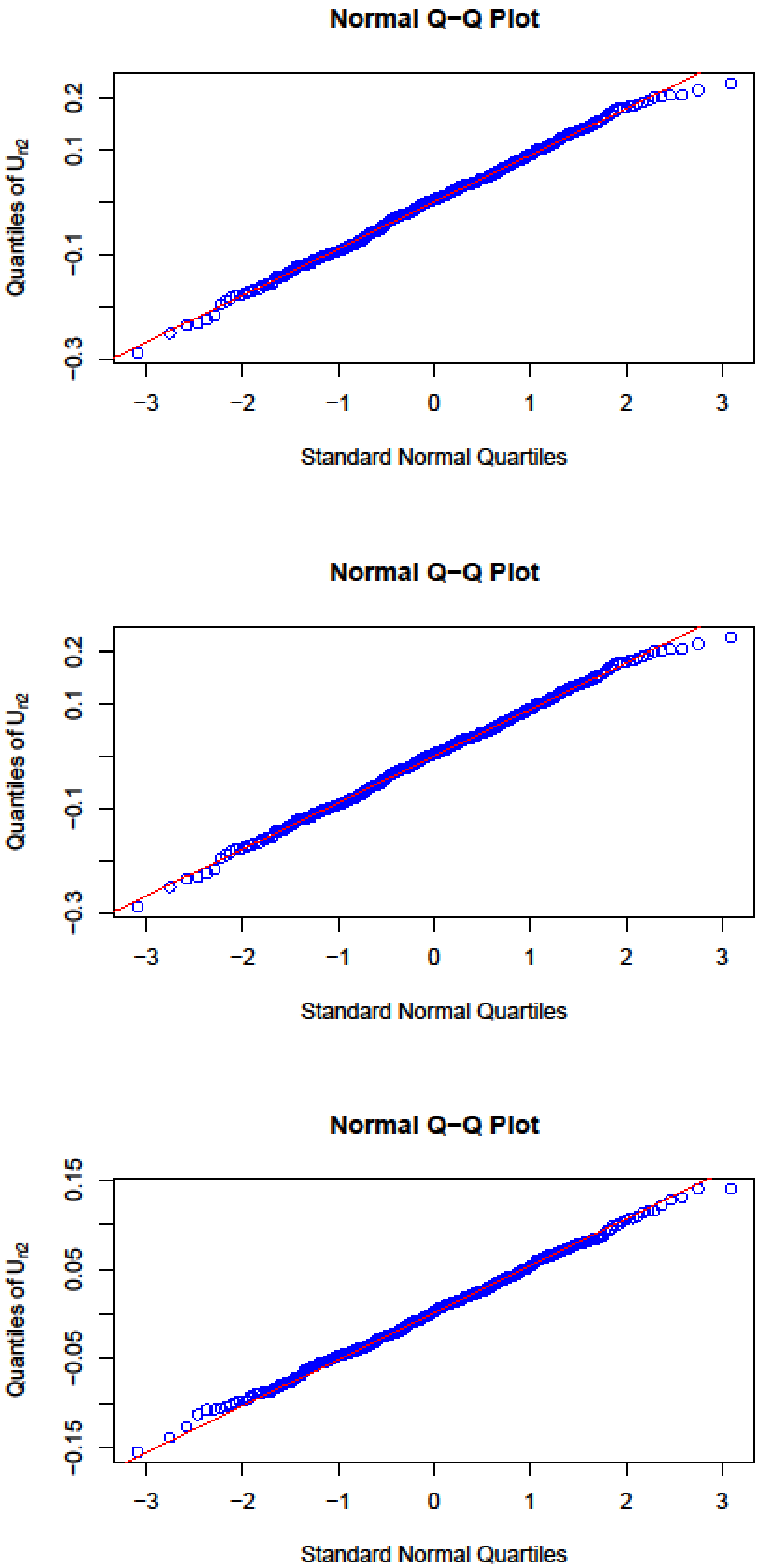


**Figure 2.** Q-Q plot of $U_{n2}$ with $n$ =50, 100 and 150, respectively

## 4. Proofs of the Main Results

We first introduce several lemmas which will be used to prove the main results of the paper.

**Lemma 3.1.** (cf. Yu, 2016) Let $\{X_i, i \geq 1\}$ be a sequence of $(\alpha, \beta)$-mixing random variables with $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and $\sum_{n=1}^{\infty} (\lambda(n))^{(1/2\alpha)\wedge(1/2\beta)} < \infty$, where $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers. Then there exists a positive constant $C$ depending only on $\alpha$, $\beta$ and $\lambda(\cdot)$ such that

$$E\left|\sum_{i=1}^{n} a_{ni} X_i\right|^p \leq C\left\{\sum_{i=1}^{n} |a_{ni}|^p E|X_i|^p + \left(\sum_{i=1}^{n} a_{ni}^2 EX_i^2\right)^{p/2}\right\}.$$

**Lemma 3.2.** (Liang and Fan, 2009) Let $X$, $Y_1, \cdots, Y_m$ be random variables. For positive numbers $\omega_1, \cdots, \omega_m$, we have that

$$\sup_u \left|P\left(X + \sum_{i=1}^{m} Y_i \leq u\right) - \Phi(u)\right| \leq \sup_u |P(X \leq u) - \Phi(u)| + \sum_{i=1}^{m} \frac{\omega_i}{\sqrt{2\pi}} + \sum_{i=1}^{m} P(|Y_i| > \omega_i).$$

**Lemma 3.3.** (Lu and Lin (1997)) Let $\{X_n, n \geq 1\}$ be a sequence of $(\alpha, \beta)$-mixing random variables. Suppose that $X \in L_p(\mathcal{F}_{-\infty}^{k})$ and $Y \in L_q(\mathcal{F}_{k+n}^{\infty})$, where $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$|EXY - EXEY| \leq 4\lambda^{(1/2\alpha)\wedge(1/2\beta)}(n) \|X\|_p \|Y\|_q.$$

**Lemma 3.4.** Let $\{X_n, n \geq 1\}$ be a sequence of $(\alpha, \beta)$-mixing random variables. Suppose that $p$ and $q$ are two positive integers. Let $\eta_l = \sum_{j=(l-1)(p+q)+1}^{(l-1)(p+q)+p} X_j$ for $1 \leq l \leq k$. Then for any $t \in \mathbb{R}$

$$\left|E\exp\left(it\sum_{l=1}^{k} \eta_l\right) - \prod_{l=1}^{k} E\exp(it\eta_l)\right| \leq C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q) \sum_{l=1}^{k} \|\eta_l\|_2.$$

**Proof.** It is easily checked that

$$\begin{aligned}
\left|E\exp\left(it\sum_{l=1}^{k} \eta_l\right) - \prod_{l=1}^{k} E\exp(it\eta_l)\right| &\leq \left|E\exp\left(it\sum_{l=1}^{k} \eta_l\right) - E\exp\left(it\sum_{l=1}^{k-1} \eta_l\right) E\exp(it\eta_l)\right| \\
&\quad + \left|E\exp\left(it\sum_{l=1}^{k-1} \eta_l\right) - \prod_{l=1}^{k-1} E\exp(it\eta_l)\right| \\
&\doteq J_1 + J_2. \qquad (4.1)
\end{aligned}$$

Noting that $e^{ix} = \cos x + i\sin x$, we have

$$\begin{aligned}
J_1 &\leq \left|Cov\left(\cos\left(t\sum_{l=1}^{k-1} \eta_l\right), \cos(t\eta_k)\right)\right| + \left|Cov\left(\sin\left(t\sum_{l=1}^{k-1} \eta_l\right), \sin(t\eta_k)\right)\right| \\
&\quad + \left|Cov\left(\sin\left(t\sum_{l=1}^{k-1} \eta_l\right), \cos(t\eta_k)\right)\right| + \left|Cov\left(\cos\left(t\sum_{l=1}^{k-1} \eta_l\right), \sin(t\eta_k)\right)\right| \\
&\doteq J_{11} + J_{12} + J_{13} + J_{14}. \qquad (4.2)
\end{aligned}$$

It follows from Lemma 3.3 and $|\sin x| \leq |x|$ that

$$J_{14} \le C\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(n)\left\|\cos\left(t\sum_{l=1}^{k-1}\eta_l\right)\right\|_2 \|\sin(t\eta_k)\|_2$$

$$\le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2 \tag{4.3}$$

and

$$J_{12} \le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2. \tag{4.4}$$

Noting that $\cos(2x) = 1-2\sin^2 x$, and hence applying Lemma 3.3 and invoking again the inequality $\sin^2 x \le |\sin x| \le |x|$, we find that

$$J_{13} = 2\left|Cov\left(\sin(t\sum_{l=1}^{k-1}\eta_l), \sin^2(\frac{t\eta_k}{2})\right)\right| \le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(n)\|\eta_k\|_2, \tag{4.5}$$

and

$$J_{11} \le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2. \tag{4.6}$$

By (3.1)-(3.6), we have

$$\left|E\exp\left(it\sum_{l=1}^{k}\eta_l\right) - \prod_{l=1}^{k}E\exp(it\eta_l)\right| \le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2 + J_2.$$

Proceeding in this manner, we obtain

$$\left|E\exp\left(it\sum_{l=1}^{k}\eta_l\right) - \prod_{l=1}^{k}E\exp(it\eta_l)\right| \le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_k\|_2 + C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\|\eta_{k-1}\|_2$$

$$+\left|E\exp\left(it\sum_{l=1}^{k-2}\eta_l\right) - \prod_{l=1}^{k-2}E\exp(it\eta_l)\right|$$

$$\le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{l=1}^{k}\|\eta_k\|_2.$$

This completes the proof of the lemma.

**Lemma 3.5.** Let $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers such that $\sum_{i=1}^{n} b_{ni}^2 = O\sum(\Delta_n)$ and $\max_{1\le i\le n} b_{ni}^2 = O(B_n)$, where $\Delta_n$ and $B_n$ are some positive numbers. Suppose that $E|e_i|^{2+\delta} < \infty$ for some $\delta > 0$ and $\alpha(n) = O(n^{-\lambda}$ for some $\lambda > (2+\delta)/\delta$. Then for any $n \ge 1$,

$$P\left(|\sum_{i=1}^{n} b_{ni}\varepsilon_i| > \gamma_n + l_n\right) \le C\left\{\gamma_n + \left(l_n^{-1}\sum_{i=1}^{n}|b_{ni}|\sum_{|j|>n}|\phi_j|\right)^2\right\},$$

and

$$P\left(\left|\sum_{i=1}^{n} b_{ni}e_i\right| > \gamma_n\right) \le C\gamma_n,$$

where $l_n$ are positive numbers and $\gamma_n = (n^{\epsilon}\Delta_n B_n^{\tau} + \Delta_n^{1+\tau})^{1/(3+2\tau)}$ for some $0 < \tau \le (\psi\delta - \delta - 2)/(2\psi + \delta + 2)$ and any $\epsilon > 0$.

**Proof.** We only prove the first inequality, and the second one is completely analogous. According to the definition of $\varepsilon_i$, we have that

$$P\left(|\sum_{i=1}^{n} b_{ni}\varepsilon_i| > \gamma_n + l_n\right) \le P\left(\left|\sum_{i=1}^{n} b_{ni}\sum_{|j|\le n}\phi_j e_{i-j}\right| > \gamma_n\right) + P\left(\left|\sum_{i=1}^{n} b_{ni}\sum_{|j|>n}\phi_j e_{i-j}\right| > \gamma_n\right).$$

By $Ee_i^2 < \infty$ and Markov's inequality, we have

$$P\left(\left|\sum_{i=1}^{n} b_{ni}\sum_{|j|>n}\phi_j e_{i-j}\right| > l_n\right) \le l_n^{-2} E\left(\sum_{i=1}^{n} b_{ni}\sum_{|j|>n}\phi_j e_{i-j} > l_n\right)^2$$

$$\le l_n^{-2} E\left(\sum_{i_1=1}^{n} |b_{ni_1}| \sum_{i_2=1}^{n} |b_{ni_2}| \left|\sum_{|j_1|>n}\phi_{j_1} e_{i_1-j_1} \cdot \sum_{|j_2|>n}\phi_{j_2} e_{i_2-j_2}\right|\right)$$

$$\le \left(l_n^{-1}\sum_{i=1}^{n} |b_{ni}| \sum_{|j|>n} |\phi_j|\right)^2$$

Note that

$$\sum_{i=1}^{n} b_{ni}\sum_{|j|>n}\phi_j e_{i-j} = \sum_{l=1-n}^{2n}\sum_{i=\max(1,l-n)}^{\min(n,n+l)} b_{ni}\phi_{i-l} e_l .$$

Applying Lemma 3.1 with $r = 2+2\tau$, $\eta = \delta - 2\tau$. Noting that $E|e_i|^{2+\delta} < \infty$ and $0 < \tau \le (\lambda\delta - \delta - 2)/(2\psi + \delta + 2) < \delta/2$ implies $(2+\delta)/\delta < (2+\delta)(1+\tau)/(\delta - 2\tau) = r(r+\eta)/(2\eta) < \psi$,

we have

$$P\left(\left|\sum_{i=1}^{n} b_{ni}\sum_{|j|>n}\phi_j e_{i-j}\right| > \gamma_n\right) \le \gamma_n^{-2-2\tau} E\left|\sum_{l=1-n}^{2n}\sum_{i=\max(1,l-n)}^{\min(n,n+l)} b_{ni}\phi_{i-l} e_l\right|^{2+2\tau}$$

$$\le C\gamma_n^{-2-2\tau}\left\{\sum_{l=1-n}^{2n}\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} b_{ni}\phi_{i-l}\right|^{2+2+2\tau} + \left[\sum_{l=1-n}^{2n}\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} b_{ni}\phi_{i-l}\right|^2\right]^{1+\tau}\right\}$$

$$\le C\gamma_n^{-2-2\tau}\left\{B_n^{\tau}\sum_{l=1-n}^{2n}\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} b_{ni}\phi_{i-l}\right|^{2+2+2\tau} + \left[\sum_{l=1-n}^{2n}\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} b_{ni}\phi_{i-l}\right|^2\right]^{1+\tau}\right\}$$

$$\le C\gamma_n^{-2-2\tau}\left\{B_n^{\tau}\sum_{l=1-n}^{2n} b_{ni}^2 + \left(\sum_{l=1-n}^{2n} b_{ni}^2\right)^{1+\tau}\right\} \le C\gamma_n,$$

Where the inequality in the last line above follows from $(\sum_i b_{ni}\phi_{i-l})^2 \le \sum_i |\phi_{i-l}|$ and changing the order of simulation. The proof is completed.

**Lemma 3.6.** Under the assumptions of Theorem 2.1, the following statements hold.

(i) $E(D_n'')^2 \le C\lambda_{1n}$, $E(D_n''')^2 \le C\lambda_{2n}$, $(\Gamma_n^{-1}A_{2n})^2 \le C\lambda_{3n}$

(ii) Let $\psi_{4n} = (n^{1/2} b_n \sum_{|j|>n|} |\phi_j|)^{2/3}$ *and* $\psi_{5n} = (\log n \sum_{|j|>n|} |\phi_j|)^{2/3}$. Then

(a) $P(|D_n''| > (n^{\epsilon-\tau}\psi_{1n} + \psi_{1n}^{1+\tau})^{1/(3+2\tau)}) \le C(n^{\epsilon-\tau}\psi_{1n} + \psi_{1n}^{1+\tau})^{1/(3+2\tau)})$ *for any* $\epsilon > 0$;

(b) $P(|D_n'''| > (n^{\epsilon-\tau}\psi_{2n} + \psi_{2n}^{1+\tau})^{1/(3+2\tau)}) \le C(n^{\epsilon-\tau}\psi_{2n} + \psi_{2n}^{1+\tau})^{1/(3+2\tau)})$ *for any* $\epsilon > 0$;

(c) $P(|\Gamma_n^{-1}I_{12n}|> 2b_n^{(2+2\tau)/(3+2\tau)}+\psi_{4n})\le C(b_n^{(2+2\tau)/(3+2\tau)}+\psi_{4n})$;

(d) $P(|\Gamma_n^{-1}I_{13n}|>(n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2}n+a_n^{\tau+1}\log^{2\tau+2}n)^{1/(3+2\tau)}+\psi_{5n})$
$\le C((n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2}n+a_n^{2\tau+1}\log^{2\tau+2}n)^{1/(3+2\tau)}+\psi_{5n}))$

(e) $P(|\Gamma_n^{-1}I_{2n}|>(n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2}n+a_n^{\tau+1}\log^{2\tau+2}n)^{1/(3+2\tau)}+2b_n^{(2+2\tau)/(3+2\tau)}+\psi_{4n}+2\psi_{5n})$
$\le C((n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2}n+a_n^{2\tau+1}\log^{2\tau+2}n)^{1/(3+2\tau)}+b_n^{(2+2\tau)/(3+2\tau)}+\psi_{4n}+\psi_{5n}))$;

(f) $P(|\Gamma_n^{-1}A_{2n}|>(\psi_{3n}^{1/3})\le C\psi_{3n}^{1/3}$;

(g) $|\Gamma_n^{-1}I_{3n}|\le c_0(b_n\log n+n^{1/2}b_n^2)$.

**Proof.** (i) Applying Lemma 3.2 with $p=q=2+\delta$ and *Hölder's* inequality, we have by $E|e_0|^{2+\delta}<\infty$, $\lambda(n)=O(n^{-\lambda})$ for $\lambda>(2+\delta)/\delta$, (A1) (iii) and (1.2) that

$$\begin{aligned}E(D_n'')^2&=E\left\{\sum_{i=1}^{k}\sum_{l=l_m}^{l_m+q+1}\left(\Gamma_n^{-1}\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-1}\right)e_l\right\}^2\\&\le\left\{1+16\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q+1}\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(l)\right\}\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q+1}\Gamma_n^{-2}\left(E\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-1}e_l\right|^{2+\delta}\right)^{2/(2+\delta)}\\&\le C\Gamma_n^{-2}\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q+1}\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-1}\right)^2\\&\le Ckq/p=C\psi_{1n}.\end{aligned}\tag{4.7}$$

Similarly, we have by Lemma 3.2 and Hölder's inequality again that

$$\begin{aligned}E(D_n''')^2&=E\left\{\sum_{l=k(p+q)+1-n}^{2n}\left(\Gamma_n^{-1}\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-1}\right)e_l\right\}^2\\&\le C\Gamma_n^{-2}\sum_{l=k(p+q)+1-n}^{2n}\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-l}\right)^2\\&\le Cp/n=C\psi_{2n}.\end{aligned}$$

Moreover, we have by $E|e_0|^{2+\delta}<\infty$ and (A1) (iii) again and $\Gamma_n^2\ge Cn$ that

$$\begin{aligned}E(\Gamma_n^{-1}A_{2n})^2&=\Gamma_n^{-2}E\left(\sum_{i=1}^{n}u_i\sum_{|j|>n}\phi_i\phi_{i-1}\right)^2\\&\le\Gamma_n^{-2}E\left(\sum_{i=1}^{n}|u_i|\sum_{i_2=1}^{n}|u_{i_2}|\left|\sum_{|j_1|>n}\phi_{i_1}\phi_{i_1-j_1}\cdot\sum_{|j_2|>n}\phi_{i_2}\phi_{i_2-j_2}\right|\right)\\&\le C\Gamma_n^{-2}\left(\sum_{i=1}^{n}|u_i|\sum_{|j|>n}|\phi_i|\right)^2\\&\le Cn\left(\sum_{|j|>n}|\phi_i|\right)^2=C\phi_{3n}\end{aligned}$$

Hence (i) has been proved.

(ii) The inequalities (a)-(e) can be proved by applying Lemma 3.5. Now we will verify them one by one.

(a) Noting by (A1) (iii) that $\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q+l}(\Gamma_n^{-1}\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-l})^2 \le Cq/p = O(\psi_{1n})$ and $\max_l(\Gamma_n^{-1}\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-l})^2 = O(n^{-1})$, the result follows immediately form Lemma 3.5.

(b) Similarly, noting additionally that $\sum_{l=k(p+q)+1-n}^{2n}(\Gamma_n^{-1}\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-1})^2 \le Cp/n = O(\psi_{2n})$, we have by Lemma 3.5 again that the result follows.

(c) Note that $\Gamma_n^{-1}I_{12n} = \sum_{i=1}^{n}\Gamma_n^{-1}\tilde{h}_i\varepsilon_i$. Therefore, we have

$\max_{1\le i\le n}(\Gamma_n^{-1}\tilde{h}_i)^2 = O(n^{-1}b_n^2)^2$, $\sum_{i=1}^{n}(\Gamma_n^{-1}\tilde{h}_i)^2 = O(b_n^2)$, and $\sum_{i=1}^{n}|\Gamma_n^{-1}\tilde{h}_i| = O(n^{1/2}b_n^2)$.

Let $0<\epsilon\le\tau$, we have Lemma 3.5 again that

$$P(|\Gamma_n^{-1}I_{12n}|> 2b_n^{(2+2\tau)/(3+2\tau)}+\psi_{4n}) \le P(|\Gamma_n^{-1}I_{12n}|> (n^{\epsilon-\tau}b_n^{2+2\tau}+b_n^{2+2\tau})^{1/(3+2\tau)}+\psi_{4n})$$
$$\le C(b_n^{(2+2\tau)/(3+2\tau)}+\psi_{4n}).$$

(d) Observe that $\Gamma_n^{-1}I_{13n} = \sum_{i=1}^{n}(\Gamma_n^{-1}\sum_{j=1}^{n}W_{nj}(t_i)u_j)\varepsilon_i = \sum_{i=1}^{n}b_{1ni}\varepsilon_i$. Utilizing the Abel Inequality (see Mitrinovic, 1970, Theorem 1, p.32), we have by (A1) and (A3) that

$$\max_{1\le i\le n} b_{1ni}^2 \le C\Gamma_n^{-2}\max_{1\le i,j\le n}W_{nj}^2(t_i)\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}u_{j_i}\right|\right)^2 = O(a_n^2\log^2 n),$$

$$\sum_{i=1}^{n}b_{1ni}^2 \le C\Gamma_n^{-2}\max_{1\le i,j\le n}W_{nj}(t_i)\max_{1\le j\le n}\sum_{i=1}^{n}W_{nj}(t_i)\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}u_{j_i}\right|\right)^2 = O(a_n\log^2 n),$$

and

$$\sum_{i=1}^{n}b_{1ni}^2 \le C\Gamma_n^{-2}\max_{1\le j\le n}\sum_{i=1}^{n}W_{nj}(t_i)\max_{1\le m\le n}\left|\sum_{i=1}^{m}u_{j_i}\right| = O(\log n).$$

Thus the result follow from Lemma 3.5 immediately.

(e) It follows $\tilde{x}_i = x_i - \sum_{j=1}^{n}W_{ni}(t_i)x_j = u_i + \tilde{h}_i - \sum_{s=1}^{n}W_{ns}(t_i)u_s$ that

$$|\Gamma_n^{-1}I_2n| \le \left|\Gamma_n^{-1}\sum_{i=1}^{n}u_i\left(\sum_{j=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j\right)\right| + \left|\Gamma_n^{-1}\sum_{i=1}^{n}\tilde{h}_i\left(\sum_{j=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j\right)\right|$$
$$+\left|\Gamma_n^{-1}\sum_{i=1}^{n}\left(\sum_{s=1}^{n}W_{ns}(t_i)u_s\right)\left(\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j\right)\right|$$
$$=\left|\sum_{i=1}^{n}\left(\Gamma_n^{-1}\sum_{j=1}^{n}W_{nj}(t_i)u_j\right)\varepsilon_j\right| + \left|\sum_{i=1}^{n}\left(\Gamma_n^{-1}\sum_{j=1}^{n}W_{nj}(t_i)\tilde{h}_i\right)\varepsilon_j\right|$$
$$+\left|\sum_{i=1}^{n}\left[\Gamma_n^{-1}\sum_{j=1}^{n}W_{nj}(t_i)\left(\sum_{s=1}^{n}W_{ns}(t_i)u_s\right)\right]\varepsilon_j\right|$$

$$=|\sum_{i=1}^{n} b_{2ni}\varepsilon_i| + |\sum_{i=1}^{n} b_{3ni}\varepsilon_i| + |\sum_{i=1}^{n} b_{4ni}\varepsilon_i|.$$

Similar to the proofs of (c) and (d), we have

$$\max_{1\le i\le n} b_{2ni}^2 = O(a_n^2 \log^2 n),\ \sum_{i=1}^{n} b_{2ni}^2 = O(a_n^2 \log^2 n),\ \sum_{i=1}^{n} b_{2ni}^2 = O(\log n);$$

$$\max_{1\le i\le n} b_{3ni}^2 = O(n^{-1} b_n^2),\ \sum_{i=1}^{n} b_{3ni}^2 = O(b_n^2),\ \sum_{i=1}^{n} b_{3ni}^2 = O(n^{-1/2} b_n);$$

$$\max_{1\le i\le n} b_{4ni}^2 = O(a_n^2 \log^2 n),\ \sum_{i=1}^{n} b_{4ni}^2 = O(a_n^2 \log^2 n),\ \sum_{i=1}^{n} |b_{4ni}^2| = O(\log n).$$

Choosing $0 < \epsilon \le \tau$, we have by Lemma 3.5 again that

$$P(|\Gamma_n^{-1} I_{2n}| > (n^{\epsilon} a_n^{2\tau+1} \log^{2\tau+2} n + a_n^{\tau+1} \log^{2\tau+2} n)^{1/(3+2\tau)} + 2b_n^{(2+2\tau)/(3+2\tau)} + \psi_{4n} + 2\psi_{5n})$$
$$\le P\left(\left|\sum_{i=1}^{n} b_{2ni}\varepsilon_i\right| > n^{\epsilon} a_n^{2\tau+1} \log^{2\tau+2} n + a_n^{2\tau+1} \log^{2\tau+2} n)^{1/(3+2\tau)} + \psi_{5n}\right)$$
$$+P\left(\left|\sum_{i=1}^{n} b_{3ni}\varepsilon_i\right| > (n^{\epsilon-\tau} b_n^{2+2\tau} + b_n^{2+2\tau})^{1/(3+2\tau)} + \psi_{5n}\right)$$
$$+P\left(\left|\sum_{i=1}^{n} b_{4ni}\varepsilon_i\right| > n^{\epsilon} a_n^{2\tau+1} \log^{2\tau+2} n + a_n^{2\tau+1} \log^{2\tau+2} n)^{1/(3+2\tau)} + \psi_{5n}\right)$$
$$\le C\left(n^{\epsilon} a_n^{2\tau+1} \log^{2\tau+2} n + a_n^{2\tau+1} \log^{2\tau+2} n)^{1/(3+2\tau)} + \psi_{5n}\right)$$

(f) The inequality (f) can be derived immediately by (i) and Markov's inequality.
(g) It follows from the Abel Inequality and Remark 2.1 again that

$$|\Gamma_n^{-1} I_{3n}| = \left|\Gamma_n^{-1}\sum_{i=1}^{n} \tilde{x}\tilde{g}_i\right| = \left|\Gamma_n^{-1}\sum_{i=1}^{n}\left(u_i + \tilde{h}_i - \sum_{s=1}^{n} W_{ns}(t_i)u_s\right)\tilde{g}_i\right|$$
$$\le C\Gamma_n^{-1}(\max_{1\le i\le n} |\tilde{g}_i| \max_{1\le i\le n}\left|\sum_{i=1}^{m} u_{j_i}\right| + n\max_{1\le i\le n} |\tilde{h}_i| \max_{1\le i\le n} |\tilde{g}_i|$$
$$+\max_{1\le i\le n} |\tilde{g}_i| \max_{1\le j\le n}\sum_{i=1}^{n} W_{nj}(t_i)\max_{1\le i\le n}\left|\sum_{i=1}^{m} u_{j_i}\right|)$$
$$\le c_0(b_n \log n + n^{1/2} b_n^2).$$

The prove is completed.

**Lemma 3.7** Let $s_n^2 = \sum_{m=1}^{k} Var(y_n m)$. Then under the assumption of Theorem 3.1, we have

$$|s_n^2 - 1| = O\left(\psi_{1n}^{1/2} + \psi_{2n}^{1/2} + \psi_{3n}^{1/2} + u(q)\right).$$

**Proof.** Similar to Lemma 3.6 in Liang and Fan (2009), we can complete the proof of the lemma. The details are omitted.

Assume that $\{\eta_{nm} : 1 \le m \le k\}$ are independent random variables and $\eta_{nm}$ has the same distribution as that of $y_{nm}$ for each $m = 1, 2, \cdots, k$. Let $T_n = \sum_{m=1}^{k} \eta_{nm}$, then $\sum_{m=1}^{k} Var(\eta_{nm}) = \sum_{m=1}^{k} Var(y_{nm}) = s_n^2$.

**Lemma 3.7** Under the assumptions of Theorem 3.1, we have

$$\sup_u |P(T_n / s_n \le u) - \Phi(u)| = O(\psi_{2n}^{\tau}).$$

**Proof.** We have by Berry-Esseen inequality (Petrov, 1995, p.154, Theorem 5.7) that

$$\sup_u |P(T_n / s_n \le u) - \Phi(u)| \le C \sum_{m=1}^{k} E|\eta_{nm}|^{2+2\tau} / s_n^{2+2\tau}.$$

Applying Lemma 3.3 with $r = 2+2\tau$, $\eta = \delta - 2\tau$ and noting that $E|e_0|^{2+\delta} \le \infty$, we have by choosing $\epsilon \in (0,\tau]$ that

$$\sum_{m=1}^{k} E|\eta_{nm}|^{2+2\tau} = \sum_{m=1}^{k} E|y_{nm}|^{2+2\tau}$$

$$\le C\Gamma_n^{-2-2\tau} \sum_{m=1}^{k} \left\{ p^{\epsilon} \sum_{l=k_m}^{k_m+p-1} \left| \sum_{i=\max(1,l-n)}^{\min(n,n+l)} u_i \phi_{i-l} \right|^{2+2\tau} + \left[ \sum_{l=k_m}^{k_m+p-1} \left| \sum_{i=\max(1,l-n)}^{\min(n,n+l)} u_i \phi_{i-l} \right|^2 \right] \right\}$$

$$\le C\Gamma_n^{-2-2\tau} \sum_{m=1}^{k} (p^{1+\epsilon} + p^{1+\tau}) \le Cn^{-1-\tau} kp^{1+\tau} = O(\psi_{2n}^{\tau}),$$

which together with $s_n^2 \to 1$ from Lemma 3.7 yields that

$$\sup_u |P(T_n / s_n \le u) - \Phi(u)| = O(\psi_{2n}^{\tau}).$$

The prove is completed.

**Lemma 3.8** Under the assumptions of Theorem 3.1, we have

$$\sup_u |P(D_n' \le u) - P(T_n \le u)| = O(\psi_{2n}^{\tau} + (np^{-1}\lambda(q))^{1/4}).$$

**Proof.** Suppose that $\varphi(t)$ and $\Psi(t)$ are characteristic functions of $D_n'$ and $T_n$, respectively.

We have by Esseen inequality (Petrov, 1995, p.146, Theorem 5.3) that for any $T > 0$,

$$\sup_u |P(D_n' \le u) - P(T_n \le u)| \le \int_T^T \left| \frac{\varphi(t) - \Psi(t)}{t} \right| dt + T \int_{|y| \le C/T} |P(T_n \le u+y) - p(T_n \le u)|\, dy$$

$$= A_{1n} + A_{2n}.$$

Applying Lemma 3.4 with $r = s = 2$ and Lemma 3.2 with $p = q = 2+\delta$, and similar to the proof of (3.2), we have by $E|e_0|^{2+\delta} < \infty$ that

$$|\varphi(t) - \Psi(t)| = \left| E\exp\{it \sum_{m=1}^{k} y_{nm}\} - \prod_{m=1}^{k} E\exp\{ity_{nm}\} \right|$$

$$\le C|t| \lambda^{(\frac{1}{2\alpha}) \wedge (\frac{1}{2\beta})}(q) \sum_{l=1}^{k} \| y_{nm} \|_2$$

$$\le C|t| \lambda^{(\frac{1}{2\alpha}) \wedge (\frac{1}{2\beta})}(q) \sum_{l=1}^{k} (Ey_{nm}^2)^2$$

$$\le C|t| \Gamma_n^{-1} \lambda^{(\frac{1}{2\alpha}) \wedge (\frac{1}{2\beta})}(q) \sum_{l=1}^{k} \left[ \sum_{l=k_m}^{k_m+p-1} \left( E \left| \sum_{i=\max(1,l-n)}^{\min(n,n+l)} u_i \phi_{i-l} e_l \right|^{2+\tau} \right)^{2/(2+\delta)} \right]^{1/2}$$

$$\le C|t| n^{-1/2} \lambda^{(\frac{1}{2\alpha}) \wedge (\frac{1}{2\beta})}(q) \sum_{l=1}^{k} \left[ \sum_{l=k_m}^{k_m+p-1} \left( \sum_{i=\max(1,l-n)}^{\min(n,n+l)} u_i \phi_{i-l} e_l \right)^2 \right]^{1/2}$$

$$\le C|t|n^{-1/2}\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)kp^{1/2}$$
$$\le C|t|(np^{-1}\lambda(q))^{1/2},$$

which implies that $A_{1n} \le CT(np^{-1}\lambda(q))^{1/2}$. On the other hand, from Lemma 3.7 and we have

$$\begin{aligned}\sup_u |P(T_n \le u+y)-P(T_n\le u)| &\le \sup_u |P(T_n/s_n \le (u+y)/s_n)-\Phi((u+y)/s_n)| \\ &+\sup_u |P(T_n/s_n\le u/s_n)-\Phi(u/s_n)| \\ &+\sup_u|\Phi((u+y)/s_n)-\Phi(u/s_n)| \\ &\le C(\psi_{2n}^{\tau}+|y|),\end{aligned}$$

which derives that $A_{2n} \le C(\psi_{2n}^{\tau}+1/T)$. By choosing $T=(np^{-1}\lambda(q))^{-1/4}$, we can obtain that

$$\sup_u |P(D_n' \le u)-P(T_n\le u)| = O(\psi_{2n}^{\tau}+(np^{-1}\lambda(q))^{1/4}).$$

This completes the proof of the lemma.

**Proof of Theorem 3.1** We can observe that

$$S_n^2(\hat\beta_n-\beta)=\sum_{i=1}^n \tilde x_i\varepsilon_i-\sum_{i=1}^n\tilde x_i\left(\sum_{i=1}^n W_{nj}(t_i)\varepsilon_j\right)+\sum_{i=1}^n\tilde x_i\tilde g_i := I_{1n}+I_{2n}+I_{3n}.$$

It is easy to show that

$$I_{1n}=\sum_{i=1}^n u_i\varepsilon_i+\sum_{i=1}^n\tilde h_i\varepsilon_i-\sum_{i=1}^n\left(\sum_{i=1}^n W_{nj}(t_i)u_j\right)\varepsilon_j = I_{11n}+I_{12n}+I_{13n},$$

and

$$I_{11n}=\sum_{i=1}^n u_i\sum_{j=-n}^n \Psi_j e_{i-j}+\sum_{i=1}^n u_i\sum_{|j|>n}\Psi_j e_{i-j} := A_{1n}+A_{2n}.$$

By changing the order of summation, we obtain that

$$\Gamma_n^{-1}A_{1n}=\Gamma_n^{-1}\sum_{i=1}^n u_i\sum_{j=-n}^n\Psi_j e_{i-j}=\Gamma_n^{-1}\sum_{l=1-n}^{2n}\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)} u_i\phi_{i-l}\right)e_l=\sum_{l=1-n}^{2n} Z_{nl}.$$

Let $k=\lfloor 3n/(p+q)\rfloor$, $k_m=(m-1)(p+q)+1-n$ and $l_m=k_m+p$, $m=1,2,\cdots,k$. Denote

$$y_{nm}=\sum_{l=k_m}^{k_m+p-1} Z_{nl},\ y_{nm}'=\sum_{l=k_m}^{k_m+p-1} Z_{nl},\ y_{nk}''=\sum_{l=k(p+q)+1-n}^{2n} Z_{nl},$$

$$D_n'=\sum_{m=1}^k y_{nm},\ D_n''=\sum_{m=1}^k y_{nm}'\ \ D_n'''=y_{nk}''.$$

The we have that

$$\frac{S_n^2(\hat\beta_n-\beta)}{\Gamma_n}=D_n'+D_n''+D_n'''+\Gamma_n^{-1}(A_{2n}+I_{12n}+I_{13n}+I_{2n}+I_{3n}).$$

It follows from Lemma 3.7-3.9 that

$$\sup_n |P(D_n^{'} \le u) - \Phi(u)| \le \sup_u |P(D_n^{'} \le u) - P(T_n \le u)|$$
$$+\sup_u |P(T_n \le u) - \Phi(u/s_n)| + \sup_u |\Phi(u/s_n) - \Phi(u)|$$
$$\le C[\psi_{2n}^{\tau} + (np^{-1}\lambda(q))^{1/4} + \psi_{2n}^{\tau} + |s_n^2 - 1|/s_n^2]$$
$$\le C[\psi_{2n}^{\tau} + (np^{-1}\lambda(q))^{1/4} + \psi_{1n}^{\tau} + \psi_{2n}^{\tau} + \psi_{3n}^{\tau} + u(q)].$$

Consequently, from (3.1), Lemma 3.1 and Lemma 3.6, we have

$$\sup_u \left| P\left( \frac{S_n^2(\beta_n - \beta)}{\Gamma_n} \le u \right) - \Phi(u) \right|$$
$$= \sup_u |P(D_n^{'} + D_n^{''} + D_n^{'''} + \Gamma_n^{-1}(A_{2n} + I_{12n} + I_{13n} + I_{2n} + I_{3n}) \le u) - \Phi(u)|$$
$$\le \sup_u |P(D_n^{'} \le u) - \Phi(u)|$$
$$+\frac{1}{\sqrt{2\pi}}\{(n^{\epsilon-\tau}\psi_{1n} + \psi_{1n}^{1+\tau})^{1/(3+2\tau)} + (n^{\epsilon-\tau}\psi_{2n} + \psi_{2n}^{1+\tau})^{1/(3+2\tau)}) + 4b_n^{(2+2\tau)/(3+2\tau)}$$
$$+3(n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2} n + a_n^{2\tau+1}\log^{2\tau+2} n)^{1/(3+2\tau)} + \psi_{3n}^{1/3} + \psi_{4n} + 3\psi_{5n} + c_0(b_n \log n + n^{1/2}b_n^2)\}$$
$$+P(|D_n^{''} + D_n^{'''} + \Gamma_n^{-1}(A_{2n} + I_{12n} + I_{13n} + I_{2n} + I_{3n})|$$
$$> (n^{\epsilon-\tau}\psi_{1n} + \psi_{1n}^{1+\tau})^{1/(3+2\tau)} + (n^{\epsilon-\tau}\psi_{2n} + \psi_{2n}^{1+\tau})^{1/(3+2\tau)} + 4b_n^{(2+2\tau)/(3+2\tau)}$$
$$+3(n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2} n + a_n^{2\tau+1}\log^{2\tau+2} n)^{1/(3+2\tau)} + \psi_{3n}^{1/3} + \psi_{4n} + 3\psi_{5n} + c_0(b_n \log n + n^{1/2}b_n^2))$$
$$\le C(n^{\epsilon-\tau}\psi_{1n} + \psi_{1n}^{1+\tau})^{1/(3+2\tau)} + \psi_{1n}^{1/2} + (n^{\epsilon-\tau}\psi_{2n} + \psi_{2n}^{1+\tau})^{1/(3+2\tau)}) + \psi_{2n}^{1/2} + \psi_{2n}^{\tau} + \psi_{3n}^{1/3} + u(q)$$
$$+(np^{-1}\lambda(q))^{1/4} + b_n^{(2+2\tau)/(3+2\tau)} + (n^{\epsilon}a_n^{2\tau+1}\log^{2\tau+2} n + a_n^{2\tau+1}\log^{2\tau+2} n)^{1/(3+2\tau)} + b_n \log n + n^{1/2}b_n^2)$$
$$= O(\mu_{1n} + \mu_{2n}).$$

The proof of the theorem is completed.

To prove Theorem 2.2, we need the following lemmas.

**Lemma 3.8** Under the assumption of Theorem 2.2, the following statements hold.

(i) $E(J_{11n}^{''})^2 \le C\gamma_{1n}, E(J_{11n})^2 \le C(\gamma_{2n})^2 \le C\gamma_{2n}.$

(ii) $P(|J_{11n}^{''}| > (n^{\epsilon}\gamma_{1n}a_n^{\tau} + \gamma_{1n}^{1+\tau})^{1/(3+2)}) \le C(n^{\epsilon}\gamma_{1n}a_n^{\tau} + \gamma_{1n}^{1+\tau})^{1/(3+2)}$ *for any* $\epsilon > 0,$

and

$P(|J_{11n}^{'''}| > (n^{\epsilon}\gamma_{2n}a_n^{\tau} + \gamma_{2n}^{1+\tau})^{1/(3+2)}) \le C(n^{\epsilon}\gamma_{2n}a_n^{\tau} + \gamma_{2n}^{1+\tau})^{1/(3+2)}$ *for any* $\epsilon > 0.$

(iii) Let $\gamma_{4n} := n^{-1}\Theta_n^{-1}(t) + a_n^{1/2}\log n.$ Then

$|J_{3n}| \le c\gamma_{4n}; P(|J_{2n}| > \gamma_{4n}^{(2+\delta)/(3+\delta)}) \le C\gamma_{4n}^{(2+\delta)/(3+\delta)}.$

Furthermore, if $|\sum_{i=1}^{n} W_{ni}(t)x_i| = O(\varphi_n)$ for some positive numbers $\varphi_n$, then

$|J_{3n}| \le C(n^{-1}\Theta_n^{-1}(t)\varphi_n) := C\gamma_{4n}^{'},\ P(|J_{2n}| > (\gamma_{4n}^{'})^{(2+\delta)/(3+\delta)} \le C(\gamma_{4n}^{'})^{(2+\delta)/(3+\delta)}.$

**Proof.** (i) Similar to the proof of (3.2), we have by $\omega_n(t) = O(\Theta_n^2(t)),$ $(A_3)$ (ii) and $E|e_0|^{2+\delta} < \infty$ that

$$E(J_{11n}^2 = E\left\{\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q-1}\left(\Theta_n^2(t)\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right)e_l\right\}^2$$
$$\leq C\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q-1}\Theta_n^2(t)\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right)^2$$
$$\leq Ckq\omega_n(t) \leq Cnqp^{-1}a_n = C\gamma_{1n}.$$

Similarly,

$$E(J_{11n}^{'''})^2 \leq C\sum_{l=k(p+q)+1-n}^{2n}\Theta_n^2(t)\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right)^2 \leq Cpa_n = C\gamma_{2n}.$$

Noting that $\sum_{i=1}^{n} W_{ni}(t) = 1,$ analogous to the proof of (3.3) we have $E(J_{12n})^2 \leq \Theta_n^2(t)(\sum_{|j>n}|\Psi_j|)^2 = \gamma_{3n}$

(ii) Noting that

$$\sum_{m=1}^{k}\sum_{l=l_m}^{l_m+q-1}\left(\Theta_n^2(t)\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right)^2 \leq Ckqp^{-1}a_n = O(\gamma_{1n})$$

and

$$\max_l\left(\Theta_n^2(t)\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right)^2 = O(a_n),$$

the first inequality follows immediately form Lemma 3.5.

Similarly, noting that

$$\sum_{m=1}^{k}\sum_{l=k(p+q)+1-n}^{2n}\left(\Theta_n^2(t)\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right)^2 \leq Cp\omega_n(t) \leq Cpa_n = 0(\gamma_{2n}),$$

we can also get the second one from Lemma 3.5.

(iii) Observe that $a_n(\log n)^2 \to 0$ since $\mu_{2n} \to 0$ in Theorem 2.1, the proof of (iii) can be easily obtained by following the proof of Lemma 3.9 in Liang and Fan (2009). The details are omitted here.

**Lemma 3.11.** Let $s_{1n}^2 = \sum_{m=1}^{k} Var(\chi_{nm}).$ Then under the assumptions of Theorem 2.2, we have

$$|s_{1n}^2 - 1| = O(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} + u(q)).$$

**Proof.** Similar to the proof of Lemma 3.10 in Liang and Fan (2009), we can prove the lemma.
The details are omitted.

Assume that $\{k_{nm} : 1 \leq m \leq k\}$ are independent random variables and $k_{nm}$ has the same distribution as that of $\chi_{nm}$ for each $m = 1, 2, \cdots, k.$ Let $T_{1n} = \sum_{m=1}^{k}\kappa_{nm},$ then $\sum_{m=1}^{k} Var(\kappa_{nm}) = \sum_{m=1}^{k} Var(\kappa_{nm}) = s_{1n}^2.$

**Lemma 3.12** Under the assumptions of Theorem 2.2, we have

$\sup_u |P(T_{1n} / s_{1n} \leq u) - \Phi(u)| = O(\gamma_{2n}^{\tau}).$

**Proof.** Note that $\omega_n(t) = O(\Theta_n^2(t)),$ $E|e_0|^{2+\delta} < \infty$ and $\sum_{i=1}^{n} W_{ni}(t) = 1.$ Similar to the proof of Lemma 3.8, we have by Lemma 3.3, $(\sum_i W_{ni}^{1/2}(t)|\Psi_{i-l}|)^2 \leq \sum_i W_{ni}(t)|\Psi_{i-l}|\sum_i|\Psi_{i-l}|$ and changing the order of summation that

$$\sum_{m=1}^{k} E|\kappa_{nm}|^{2+2\tau} = \sum_{m=1}^{k} E|\chi_{nm}|^{2+2\tau}$$

$$\le C\Theta_n^{-2-2\tau}(t)\sum_{m=1}^{k}\left\{p^{\epsilon}\sum_{l=l_m}^{l_m+q-1}\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right|^{2+2\tau}+\left[\sum_{l=l_m}^{l_m+q-1}\left|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1}\right|^{2}\right]^{1+\tau}\right\}$$

$$\le C\sum_{n=1}^{k}\left\{p^{\epsilon}\omega_n^{\tau}(t)\sum_{l=l_m}^{l_m+q-1}|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1})^2+(p\omega_n(t))^{\tau}\sum_{l=l_m}^{l_m+q-1}|\sum_{i=\max(1,l-n)}^{\min(n,n+l)} W_{ni}(t)\Psi_{i-1})^2\right\}$$

$$\le (p\omega_n(t))^{\tau}\sum_{i=1}^{n}W_{ni}(t)\left(\sum_{j=-\infty}^{\infty}|\Psi_j|\right)^2 \le C(pa_n)^{\tau}=O(\gamma_{2n}^{\tau}),$$

which together with $s_{1n}^2 \to 1$ from Lemma 3.11 yields that

$$\sup_u |P(T_{1n}/s_{1n}\le u)-\Phi(u)| = O(\gamma_{2n}^{\tau}).$$

The proof is completed.

**Lemma 3.13** Under the assumptions of Theorem 2.2, we have

$$\sup_u |P(J_{11n}'\le u)-P(T_{1n}\le u)| = O(\gamma_{2n}^{\tau}+(n^2p^{-1}a_n\lambda(q))^{1/4}).$$

**Proof.** Suppose that $\varphi_1(t)$ and $\Psi_1(t)$ are the characteristic functions of $J_{11n}'$ and $T_{1n}$, respectively. Similar to the proof of Lemma 3.9, we have

$$|\varphi(t)-\Psi(t)| = \left|E\exp\{it\sum_{m=1}^{k}\chi_{nm}\}-\prod_{m=1}^{k}E\exp\{it\chi_{nm}\}\right|$$

$$\le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{l=1}^{k}\|\chi_{nm}\|_2$$

$$\le C|t|\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{l=1}^{k}(E\chi_{nm}^2)^2$$

$$\le C|t|\Theta_n^{-1}\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)\sum_{l=1}^{k}\left[\sum_{l=k_m}^{k_m+p-1}\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-l}e_l\right)^2\right]^{1/2}$$

$$\le Ck|t|p^{1/2}\omega^{1/2}\lambda^{(\frac{1}{2\alpha})\wedge(\frac{1}{2\beta})}(q)$$

$$\le C|t|(n^2p^{-1}a_n\lambda(q))^{1/2},$$

together with Lemma 3.11 and 3.12 yield that

$$\sup_u |P(J_{11n}'\le u)-P(T_{1n}\le u)| = O(\gamma_{2n}^{\tau}+(n^2p^{-1}a_n\lambda(q))^{1/4}).$$

This completes the proof of the lemma.

**Proof of Theorem 3.1** We have that

$$\frac{\hat{g}_n(t)-E\tilde{g}_n(t)}{\Theta_n(t)} = \Theta_n^{-1}(t)\left(\sum_{i=1}^{n}W_{ni}(t)\varepsilon_i + \sum_{i=1}^{n}W_{ni}(t)x_i(\beta-\hat{\beta}_n) + \sum_{i=1}^{n}W_{ni}(t)x_i(E\hat{\beta}_n-\beta)\right)$$
$$= J_{1n}+J_{2n}+J_{3n}.$$

Note that

$$J_{1n} = \Theta_n^{-1}(t)\sum_{i=1}^{n}W_{ni}(t)\left(\sum_{j=-n}^{n}\Psi_j e_{i-j} + \sum_{|j|>n}\Psi_j e_{i-j}\right) := J_{11n}+J_{12n},$$

and

$$J_{11n} = \Theta_n^{-1}(t)\sum_{l=1-n}^{2n}\left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)}u_i\phi_{i-l}\right)^2 e_l = \sum_{l=1-n}^{2n}\varsigma_{nl}.$$

Similar to the decomposition for $\Gamma_n^{-1}A_{1n}$, we denote

$$\chi_{nm} = \sum_{l=k_m}^{k_m+p-1}\varsigma_{nl},\ \chi_{nm}^{'} = \sum_{l=l_m}^{k_m+p-1}\varsigma_{nl},\ \chi_{nk}^{''} = \sum_{l=k(p+q)+1-n}^{2n}\varsigma_{nl},$$

$$J_{11n}^{'} = \sum_{m=1}^{k}\chi_{nm},\ J_{11n}^{''} = \sum_{m=1}^{k}\chi_{nm}^{'}\ \ J_{11n}^{'''} = \chi_{nk}^{''}.$$

Then we have that

$$\frac{\hat{g}_n(t)-E\tilde{g}_n(t)}{\Theta_n(t)} = J_{11n}^{'}+J_{11n}^{''}+J_{11n}^{'''}+J_{12n}+J_{2n}+J_{3n}.$$

we have by Lemma 3.11-3.13 that

$$\sup_n |P(J_{11n}^{'}\le u)-\Phi(u)| \le C[\gamma_{2n}^{\tau}+(n^2p^{-1}a_n\lambda(q))^{1/4}+\gamma_{1n}^{\tau}+\gamma_{2n}^{\tau}+\gamma_{3n}^{\tau}+u(q)].$$

which together with (3.4), Lemma 3.1 and Lemma 3.10 yields that

$$\sup_u\left|P\left(\frac{\hat{g}_n(t)-E\tilde{g}_n(t)}{\Theta_n(t)}\le u\right)-\Phi(u)\right|$$
$$\le \sup_u|P(J_{11n}^{'}\le u)-\Phi(u)| + \frac{1}{\sqrt{2\pi}}\{(n^{\epsilon}\gamma_{1n}a_n^{\tau}+\gamma_{1n}^{1+\tau})^{1/(3+2\tau)}$$
$$+(n^{\epsilon}\gamma_{2n}a_n^{\tau}+\gamma_{2n}^{1+\tau})^{1/(3+2\tau)}+\gamma_{3n}^{1/3}+c\gamma_{4n}+\gamma_{4n}^{(2+2\tau)/(3+2\tau)}\}+P(|J_{11n}^{''}|>(n^{\epsilon}\gamma_{1n}a_n^{\tau}+\gamma_{1n}^{1+\tau})^{1/(3+2\tau)})$$
$$+P(|J_{11n}^{'''}|>(n^{\epsilon}\gamma_{2n}a_n^{\tau}+\gamma_{2n}^{1+\tau})^{1/(3+2\tau)})+P(|J_{12n}|>\gamma_{3n}^{1/3})+P(|J_{2n}|>\gamma_{4n}^{(2+\delta)/(3+\delta)})$$
$$\le C(\gamma_{1n}^{1/2}+(n^{\epsilon}\gamma_{1n}a_n^{\tau}+\gamma_{1n}^{1+\tau})^{1/(3+2\tau)}+\gamma_{2n}^{\tau}+\gamma_{2n}^{1/2}+(n^{\epsilon}\gamma_{2n}a_n^{\tau}+\gamma_{2n}^{1+\tau})^{1/(3+2\tau)}$$
$$+\gamma_{3n}^{1/3}+u(q)+(n^2p^{-1}a_n\lambda(q))^{1/4}+\gamma_{4n}^{(2+\delta)/(3+\delta)})$$
$$=O(\mu_{1n}+\mu_{2n}).$$

The proof is completed.

## REFERENCES


[1] Amini M, Roozbeh M., 2016. Least trimmed squares ridge estimation in partially linear regression models. J Stat Comput Simul. 86(14): 2766-2780.

[2] Amini M, Roozbeh M., 2019. Improving the prediction performance of the LASSO by subtracting the additive structural noises. Comput Stat.34(1):415-432.

[3] Bradley R.C, Bryc W., 1985. Multilinear forms and measures of dependence between random variables. Journal of Multivariate Analysis, 16, 335-367.

[4] Cai Z.W., 1991. Strong consistency and rates for recursive nonparametric conditional probability density estimates under ($\alpha$, $\beta$)-mixing conditions. Stochastic Processes and

Their Applications, 38, 323-333.

[5] Carroll R.J., 1982. Adapting for heteroscedasticity in linear models. Annals of Statistics, 10(4), 1224-1232.

[6] Carroll R.J., Härdle W., 1989. Second order effects in semiparametric weighted least squares regression. Statistics, 2, 179-186.

[7] Chen H., 1988. Convergence rates for parametric components in a partial linear model. Annals of Statistics, 16, 136-146.

[8] Chen M.H., Ren, Z., Hu S.H., 1998. Strong consistency of a class of estimators in partial linear model. Acta Mathematica Sinica, 41(2), 429-439.

[9] Engle R.F, Granger C.W.J., Rice J., Weiss G.H., 1986. Semiparametric estimates of the relation between weather and electricity sales. Journal of the American Statistical Association, 81(394), 310-320.

[10] Fan J., Gijbels I., 1996. Local Polynomial Modelling and Its Applications. Chapman and Hall, London.

[11] Gao P., 2016. Strong stability of ($\alpha$, $\beta$)-mixing sequences. Applied Mathematics-A Journal of Chinese Uni- versities, Series B, 31(4), 405-412.

[12] Gao J.T., Chen X.R., Zhao L.C., 1994. Asymptotic normality of a class of estimators in partial linear models. Acta Mathematica Sinica, 37(2), 256-268.

[13] Härdle W., Liang H., Gao J., 2000. Partially linear Models. Springer Verlag.

[14] Heckman N.E., 1986. Spline smoothing in a partly linear model. Journal of the Royal Statistical Society, Series B, Statistical Methodology, 48, 244-248.

[15] Horowitz JL., 2009. Semiparametric and nonparametric methods in econometrics: Springer series in statistics. New York: Springer-Verlag.

[16] Liang H., Härdle W., 1997. Asymptotic properties of parametric estimation in partially linear heteroscedastic models. Technical Report no 33. Humboldt-Universität zu Berlin.

[17] Liang H.Y., Jing B.Y., 2009. Asymptotic normality in partially linear models based on dependent errors. Journal of Statistical Planning and Inference, 139, 1357-1371.

[18] Lu C.R., Lin Z.Y., 1997. Limit theory for mixed dependent variables. Science Press of China, Beijing.

[19] Liu X, Wang L, Liang H., 2011. Variable selection and estimation for semiparametric additive partial linear models. Stat Sin. 21:1225-1248.

[20] Opsomer JD, Ruppert D., 1999. A root-n consistent backfitting estimator for semiparametric additive mod- eling. J Comput Graph Stat. 8:715-732.

[21] Rice J., 1986. Convergence rates for partially linear spline models. Statistics and Probability Letters, 4, 203-208.

[22] Robinson P.M., 1987. Asymptotically efficiency estimation in the presence of heteroscedasticity of unknown form. Econometrica, 55, 875-891.

[23] Roozbeh M, Arashi M, Gasparini M., 2012. Seemingly unrelated ridge regression in semiparametric models. Commun Stat Theory Methods. 41(8):1364-1386

[24] Roozbeh M, Arashi M., 2016. Shrinkage ridge regression in partial linear models. Commun Stat Theory Methods. 45(20): 6022-6044.

[25] Roozbeh M., 2018. Optimal QR-based estimation in partially linear regression models with correlated errors using GCV criterion. Comput Stat Data Anal. 117:45-61.

[26] Schick A., 1996. Root-*n* consistent estimation in partly linear regression models. Statistics and Probability Letters, 28, 353-358.

[27] Speckman P., 1988. Kernel smoothing in partial linear models. Journal of the Royal Statistical Society, Series B, Statistical Methodology, 50, 413-436.

[28] Shao Q.M., 1989. Limit theorems for the partial sums of dependent and independent random variables. University of Science and Technology of China, 1-309, Hefei.

[29] Shen Y., Zhang Y.J., 2011. Strong limit theorems for ($\alpha$, $\beta$)-mixing random variable sequences. Journal of University of Science and Technology of China, 41(9), 778-795.

[30] You J., Chen G., 2007. Semiparametric generalized least squares estimation in partially linear regression models with correlated errors. Journal of Statistical Planning and Inference, 137, 117-132.

[31] Yu C.Q., 2016. Convergence theorems of weighted sum for ($\alpha$, $\beta$)-mixing sequences. Journal of Hubei Uni- versity (Natural Science), 38(6), 477-487.