

Interval Estimation for Burr Type-XII Model Based on the Generalized Order Statistics

M. Maswadah, Seham M.*

Department of Mathematics, Faculty of Science, Aswan University, Egypt

Abstract In this paper, the conditional inference was applied to study some statistical properties for the confidence intervals of the Burr type-XII distribution parameters based on generalized order statistics. To measure the performance of this approach compared to the Asymptotic maximum likelihood estimates, simulation studies were performed for different values of sample sizes and shape parameters. The simulation results indicated that the conditional intervals possess good statistical properties and can perform well even when the sample size is small compared to the classical intervals. Finally, real data sets are given to illustrate the confidence intervals that were developed in this paper.

Keywords Asymptotic maximum likelihood estimates, Covering percentage, Conditional inference, Progressive type-II censored samples with binomial random removals

1. Introduction

The Burr type-XII distribution has been extensively studied in the literature as an appropriate and useful failure model for the applied statistics. Several studies were performed on Burr type-XII distribution, which includes Evans and Simon (1975) who discussed the Burr distribution as a failure model and derived the maximum likelihood estimators (MLEs) for the parameters. Evans and Ragab (1983) discussed the Bayesian estimators for Burr distribution parameters based on type-II censored samples. Wu and Yu (2005) proposed some pivotal quantities to test shape parameters and established their confidence intervals based on the censored data. Soliman (2005) derived the MLE and Bayesian estimates of the parameters and some lifetime parameters. Xiuchun et al. (2007) derived the empirical Bayesian estimator for the reliability based on progressively type-II censored samples, and Liang and Yimin (2010) derived the empirical Bayesian estimators based on record values. For a comprehensive review of the Burr type-XII distribution, see Rodriguez (1977).

However, in recent years, this distribution has been used in a variety of fields such as business, see Rodriguez and Taniguchi (1980). Economics and Finance see, McDonald and Richards (1987). Hydrology, see Mielke and Johanson (1974). Quality assurance, see Burr (1976). Medicine see Wingo (1983) and Mineralogy, see Cook and Johnson (1986). Wingo (1993) derived the MLEs of the parameters

under type-II censoring. Wingo (1993) derived the MLEs for fitting the Burr type-XII model based on multiply censored data. Zimmer et al. (1998) studied the reliability analysis of the Burr type-XII distribution. Wu and Yu (2005) proposed pivotal quantities to test the shape parameter and establish confidence interval of the shape parameter for the Burr type-XII distribution under the failure-censored plan. Malinawska et al. (2006) derived the estimation parameters of the Burr type-XII model using generalized order statistics. Lio et al. (2007) derived the parameter estimates based on the progressive type-II censoring. Li et al. (2007) proposed the empirical Bayes estimators of reliability performances for the Burr type-XII model using LINEX loss function based on progressively type-II censored samples. Wu et al. (2007) derived the MLE estimates for the Burr type-XII distribution parameters. Asgharzadeh and Valiollahi (2008) derived the parameter estimates based on progressive type-II censored data. Wing (2008) studied the statistical inference of the Burr type-XII distribution. Lee et al. (2009) obtained Bayes and empirical Bayes estimators for reliability. Asgharzadeh and Abdi (2012) derived the confidence intervals of the Burr type-XII model parameters based on records. Soliman et al. (2012) studied the Bayesian statistical inference and prediction for the Burr type-XII model based on the progressive first inference censored samples. Kumar (2017) presented some statistical properties for this distribution. Amal et al. (2018) derived the Bayesian estimate of the reliability based on a new numerical loss function. Hakim et al. (2021) presented some characteristics of this distribution and its application to heavy tailed survival time data. Thus, many aspects of the distribution have been covered by the researchers, but there is still a lack of comprehensive statistical analysis of the distribution. In this paper, the

* Corresponding author:

seham_mohamed39@yahoo.com (Seham M.)

Received: Apr. 8, 2021; Accepted: May 12, 2021; Published: Jun. 15, 2021

Published online at <http://journal.sapub.org/ijps>

conditional inference was applied to construct confidence intervals for the Burr type-XII model parameters based on generalized order statistics. However, the conditional inference as suggested by Sir Fisher (1934) has been applied to many lifetime distributions belonging to the location-scale family, see Lawless (1973-1982) or those that can be transferred to this family, see Maswadah (2003, 2005), and Maswadah and EL-Faheem (2018). Thus, as a new application of the conditional approach, the conditional confidence intervals were generated for the Burr Type-XII model parameters based on generalized order statistics.

The Burr type-XII model was considered by Burr (1942), as a new lifetime distribution with the cumulative distribution function (cdf) and the probability density function (pdf), which are presented respectively by:

$$F(x; \alpha, \beta) = 1 - (1 + x^\alpha)^{-\beta}, \quad \alpha, \beta > 0, x > 0, \quad (1.1)$$

$$f(x; \alpha, \beta) = \alpha \beta x^{\alpha-1} (1 + x^\alpha)^{-\beta-1}, \quad \alpha, \beta > 0, x > 0, \quad (1.2)$$

α and β are two shape parameters.

Thus, for the significance of this distribution, the confidence intervals were derived based on the conditional and the classical inferences based on generalized order statistics (GOS) that introduced by Kamps (1995) as a unified approach to ordinary OS, type-II progressive order statistics, record values and k-th record values.

Definition.

Let $n \in \mathbb{N}$, $K > 0$, $\tilde{R} = (R_1, R_2, \dots, R_{n-1}) \in \mathbb{R}^{n-1}$ be parameters such that

$$\gamma_i = k + n - i + M_i > 0, \quad M_i = \sum_{j=i}^{n-1} R_j, \quad \text{for all } i \in \{1, 2, 3, \dots, n\}.$$

The random variables $X(1, n, \tilde{R}, k), \dots, X(n, n, \tilde{R}, k)$ are called GOS, with noting that $X(0, n, \tilde{R}, k) = 0$, if their joint pdf can be written in the form:

$$f(x_1, x_2, \dots, x_n) = C \prod_{i=1}^{n-1} f(x_i) [1 - F(x_i)]^{R_i} [1 - F(x_n)]^{k-1} f(x_n), \quad (1.3)$$

On the cone $F^{-1}(0) < x_1 < \dots < x_n < F^{-1}(1)$ of \mathbb{R}^n ,

where $C = \prod_{i=1}^n \gamma_i$, $\gamma_n = k > 0$.

2. Conditional Inference Methodology

For the first time, in this section, we will provide an outline for the conditional inference to the Burr Type-XII distribution based on the generalized order statistics.

Given a set of n GOS $X(1, n, \tilde{R}, k), \dots, X(n, n, \tilde{R}, k)$ with sampling density function (1.3), thus by substituting (1.1) and (1.2) in (1.3) we can derive the joint pdf as follows:

$$f(x_1, \dots, x_n) = C \alpha^n \beta^n \prod_{i=1}^n x_i^{\alpha-1} \left[\exp \left[-\beta \left(\sum_{i=1}^n (1 + R_i) \ln(1 + x_i^\alpha) + (k - R_n - 1) \ln(1 + x_n^\alpha) \right) \right] \right]. \quad (2.1)$$

For (1.1), if $\hat{\alpha}$ and $\hat{\beta}$ be any equivariant estimators such as the MLEs for α and β , then $Z_1 = \alpha / \hat{\alpha}$ and $Z_2 = \beta^{1/z_1} / \hat{\beta}$ are pivotal quantities and $a_i = \hat{\beta} [\ln(1 + x_i^\alpha)]^{\hat{\alpha}} = \hat{\beta} g^{\hat{\alpha}}(x_i)$, $i = 1, 2, \dots, n$ form a set of ancillary statistics. With noting that

$$\beta g^\alpha(x_i) = \left(g^{\hat{\alpha}}(x_i) \hat{\beta} \cdot \frac{\beta^{\hat{\alpha}/\alpha}}{\hat{\beta}} \right)^{\alpha/\hat{\alpha}} = (a_i z_2)^{Z_1}.$$

Thus, based on the following theorem, we can derive the conditional densities of the pivotal quantities conditional on the ancillary statistics and the confidence intervals can be constructed and converted to α and β fiducially.

Theorem:

Let $\hat{\alpha}$ and $\hat{\beta}$ be any equivariant estimators of α and β , based on the generalized order statistics $X(1, n, \tilde{R}, k), \dots, X(n, n, \tilde{R}, k)$. Then the conditional pdf of Z_1 and Z_2 given $A = (a_1, a_2, \dots, a_{n-2})$ can be derived in the form

$$g(z_1, z_2 | A) = D \cdot z_1^{n-1} z_2^{nz_1-1} \prod_{i=1}^n a_i^{z_1-1} \exp(-z_2^{z_1} U), \quad (2.2)$$

D is a normalizing constant depends on A only, and $U = \sum_{i=1}^n (1 + R_i) a_i^{z_1} + (k - R_n - 1) a_n^{z_1}$.

Proof: see Maswadah (2003).

3. Confidence Intervals Procedures

3.1. Conditional Confidence Intervals

The marginal density of Z_1 and the distribution function of Z_2 can be derived from (2.2) respectively as:

$$g_1^*(z_1 | A) = D \Gamma(n) z_1^{n-2} \prod_{i=1}^n a_i^{z_1-1} U^{-n}, \quad (3.1)$$

$$G_{z_2}^*(t | A) = D \Gamma(n) \int_0^\infty z_1^{n-2} \prod_{i=1}^n a_i^{z_1-1} U^{-n} \left(1 - \exp(-t^{z_1} U) \sum_{j=0}^{n-1} \frac{(t^{z_1} U)^j}{j!} \right) dz_1. \quad (3.2)$$

D is a normalizing constant does not depend on Z_1 and Z_2 , that can be derived as:

$$D^{-1} = \Gamma(n) \int_0^\infty z_1^{n-2} \prod_{i=1}^n a_i^{z_1-1} U^{-n} dz_1.$$

To obtain confidence intervals for α (say), from (3.1) the probability statement for Z_1 can be obtained as $P(L \leq Z_1 \leq R) = 1 - \gamma$, which is the $100(1 - \gamma)\%$ confidence interval for Z_1 and then transformed fiducially to α as $P(\hat{\alpha}L \leq \alpha \leq \hat{\alpha}R) = 1 - \gamma$. This interval is not unique, and therefore using the tails of symmetric probability, the lower (L) and upper (R) limits of such an interval are solutions of $P(0 \leq Z_1 \leq L) = \gamma/2$ and $P(0 \leq Z_1 \leq R) = 1 - \gamma/2$ respectively. Similarly, the confidence interval for β can be constructed from (3.2).

3.2. Asymptotic Confidence Intervals

In this subsection, we obtained the Fisher information matrix to compute 95% asymptotic confidence intervals for the Burr type-XII distribution parameters based on maximum likelihood estimators (MLEs). The Fisher information matrix can be obtained by using loglikelihood function of (2.1). Thus, we have

$$I(\hat{\alpha}, \hat{\beta}) = \begin{bmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\beta}) \\ \text{cov}(\hat{\beta}, \hat{\alpha}) & \text{var}(\hat{\beta}) \end{bmatrix} \equiv \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} \\ I_{\beta\alpha} & I_{\beta\beta} \end{bmatrix}_{(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})}^{-1}.$$

Where,

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \frac{x_i^\alpha \ln x_i}{1 + x_i^\alpha} - \beta \left[\sum_{i=1}^m (1 + R_i) \frac{x_i^\alpha \ln x_i}{1 + x_i^\alpha} + (k - R_n - 1) \frac{x_n^\alpha \ln x_n}{1 + x_n^\alpha} \right],$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n (1 + R_i) \ln(1 + x_i^\alpha) - (k - R_n - 1) \ln(1 + x_n^\alpha),$$

$$I_{\alpha\alpha} = \frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^n x_i^\alpha (\ln x_i)^2 - \sum_{i=1}^m \frac{(1 + x_i^\alpha) x_i^\alpha (\ln x_i)^2 - (x_i^\alpha \ln x_i)^2}{(1 + x_i^\alpha)^2} - \beta \left[\sum_{i=1}^m (1 + R_i) \frac{(1 + x_i^\alpha) x_i^\alpha (\ln x_i)^2 - (x_i^\alpha \ln x_i)^2}{(1 + x_i^\alpha)^2} + (k - R_n - 1) \frac{(1 + x_n^\alpha) x_n^\alpha (\ln x_n)^2 - (x_n^\alpha \ln x_n)^2}{(1 + x_n^\alpha)^2} \right],$$

$$I_{\beta\beta} = \frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n}{\beta^2},$$

$$I_{\alpha\beta} = \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = -\sum_{i=1}^n (1 + R_i) \frac{x_i^\alpha \ln x_i}{1 + x_i^\alpha} - (k - R_n - 1) \frac{x_n^\alpha \ln x_n}{1 + x_n^\alpha}.$$

Thus, the approximate $100(1 - \gamma)\%$ two sided confidence intervals for α and β can be obtained respectively by

$$\hat{\alpha} \pm Z_{\gamma/2} \sigma_{\hat{\alpha}} \text{ and } \hat{\beta} \pm Z_{\gamma/2} \sigma_{\hat{\beta}},$$

where $Z_{\gamma/2}$ is the upper $\gamma/2$ -th percentile of a standard normal distribution, $\sigma_{\hat{\alpha}}$ and $\sigma_{\hat{\beta}}$ are the standard deviations of the MLEs of the parameters α and β respectively, where they are elements of the following AVC matrix:

$$AVC = \begin{bmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\beta}) \\ \text{cov}(\hat{\beta}, \hat{\alpha}) & \text{var}(\hat{\beta}) \end{bmatrix} \equiv \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} \\ I_{\beta\alpha} & I_{\beta\beta} \end{bmatrix}_{(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})}^{-1}.$$

4. Simulation Studies

In this section we mainly present some results based on the Monte Carlo simulation, to measure the performance of the conditional confidence intervals compared to the AMLE intervals in terms of the following criteria:

- 1- Covering percentage (CP).
- 2- Mean length of intervals ($MLIs$).

Comparative results, based on 1000 Monte Carlo simulation trials are given for sample sizes $n = 20, 40, 60, 80$ and 100 with censoring levels 0.0%, 0.25% and 0.50%, that have been generated from the Burr type-XII model for shape parameter values $\alpha = 0.5, 1, 2$ and the parameter $\beta = 2$ and 3. For the progressive type-II censoring sampling that are carried out with binomial random removals with probability $P = 0.5$, which means the number of units removed at each failure time follows a binomial distribution with probability P , where different values of P do not affect the calculations.

From the simulation results that reported in Tables 2 to 7, we can summarize the following main points:

- i. It is worthwhile to note that for different values of α , the CPs are the same for the pivotal z_1 as expected because its distribution is independent from the parameter α . However, the values of MLI for the parameter α increases when α increases. On the contrary the values of the CPs and $MLIs$ for the pivotal z_2 are the same for all values of α as expected.
- ii. Generally, the values of $MLIs$ decrease, the CPs almost getting increase as the sample size increases for both parameters α and β .
- iii. For the parameter α , the values of $MLIs$ and CPs decrease and the values of $SDEs$ getting increase for increasing the value of β , but it is not the case for highly type-II censoring, where the values of MLI getting increase. On the contrary, for the parameter β , the values of MLI and CPs increase for increasing β as expected.
- iv. The values of $MLIs$ for α based on the conditional inference are smaller than those based on the AMLEs inference, in spite of they have almost higher CPs based on complete and censored samples. However

the values of $MLIs$ for β based on the AMLEs inference are almost equal for two decimal places to those based on the conditional inference, but it is not the case for highly type-II censored samples and small sample sizes.

- v. Generally, the results based on the type-II progressive censored samples are better than those based on the type-II censored samples, in which they have smaller $MLIs$ and higher CPs .

Thus the simulation results indicated that the conditional confidence intervals possess good statistical properties and they can perform quite well even when the sample size is extremely small. However, the AMLE approach turns out to be imprecise or even unreliable for small or highly type-II censored samples.

5. Real Data Analysis

5.1. Application Data for Joint Patients

Consider data in Wingo (1983) representing the relief times in hours of 50 arthritics patients receiving a fixed dose of analgesic, which fits the Burr type-XII distribution.

0.70, 0.84, 0.58, 0.50, 0.55, 0.82, 0.59, 0.71, 0.72, 0.61,

0.62, 0.49, 0.54, 0.72, 0.36, 0.71, 0.35, 0.64, 0.85, 0.55, 0.59, 0.29, 0.75, 0.53, 0.46, 0.60, 0.60, 0.36, 0.52, 0.68, 0.80, 0.55, 0.84, 0.70, 0.34, 0.70, 0.49, 0.56, 0.71, 0.61, 0.57, 0.73, 0.75, 0.58, 0.44, 0.81, 0.80, 0.87, 0.29, 0.50

Thus, for comparison purposes, the 90% and 95% confidence intervals for the parameters α and β are derived based on the conditional and the AMLEs approaches. The results in Table 1 indicated that the length of intervals for the parameters α and β based on the conditional approach are smaller than those based on the AMLEs approach that ensures the simulation results.

5.2. Application Data for Failure Components

Consider the data in Wingo (1993) representing the failure times for 30 specific electronic components included in the life test, which are censored after 20 failures using type-II censoring. These failure times (in months) are:

0.1, 0.1, 0.2, 0.3, 0.4, 0.5, 0.5, 0.6, 0.7, 0.8, 0.9, 0.9, 1.2, 1.6, 1.8, 2.3, 2.5, 2.6, 2.9, 3.1

The AMLEs for the parameters α and β are 1.29118 and 0.63779 respectively. Wingo (1993) derived the 0.90% confidence intervals for the parameters based on the pivotal quantities.

Table 1. The Lower (LL), the Upper limits (UL) and the interval lengths of the 90% and 95% confidence intervals (CI) for the parameters α, β based on the Conditional and the AMLEs approaches for complete, Type-II censored and Type-II progressive censored samples with binomial random removal with probability $P = 0.5$ for the arthritics patients data

Meth.	Conditional CIs					AMLEs CIs			
	CI	90%		95%		90%		95%	
	Par.	LL	UL	LL	UL	LL	UL	LL	UL
Complete	α	4.1049 (1.7151)	5.8199	3.9578 (2.0442)	6.0019	4.1366 (1.7281)	5.8646	3.9742 (2.0528)	6.0269
	β	6.3974 (4.4614)	10.8589	6.0944 (5.3934)	11.4878	5.4125 (5.7109)	11.1234	4.8760 (6.7839)	11.6599
Censored 50%	α	4.6213 (2.3815)	7.0028	4.4187 (2.8407)	7.2593	3.8526 (2.4600)	6.3126	3.6215 (2.9222)	6.5437
	β	5.2649 (4.2801)	9.5450	4.9407 (5.1606)	10.1013	4.8618 (10.8332)	15.6950	3.8442 (12.8685)	16.7127
Censored 25%	α	4.3738 (1.9929)	6.3667	4.2027 (2.3754)	6.5781	4.0195 (2.0055)	6.0249	3.8311 (2.3823)	6.2134
	β	5.9767 (4.2794)	10.2562	5.6672 (5.1641)	10.8313	5.3744 (7.9870)	13.3614	4.6241 (9.4876)	14.1117
Prog.Cen. 50%	α	3.7865 (2.4309)	6.2174	3.5870 (2.8975)	6.4845	3.8526 (2.4600)	6.3126	3.6215 (2.9222)	6.5437
	β	6.0758 (6.8554)	12.9311	5.6635 (8.4113)	14.0748	4.4991 (9.0974)	13.5965	3.6446 (10.8066)	14.4511
Prog.Ce. 25%	α	3.9758 (1.9873)	5.9630	3.8083 (2.3682)	6.1765	4.0195 (2.0055)	6.0249	3.8311 (2.3823)	6.2134
	β	6.5390 (5.5836)	12.1227	6.1775 (6.7846)	12.9622	5.2354 (7.3184)	12.5538	4.5479 (8.6934)	13.2413

(The values in parentheses are the length of intervals)

Table 2. The mean length (MLIs) and the coverage percentages (CPs) for the conditional and AMLEs methods with the nominal level 95% for the parameter α with $\beta = 2$ based on the complete and censored samples with censored levels (50%, 25%)

Methods		Conditional				AMLEs			
n	m	MLI, α			CP	MLI, α			CP
		0.5	1.0	2.0		0.5	1.0	2.0	
20	10	0.6016	1.2031	2.4063	0.943	0.6230	1.2460	2.4920	0.958
	15	0.4432	0.8864	1.7728	0.958	0.4515	0.9030	1.8061	0.959
	20	0.3643	0.7285	1.4570	0.940	0.3681	0.7362	1.4725	0.942
40	20	0.3914	0.7829	1.5657	0.947	0.3978	0.7955	1.5911	0.950
	30	0.2960	0.5921	1.1818	0.948	0.2987	0.5974	1.1949	0.950
	40	0.2479	0.4959	0.9919	0.945	0.2492	0.4984	0.9968	0.944
60	30	0.3133	0.6266	1.2532	0.942	0.3166	0.6332	1.2663	0.946
	45	0.2388	0.4775	0.9550	0.946	0.2402	0.4803	0.9606	0.946
	60	0.2001	0.4002	0.8004	0.949	0.2007	0.4014	0.8029	0.946
80	40	0.2675	0.5349	1.0699	0.953	0.2695	0.5391	1.0782	0.955
	60	0.2054	0.4108	0.8216	0.958	0.2063	0.4126	0.8251	0.961
	80	0.1728	0.3457	0.6913	0.956	0.1732	0.3463	0.6926	0.955
100	50	0.2392	0.4783	0.9567	0.950	0.2406	0.4812	0.9624	0.952
	75	0.1838	0.3676	0.7352	0.954	0.1844	0.3687	0.7375	0.954
	100	0.1546	0.3093	0.6185	0.960	0.1548	0.3096	0.6191	0.959

Table 3. The mean length (MLIs) and the coverage percentages (CPs) for the conditional and AMLEs approaches with the nominal level 95% for the parameter α with $\beta = 3$ based on the complete and censored samples with censored levels (50%, 25%)

Methods		Conditional				AMLEs			
n	m	MLI, α			CP	MLI, α			CP
		0.5	1.0	2.0		0.5	1.0	2.0	
20	10	0.6112	1.2225	2.4449	0.942	0.6352	1.2704	2.5407	0.961
	15	0.4426	0.8852	1.7704	0.958	0.4508	0.9016	1.8031	0.959
	20	0.3464	0.6927	1.3854	0.938	0.3497	0.6994	1.3988	0.942
40	20	0.3968	0.7936	1.5872	0.947	0.4037	0.8075	1.6149	0.951
	30	0.2945	0.5891	1.1781	0.951	0.2971	0.5942	1.1883	0.952
	40	0.2354	0.4709	0.9418	0.944	0.2365	0.4729	0.9459	0.943
60	30	0.3173	0.6347	1.2693	0.942	0.3208	0.6418	1.2836	0.945
	45	0.2373	0.4747	0.9493	0.945	0.2387	0.4774	0.9547	0.946
	60	0.1898	0.3797	0.7594	0.944	0.1903	0.3807	0.7614	0.944
80	40	0.2707	0.5415	1.0829	0.953	0.2729	0.5459	1.0917	0.958
	60	0.2040	0.4080	0.8161	0.958	0.2048	0.4097	0.8194	0.961
	80	0.1639	0.3278	0.6556	0.951	0.1641	0.3283	0.6566	0.952
100	50	0.2420	0.4840	0.9681	0.949	0.2435	0.4871	0.9743	0.953
	75	0.1825	0.3649	0.7299	0.955	0.1830	0.3660	0.7320	0.957
	100	0.1465	0.2930	0.5860	0.961	0.1466	0.2932	0.5864	0.958

For α , the confidence interval is (0.8528, 1.7316) with a length of 0.8788. For β the 0.90% the confidence interval is (0.44221, 0.73692) with a length of 0.29471. For the purpose of comparison, the 90% the confidence intervals of the parameters α and β are derived based on the conditional approach as follows:

For α , the confidence interval is (0.8455, 1.7020) with length 0.8566, which is shorter than Wingo (1993) interval.

For β the confidence interval is (0.03089, 0.7468) with a length of 0.7159 which is longer than Wingo interval. The confidence intervals based on the AMLEs approach are: For α the CI is (0.8525, 1.7299) with a length of 0.8774, which is longer than the conditional ones and shorter than Wingo interval.

For β the confidence interval is (0.37895, 0.89663) with a length of 0.51768, which is longer than Wingo interval.

The results of this data indicate that the confidence interval based on the conditional for α is shorter than Wingo (1993) and the AMLE confidence intervals. On the contrary, for

Wingo (1993) the confidence interval for β is very short because the data is not good fit for the Burr type-XII distribution based on the type-II censored sample.

Table 4. The mean length (MLIs) and the coverage percentages (CPs) for the conditional and AMLEs methods when the nominal level 95% for the parameter α with $\beta = 2$ based on the progressive type-II censoring with binomial random removal with probability $P = 0.5$ and censored levels (50% and 25%)

Methods		Conditional				AMLEs			
n	m	MLI, α			CP	MLI, α			CP
		0.5	1.0	2.0		0.5	1.0	2.0	
20	10	0.5491	1.0983	2.1966	0.953	0.5619	1.1237	2.2475	0.955
	15	0.4278	0.8556	1.7111	0.959	0.4340	0.8681	1.7362	0.953
40	20	0.3624	0.7247	1.4494	0.942	0.3662	0.7325	1.4649	0.945
	30	0.2888	0.5776	1.1552	0.950	0.2908	0.5816	1.1631	0.950
60	30	0.2871	0.5743	1.1485	0.956	0.2891	0.5781	1.1563	0.954
	45	0.2321	0.4643	0.9285	0.950	0.2331	0.4663	0.9325	0.949
80	40	0.2482	0.4963	0.9927	0.952	0.2494	0.4988	0.9976	0.950
	60	0.2002	0.4003	0.8006	0.950	0.2008	0.4016	0.8031	0.948
100	50	0.2200	0.4400	0.8801	0.950	0.2209	0.4417	0.8834	0.951
	75	0.1786	0.3573	0.7145	0.953	0.1790	0.3580	0.7160	0.954

Table 5. The mean length (MLIs) and the coverage percentages (CPs) for the conditional and AMLEs methods with the nominal level 95% for the parameter α with $\beta = 3$, based on the progressive type-II censoring with binomial random removal with probability $P = 0.5$ and censored levels (50% and 25%)

Methods		Conditional				AMLEs			
n	m	MLI, α			CP	MLI, α			CP
		0.5	1.0	2.0		0.5	1.0	2.0	
20	10	0.5233	1.0465	2.0931	0.951	0.5342	1.0684	2.1369	0.955
	15	0.4068	0.8136	1.6273	0.956	0.4123	0.8245	1.6489	0.948
40	20	0.3442	0.6884	1.3768	0.943	0.3475	0.6950	1.3901	0.945
	30	0.2742	0.5484	1.0968	0.947	0.2758	0.5518	1.1036	0.943
60	30	0.2725	0.5449	1.0899	0.953	0.2741	0.5483	1.0965	0.952
	45	0.2202	0.4403	0.8807	0.953	0.2210	0.4420	0.8841	0.954
80	40	0.2357	0.4714	0.9427	0.950	0.2367	0.4735	0.9469	0.952
	60	0.1899	0.3798	0.7597	0.950	0.1904	0.3808	0.7616	0.950
100	50	0.2086	0.4172	0.8344	0.950	0.2093	0.4186	0.8372	0.951
	75	0.1693	0.3385	0.6771	0.953	0.1696	0.3391	0.6783	0.954

Table 6. The mean length (MLIs) and the coverage percentages (CPs) for the conditional and AMLEs methods with the nominal level 95% for the parameter β with $\alpha = 2$ based on the type-II censored and type-II progressively censoring with binomial random removal with probability $P = 0.5$ and censored levels (50%, 25%)

METHODS		CONDITIONAL			AMLI	
	N	m	MLI	CP	MLI	CP
Type-II Censored Samples	20	10	3.2015	0.965	9.4441	0.971
		15	2.8601	0.957	2.7416	0.973
		20	2.6664	0.939	1.9443	0.956
	40	20	2.0775	0.967	3.0535	0.967
		30	1.7935	0.967	1.6501	0.971
		40	1.6504	0.947	1.2927	0.953
	60	30	1.6594	0.954	2.2459	0.968
		45	1.4167	0.958	1.3149	0.957
		60	1.2989	0.934	1.0441	0.939

METHODS			CONDITIONAL		AMLI	
	N	m	MLI	CP	MLI	CP
	80	40	1.4122	0.967	1.8059	0.966
		60	1.1994	0.953	1.1134	0.963
		80	1.0960	0.931	0.8973	0.945
	100	50	1.2439	0.965	1.5829	0.962
		75	1.0524	0.960	0.9856	0.955
		100	0.8554	0.940	0.7959	0.950
Type-II Progressive Censored Samples	20	10	5.0204	0.948	3.1473	0.959
		15	3.3864	0.943	2.3234	0.961
	40	20	2.6639	0.952	1.9181	0.954
		30	1.9815	0.947	1.5123	0.961
	60	30	1.9974	0.957	1.5108	0.950
		45	1.5429	0.943	1.2112	0.942
	80	40	1.6406	0.945	1.2922	0.950
		60	1.2942	0.933	1.0427	0.945
	100	50	1.4359	0.945	1.1429	0.955
		75	1.1313	0.942	0.9239	0.947

Table 7. The mean length (MLIs) and the coverage percentages (CPs) for the conditional and AMLEs methods with the nominal level 95% for the parameter β with $\alpha = 3$ based on the type-II censored and type-II progressively censoring with binomial random removal with probability $P = 0.5$ and censored levels (50%, 25%)

Methods			CONDITIONAL		AMLI	
	n	M	MLI	CP	MLI	CP
Type-II Censored Samples	20	10	4.7584	0.964	7.8689	0.967
		15	4.2307	0.957	5.5008	0.978
		20	4.0483	0.945	3.2563	0.971
	40	20	3.1056	0.970	6.4169	0.966
		30	2.6704	0.970	2.9937	0.977
		40	2.5400	0.953	2.0695	0.952
	60	30	2.4862	0.954	4.4525	0.958
		45	2.1135	0.960	2.3414	0.968
		60	2.0081	0.940	1.6539	0.940
	80	40	2.1181	0.966	3.4614	0.960
		60	1.7914	0.956	1.9577	0.967
		80	1.6988	0.939	1.4145	0.948
	100	50	1.8674	0.965	3.0164	0.968
		75	1.5731	0.962	1.7275	0.963
		100	1.2155	0.964	1.2515	0.954
Type-II Progressive Censored Samples	20	10	7.4487	0.954	6.0762	0.970
		15	5.1001	0.947	3.9954	0.973
	40	20	4.0476	0.956	3.1888	0.971
		30	3.0373	0.956	2.4437	0.963
	60	30	3.0624	0.959	2.4374	0.953
		45	2.3789	0.954	1.9273	0.947
	80	40	2.5254	0.956	2.0697	0.952
		60	2.0012	0.938	1.6509	0.954
	100	50	2.2177	0.950	1.8144	0.955
		75	1.7537	0.953	1.4566	0.950

6. Conclusions

In this paper, a new application for the conditional inference has been introduced to inference on Burr type-XII distribution based on generalized order statistics. Moreover, for purpose of comparison the asymptotic maximum likelihood estimate has been applied to measure the performance of the proposed approach based on the Monte Carlo simulations that indicated the conditional approach possess good statistical properties and it can perform quite well even when the sample size is extremely small. However, the AMLEs turn out to be imprecise or even unreliable for small or highly type-II censored samples.

REFERENCES

- [1] Asgharzadeh A. and Valiollahi R. (2008). Estimation Based on Progressively Censored Data from the Burr Model. *International Mathematical Forum*, Vol. 3, No. 43, 2113-2121.
- [2] Asgharzadeh A. and Abdi M. (2012). Confidence Intervals for the Parameters of the Burr Type XII Distribution Based on Records. *International Journal of Statistics and Economics*. Vol. 8, No. S12.
- [3] Burr, I.W. (1942). Cumulative frequency functions. *Annals of Mathematics*. Vol. 13, 215-232.
- [4] Burr, I.W. (1976). The effect of non-normality on constants for \bar{X} and R charts. *Industrial Quality Control*, 567-569.
- [5] Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub-shape or increasing failure rate function. *Statistics & Probability Letters*. Vol. 49, 155-161.
- [6] Cook, D.R. and Johnson, E.S. (1986). Generalized Burr-Pareto-Logestic distributions with applications to a uranium exploration data set. *Technometrics*, Vol. 28, 123-131.
- [7] Evans, R.A. and Simons, G. (1975). Research on statistical procedures in reliability engineering, ARL, TR. 75-0154, AD A013687.
- [8] Evans, L.G. and Ragab, A.S. (1983). Bayesian Inferences Given a type-II Censored Sample from a BURR Distribution. *Comm. in Statist. Theory Meth.* A12, 1569-1580.
- [9] Fisher, R.A. (1934). Two New Properties of Mathematical Likelihood. *Proc. R. Soc. A* 144, 285-307.
- [10] Hakim, A.R., Fithriani, I., and Novita, M. (2021). Properties of Burr distribution and its application to heavy tailed survival time data. *Journal of Physics: Conference Series* 1725. doi: 10.1088/1742-6596/1725/1/012016.
- [11] Kamps, U. A. (1995). Concept of generalized order statistics. *J. Statistical Planning & Inference*, Vol. 48, 1-23.
- [12] Lai, C.D., Xie, M. and Murtly, D. N.P. (2003). Bathub-shaped failure rate life distribution. IN: Balakrishnan, N., Rao, C.R. eds. *Handbokof Statistics: Advances in Reliability*: Vol. 20. London: Elsevier, 69-104.
- [13] Lawless, J.F. (1972). Confidence interval estimation for the parameters of the Weibull distribution. *Utilitas Mathematica*, Vol. 2, 71- 87.
- [14] Lawless, J.F. (1973). Conditional versus Unconditional Confidence Intervals for the Parameters of the Weibull Distribution. *J. Amer. Statist. Assoc.*, Vol. 68, 669-679.
- [15] Lawless, J.F. (1974). Approximations to Confidence Intervals in the Extreme Value and Weibull Distributions. *Biometrika*, Vol. 61, 123-129.
- [16] Lawless, J.F. (1975). Construction of Tolerance bounds for the Extreme Value and Weibull Distributions. *Technometrics*, Vol. 71, 255-261.
- [17] Lawless, J.F. (1978). Confidence interval Estimation for the Weibull and Extreme value Distributions. *Technometrics*, Vol. 20, 355-364.
- [18] Lawless, J. F. (1980). Inference in the Generalized Gamma and log Gamma Distributions. *Technometrics*, Vol. 22, 3, 409-419.
- [19] Lawless, J.F. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons. New York.
- [20] Lee, W.C., Wu, J.W. and Hong, C.W. (2009). Assessing the lifetime performance index of products from progressively type-II censored data using Burr XII model, *Mathematics and computers in simulations*. Vol. 79, 2167-2179.
- [21] Li, X., Shi, Y., Wei, J. and Chai, J. (2007). Empirical Bayes estimators of reliability performances using LINEX loss under progressively type-II censored samples. *Mathematics and Computers in Simulation*. Vol.73 No. 5, 320-326.
- [22] Liang, W. and Yimin, S. (2010). Empirical Inference for the Burr Model based on Records. *Applied Mathematical Sciences*, Vol. 4, No. 34, 1663-1670.
- [23] Lio, Y. L., Tasi, Tzong-Ru and Chiang, Jyun-You. (2007). Parameter estimations for the Burr type-XII distribution under progressive type-II censoring. *ICIC Express Letters*. Vol. 1, No. 1, 1-6.
- [24] Kumar, D. (2017). The Burr type-XII distribution with some statistical properties. *Journal of Data Science*, Vol. 16, 509-534.
- [25] McDonald, J.B. and Richards, D.O. (1987). Model selection, some generalized distributions. *Commun. Stat. Theory and Mthods*, Vol. 16, 4, 1049-1047.
- [26] Malinowska I., Pawlas P. and Szydal D. (2006). Estimation of location and scale parameters for the Burr-XII distribution using generalized order statistics. *Linear Algebra and its Applications*. Vol. 417, 150-162.
- [27] Maswadah, M. (2003). Conditional Confidence Interval Estimation For The Inverse Weibull distribution Based on Censored Generalized Order Statistics. *Journal of Statistical Computation and Simulation*, Vol. 73, No. 12, 887-898.
- [28] Maswadah, M.(2005). Conditional Confidence Interval Estimation For The Type-II Extreme Value Distribution Based on Censored Generalized Order Statistics *Journal of Applied Statistical Science*, Vol. 14, No. 1/2, 71-84.
- [29] Maswadah, M, and EL-Faheem, A.A. (2018). Conditional Inference for the Weibull Extension Model Based on the

Generalized Order Statistics. Pak.j.stat.oper.res. Vol. XIV No.200 2, P. 119-214.

- [30] Mielke, P.W.Jr. and Johnson, E.S. (1974). Some generalized beta distributions of the second kind having desirable applications features in hydrology and meteorology. Water resources Research, Vol. 10, 2, 223-226.
- [31] Amal A. Mohammed, A.A., Abraheem, S.K., and Al-Obedy, N.J.F. (2018). Bayesian Estimation of Reliability Burr Type-XII Under Al-Bayyatis' Suggest Loss Function with Numerical Solution. Journal of Physics: Conf. Series 1003. doi:10.1088/1742-6596/1003/1/012041.
- [32] Rodriguez R.N. (1977). Aguide to the Burr type-XII distributions. Biometrika, Vol. 64, 129-134.
- [33] Soliman, A.S. (2002). Reliability Estimations in a Generalized Life-Model with application to the Burr-XII. IEEE Trans. On Reliab. Vol. 51, 3, 337-343.
- [34] Soliman, A. S., Abd Ellah, A.H., Abou-Elheggag, N.A. and Modhesh, A. A. (2012). Bayesian Inference and Prediction of Burr Type XII Distribution for Progressive First Failure Censored Sampling. Intelligent Information Management, 2011, 3, 175-185
- [35] Wang, B. (2008). Statistical inference for the Burr type-XII distribution. Acta Mathematica Scientia (in Chinese), Vol. 28, 1103-1108.
- [36] Wu, J., Lu, H., Chen, C. and Wu, C. (2004). Statistical Inference about shape parameter of the new two-parameter Bathtub-shaped lifetime distribution. Quality and Reliability Engineering International. Vol. 20, 607-616.
- [37] Wu, J.W. and Yu, H.Y. (2005). Statistical Inference about shape parameter of the new two-parameter of the Burr XII distribution under the failure- censored sampling Plan. Applied Mathematics and Computation, Vol. 163, No. 1, 443-482.
- [38] Wu, S. J., Chen, Y. J. and Chang, C. T. (2007). Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution, Journal of Statistical Computation and Simulations, Vol. 77, 19-27.
- [39] Wingo, D.R. (1983). Maximum likelihood methods for fitting the Burr type-XII distribution to life test data. Biometrical J. Vol. 25, 77-81.
- [40] Wingo, D.R. (1993). Maximum likelihood estimation of the Burr type-XII distribution parameters under type-II censoring. Microelectorn Reliab. Vol. 23, 1251-1257.
- [41] Wingo, D.R. (1993). Maximum likelihood methods for fitting the Burr type-XII distribution to multiply (progressively) censored life test data. Metrika, Vol. 40, 203-210.
- [42] Wu, J-W and Yu, H-Y. (2005). Statistical inference about the shape parameter of the Burr type XII distribution under the failure-censored sampling plan. Applied Mathematics and Computation. Vol. 163 No.1, 443-482.
- [43] Xiuchun, L., Yimin, S., Jieqiong, W. and Jian C. (2007). Empirical Bayes estimators of reliability performances using Linex loss under progressively Type-II censored samples. Mathematics and Computers in Simulations, Vol. 73, No. 5, 320-326.
- [44] Zimmer, W. J., Keats, J. B. and Wang, F. K. (1998). The Burr XII distribution in reliability analysis, Journal of Quality Technology, Vol. 30, 386-394.