International Journal of Probability and Statistics 2019, 8(1): 1-13
DOI: 10.5923/j.ijps.20190801.01

A Generalization of Two-Parameter Lindley Distribution with Properties and Applications

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Abstract In this paper, a generalization of two-parameter Lindley distribution (GTPLD), which includes one parameter exponential and Lindley distributions, two-parameter Lindley distribution (TPLD) of Shanker and Mishra (2013), Weibull distribution, gamma distribution, generalized gamma distribution of Stacy (1962) and power Lindley distribution of Ghitany *et al* (2013) as particular cases, has been proposed. Its moments, hazard rate function, mean residual life function, order statistic, Renyi entropy measure has been studied. Method of maximum likelihood estimation has been discussed for estimating its parameters. Applications of the distribution have been explained with two examples of observed real lifetime datasets.

Keywords Two-parameter Lindley distribution, Moments, Hazard rate function, Mean residual life function, Order statistic, Renyi entropy measure, Maximum likelihood estimation, Applications

1. Introduction

Shanker and Mishra (2013) introduced a two-parameter Lindley distribution (TPLD) defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f(y;\theta,\beta)=\frac{\theta^2}{\beta\theta+1}(\beta+y)e^{-\theta y} \quad ;x>0,\ \theta>0,\beta\theta>-1 \tag{1.1}$$

$$F(y;\theta,\beta)=1-\left[1+\frac{\theta x}{\beta\theta+1}\right]e^{-\theta y} \quad ;x>0,\ \theta>0,\beta\theta>-1 \tag{1.2}$$

The pdf (1.1) can be expressed as

$$f(y;\theta,\beta)=p\,g_1(y;\theta)+(1-p)\,g_2(y;2,\theta)$$

where

$$p=\frac{\beta\theta}{\beta\theta+1},$$

$$g_1(y;\theta)=\theta e^{-\theta y} \quad ;y>0,\ \theta>0$$

$$g_2(y;\theta)=\theta^2 y e^{-\theta y} \quad ;y>0,\ \theta>0\,.$$

Clearly the density (1.1) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution with shape parameter 2 and scale parameter θ, with mixing proportion $p=\frac{\beta\theta}{\beta\theta+1}$. Shanker and Mishra (2013) studied its various properties including coefficients of variation, skewness, kurtosis; hazard rate function, mean residual life

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Published online at http://journal.sapub.org/ijps

function and stochastic ordering. The estimation of its parameters using both the maximum likelihood estimation and the method of moments along with applications of TPLD to model lifetime data has also been discussed by Shanker and Mishra (2013). It can be easily shown that Lindley distribution, introduced by Lindley (1958), having pdf

$$f\left(y;\theta\right)=\frac{\theta^{2}}{\theta+1}\left(1+y\right)e^{-\theta y}\quad ;x>0,\ \theta>0 \tag{1.3}$$

is a particular case of TPLD (1.1) at $\beta=1$. Lindley distribution has been studied in detail by Ghitany et al (2008). Shanker et al (2015) have detailed and critical study on applications of exponential and Lindley distribution for modeling real lifetime data from biomedical sciences and engineering and observed that both exponential and Lindley are competing each other. Shanker et al (2016) have discussion on applications of gamma distribution and Weibull distribution for real lifetime data from engineering and biological sciences. Shanker (2016) has discussed various important statistical properties including coefficient of variation, skewness, kurtosis and index of dispersion along with various applications of generalized Lindley distribution (GLD) introduced by Zakerzadeh and Dolati (2009). Shanker and Shukla (2016) have detailed comparative study on applications of three-parameter generalized gamma distribution (GGD) and generalized Lindley distribution (GLD) and observed that these two distributions are competing each other for modeling lifetime data.

In this paper, an attempt has been made to derive a generalization of two-parameter Lindley distribution (GTPLD), which includes one parameter exponential and Lindley distributions, two-parameter Lindley distribution (TPLD) of Shanker and Mishra (2013), Weibull distribution, gamma distribution, generalized gamma distribution of Stacy (1962) and power Lindley distribution of Ghitany et al (2013) as particular cases. The moments, hazard rate function, mean residual life function, order statistic and Renyi entropy measure of the distribution have been studied. Method of maximum likelihood estimation has been discussed for estimating its parameters. Applications of the distribution have been explained with two examples of observed real lifetime datasets and its goodness of fit has been compared with other lifetime distributions.

2. A Generalization of Two-Parameter Lindley Distribution

Assuming the power transformation $X=Y^{\frac{1}{\alpha}}$ in (1.1), the pdf of X can be obtained as

$$f_{1}\left(x;\theta,\alpha,\beta\right)=\frac{\alpha\theta^{2}}{\beta\theta+1}x^{\alpha-1}\left(\beta+x^{\alpha}\right)e^{-\theta x^{\alpha}}\quad ;x>0,\ \theta>0,\alpha>0,\beta\theta>-1 \tag{2.1}$$

$$=p\,g_{1}\left(x;\theta,\alpha\right)+\left(1-p\right)g_{2}\left(x;2,\theta,\alpha\right), \tag{2.2}$$

where,

$$p=\frac{\beta\theta}{\beta\theta+1},$$

$$g_{1}\left(x;\theta,\alpha\right)=\theta\alpha\,x^{\alpha-1}e^{-\theta x^{\alpha}}\quad ;x>0,\ \theta>0,\alpha>0$$

$$g_{2}\left(x;2,\theta,\alpha\right)=\alpha\theta^{2}\,x^{2\alpha-1}e^{-\theta x^{\alpha}}\quad ;x>0,\ \theta>0,\alpha>0\ .$$

Since at $\alpha=1$, (2.1) reduces to TPLD (1.1), we would call (2.1) a generalization of two-parameter Lindley distribution (GTPLD). Further, (2.1) is also a two-component mixture of Weibull distribution with shape parameter α and scale parameter θ and a generalized gamma distribution with shape parameters $\left(2,\alpha\right)$ and scale parameter θ, with mixing proportion $p=\frac{\beta\theta}{\beta\theta+1}$. At $\alpha=1$ and $\beta=1$, (2.1) reduces to the Lindley distribution (1.3). At $\beta=1$, (2.1) reduces to the Power Lindley distribution introduced by Ghitany *et al* (2013) having pdf

$$f_{2}\left(x;\theta,\alpha\right)=\frac{\alpha\theta^{2}}{\theta+1}x^{\alpha-1}\left(1+x^{\alpha}\right)e^{-\theta x^{\alpha}}\quad ;x>0,\ \theta>0,\alpha>0 \tag{2.3}$$

The gamma $\left(2,\theta\right)$ distribution and a generalized gamma $\left(2,\alpha,\theta\right)$ distribution are also particular cases of (2.1) for $\left(\beta=0,\alpha=1\right)$ and $\beta=0$ respectively. It can be easily shown that (2.1) reduces to Weibull distribution for $\beta\to\infty$. Further, for $\alpha=1$ and $\beta\to\infty$, (2.1) reduces to exponential distribution.

The corresponding cdf of (2.1) can be obtained as

$$F_1(x;\theta,\alpha,\beta) = 1 - \left[1 + \frac{\theta x^{\alpha}}{\beta\theta + 1}\right] e^{-\theta x^{\alpha}} \quad ; x > 0,\ \theta > 0, \alpha > 0, \beta\theta > -1 \tag{2.4}$$

We use $X \sim \text{GTPLD}(\theta,\alpha,\beta)$ to denote a random variable having GTPLD (2.1) with parameters θ,α and β having pdf (2.1) and cdf (2.4).

The behavior of the pdf and the cdf of GTPLD have been shown graphically for varying values of parameters θ,α and β in figures 1 and 2 respectively.

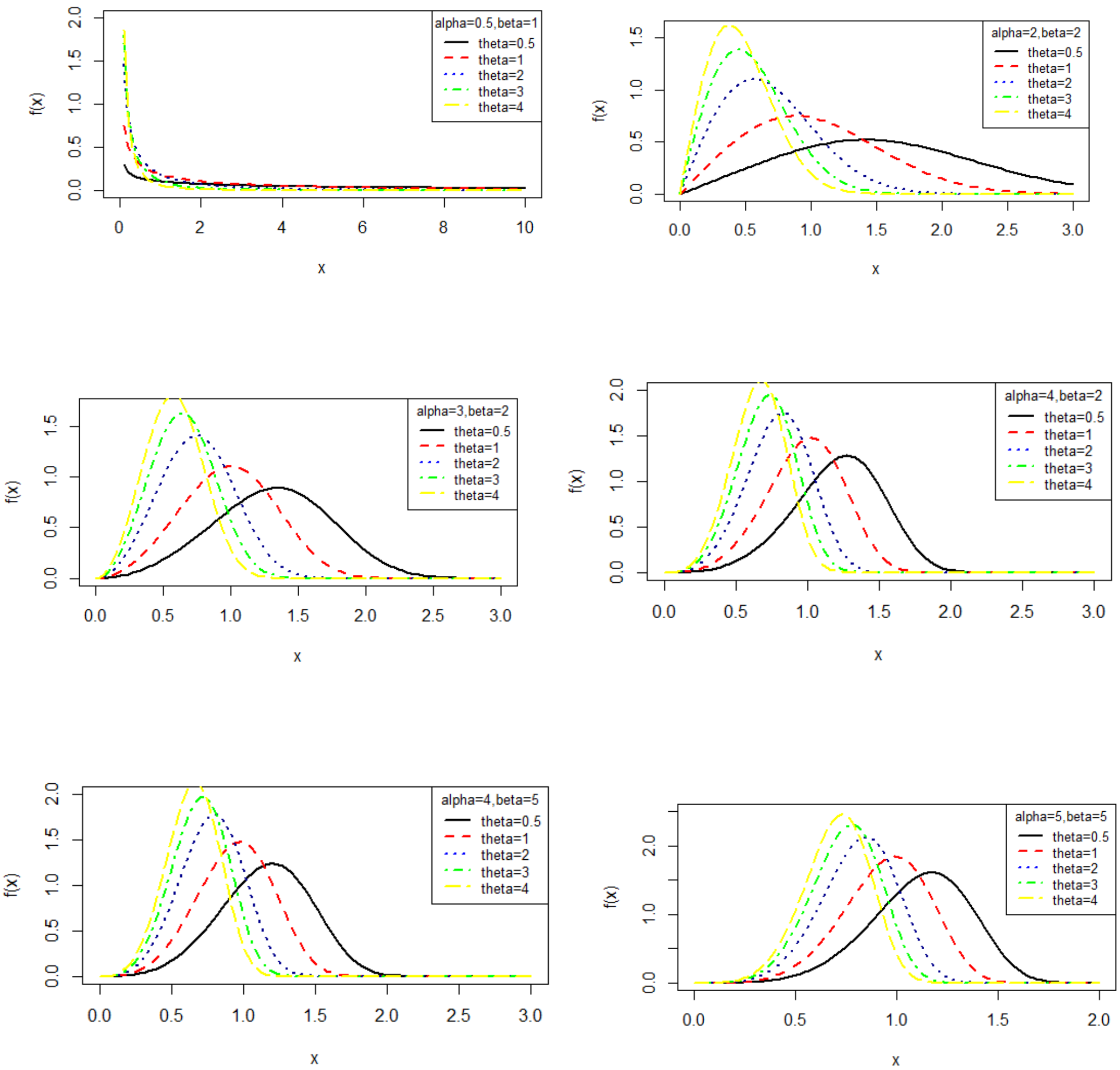

Figure 1. Behavior of the pdf of GTPLD for varying values of parameters θ,α and β

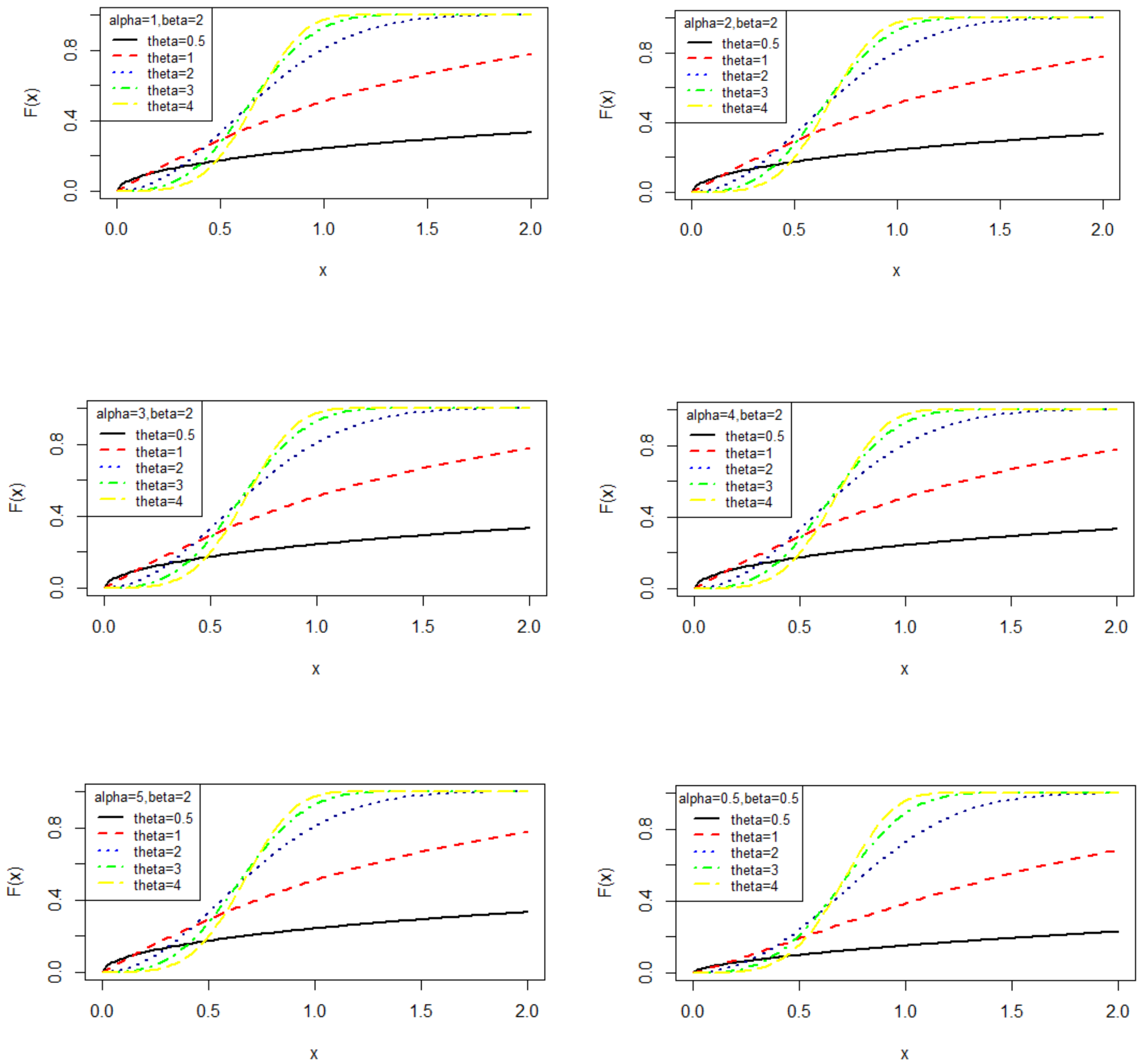

Figure 2. Behavior of the cdf of GTPLD for varying values of parameters θ, α and β

3. Reliability Properties

3.1. Survival Function and Hazard Rate Function

The survival function of GTPLD can be expressed as

$$S(x;\theta,\alpha,\beta)=1-F(x;\theta,\alpha,\beta)=\left[1+\frac{\theta x^{\alpha}}{\beta\theta+1}\right]e^{-\theta x^{\alpha}} \quad ; x>0,\ \theta>0, \alpha>0, \beta\theta>-1 .$$

Thus the hazard rate function of GTPLD can be obtained as

$$h(x;\theta,\alpha,\beta)=\frac{f(x;\theta,\alpha,\beta)}{S(x;\theta,\alpha,\beta)}=\frac{\alpha\theta^{2}x^{\alpha-1}\left(\beta+x^{\alpha}\right)}{(\beta\theta+1)+\theta x^{\alpha}}; x>0,\ \theta>0, \alpha>0, \beta\theta>-1 .$$

The behavior of hazard rate function of GTPLD for varying values of parameters $\theta, \alpha,$ and β are shown in figure 3

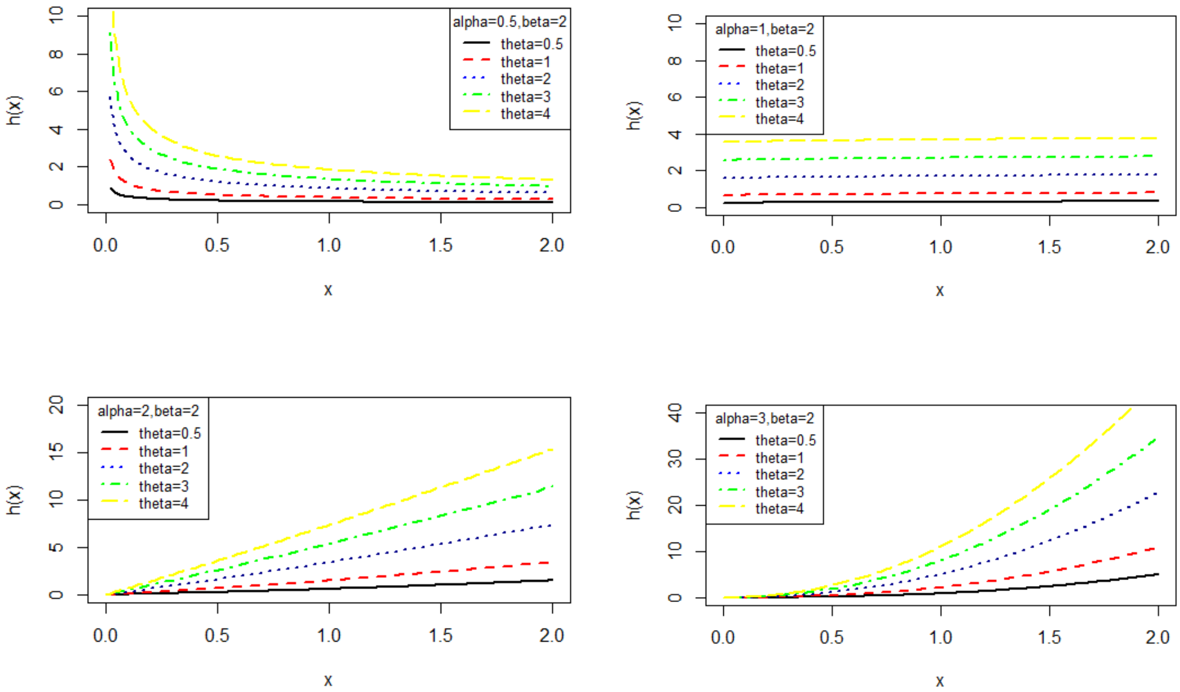

Figure 3. Behavior of the hazard rate function of GTPLD for varying values of parameters $\theta, \alpha,$ and β

3.2. Mean Residual Life Function

The mean residual life function, $m(x;\theta,\alpha,\beta)$ of GTPLD (2.1) can be obtained as

$$m(x;\theta,\alpha,\beta)=\frac{1}{S(x;\theta,\alpha,\beta)}\int_x^{\infty} t\, f(t;\theta,\alpha,\beta)\,dt-x$$

$$=\frac{\alpha\theta^2}{\left[\theta x^{\alpha}+\beta\theta+1\right]e^{-\theta x^{\alpha}}}\int_x^{\infty} t^{\alpha}\left(\beta+t^{\alpha}\right)e^{-\theta t^{\alpha}}\,dt-x$$

$$=\frac{\alpha\theta^2}{\left[\theta x^{\alpha}+\beta\theta+1\right]e^{-\theta x^{\alpha}}}\left[\beta\int_x^{\infty} e^{-\theta t^{\alpha}}t^{\alpha}\,dt+\int_x^{\infty} e^{-\theta t^{\alpha}}t^{2\alpha}\,dt\right]-x\,.$$

Taking $u=t^{\alpha}$, which gives $t=(u)^{\frac{1}{\alpha}}$ and $dt=\frac{1}{\alpha}u^{\frac{1-\alpha}{\alpha}}\,du$, we get

$$m(x;\theta,\alpha,\beta)=\frac{\theta^2}{\left[\theta x^{\alpha}+\beta\theta+1\right]e^{-\theta x^{\alpha}}}\left[\beta\int_{x^{\alpha}}^{\infty} e^{-\theta u}u^{\frac{1}{\alpha}+1-1}\,du+\int_{x^{\alpha}}^{\infty} e^{-\theta u}u^{\frac{1}{\alpha}+2-1}\,du\right]-x$$

$$=\frac{\theta^2}{\left[\theta x^{\alpha}+\beta\theta+1\right]e^{-\theta x^{\alpha}}}\left[\beta\frac{\Gamma\left(\frac{1}{\alpha}+1,\theta x^{\alpha}\right)}{\theta^{\frac{1}{\alpha}+1}}+\frac{\Gamma\left(\frac{1}{\alpha}+2,\theta x^{\alpha}\right)}{\theta^{\frac{1}{\alpha}+2}}\right]-x$$

$$= \frac{\theta\beta\Gamma\left(\frac{1}{\alpha}+1,\theta x^{\alpha}\right)+\Gamma\left(\frac{1}{\alpha}+2,\theta x^{\alpha}\right)}{\theta^{\frac{1}{\alpha}}\left[\theta x^{\alpha}+\beta\theta+1\right]e^{-\theta x^{\alpha}}}-x .$$

It can be easily verified that $m(0;\theta,\alpha,\beta)=\frac{(\alpha\beta\theta+\alpha+1)}{\alpha^2(\beta\theta+1)}\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\theta^{\frac{1}{\alpha}}}=\mu_1'$. The behavior of $m(x;\theta,\alpha,\beta)$ of GTPLD for varying values of parameters $\theta,\alpha,$ and β are shown in figure 4

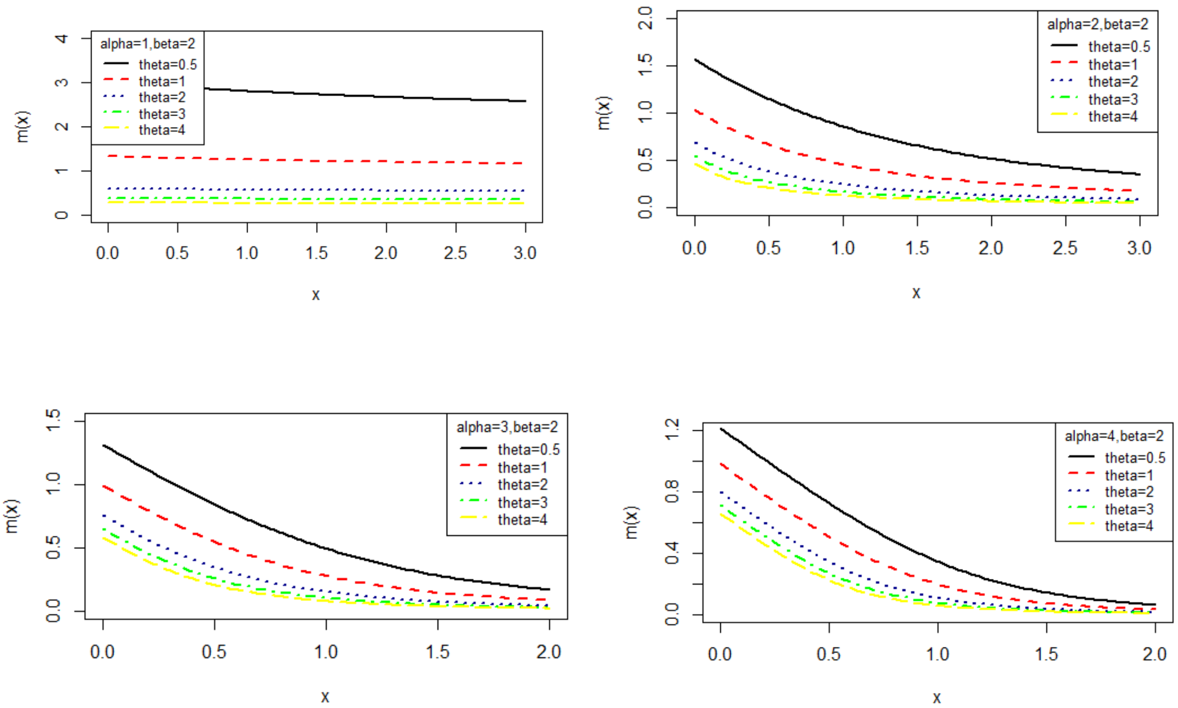

Figure 4. Behavior of the mean residual life function of GTPLD for varying values of parameters $\theta,\alpha,$ and β

4. Statistical Properties

4.1. Moments

The r th moment about origin, μ_r' of GTPLD (2.1) can be obtained as

$$\mu_r' = E\left(X^r\right)=\frac{\alpha\theta^2}{\beta\theta+1}\int_0^{\infty} x^{\alpha+r-1}\left(\beta+x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$=\frac{\alpha\theta^2}{\beta\theta+1}\left[\beta\int_0^{\infty} e^{-\theta x^{\alpha}}x^{\alpha+r-1}dx+\int_0^{\infty} e^{-\theta x^{\alpha}}x^{2\alpha+r-1}dx\right].$$

Taking $y=x^{\alpha}$, which gives $x=(y)^{\frac{1}{\alpha}}$ and $dx=\frac{1}{\alpha}y^{\frac{1-\alpha}{\alpha}}dy$, we get

$$\mu_r' = \frac{\alpha\theta^2}{\beta\theta+1}\left[\beta\int_0^\infty e^{-\theta y} y^{\frac{\alpha+r-1}{\alpha}} \frac{1}{\alpha} y^{\frac{1-\alpha}{\alpha}} dy + \int_0^\infty e^{-\theta y} y^{\frac{2\alpha+r-1}{\alpha}} \frac{1}{\alpha} y^{\frac{1-\alpha}{\alpha}} dy\right]$$

$$= \frac{\theta^2}{\beta\theta+1}\left[\beta\int_0^\infty e^{-\theta y} y^{\frac{r}{\alpha}+1-1} dy + \int_0^\infty e^{-\theta y} y^{\frac{r}{\alpha}+2-1} dy\right]$$

$$= \frac{\theta^2}{\beta\theta+1}\left[\beta\frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\theta^{\frac{r}{\alpha}+1}} + \frac{\Gamma\left(\frac{r}{\alpha}+2\right)}{\theta^{\frac{r}{\alpha}+2}}\right]$$

$$= \frac{\Gamma\left(\frac{r}{\alpha}\right)}{\theta^{\frac{r}{\alpha}}}\frac{r(\alpha\beta\theta+\alpha+r)}{\alpha^2(\beta\theta+1)}; r = 1,2,3,... \tag{4.1.1}$$

Taking $r = 1,2,3$ and 4 in (4.1.1), the first four moments about origin of GTPLD can be obtained as

$$\mu_1' = \frac{(\alpha\beta\theta+\alpha+1)}{\alpha^2(\beta\theta+1)}\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\theta^{\frac{1}{\alpha}}}$$

$$\mu_2' = \frac{2(\alpha\beta\theta+\alpha+2)}{\alpha^2(\beta\theta+1)}\frac{\Gamma\left(\frac{2}{\alpha}\right)}{\theta^{\frac{2}{\alpha}}}$$

$$\mu_3' = \frac{3(\alpha\beta\theta+\alpha+3)}{\alpha^2(\beta\theta+1)}\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\theta^{\frac{3}{\alpha}}}$$

$$\mu_3' = \frac{4(\alpha\beta\theta+\alpha+4)}{\alpha^2(\beta\theta+1)}\frac{\Gamma\left(\frac{4}{\alpha}\right)}{\theta^{\frac{4}{\alpha}}}.$$

The variance of GTPLD can thus be obtained as

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = \frac{2\alpha^2(\alpha\beta\theta+\alpha+2)\Gamma\left(\frac{2}{\alpha}\right) - (\alpha\beta\theta+\alpha+1)^2\left(\Gamma\left(\frac{1}{\alpha}\right)\right)^2}{\alpha^4\theta^{\frac{2}{\alpha}}(\beta\theta+1)^2}.$$

The higher central moments, if required, can be obtained using the relationship between central moments and raw moments. The skewness and kurtosis measures, upon substituting for the raw moments, can be obtained using the expressions

Skewness $= \dfrac{\mu_3' - 3\mu_2' + 2\left(\mu_1'\right)^3}{\sigma^3}$ and Kurtosis $= \dfrac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4}{\sigma^4}$.

4.2. Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from GTPLD (2.1). Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The pdf and the cdf of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y)=\frac{n!}{(k-1)!(n-k)!}F^{k-1}(y)\{1-F(y)\}^{n-k}\ f(y)$$
$$=\frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\ F^{k+l-1}(y)\,f(y)$$

and

$$F_Y(y)=\sum_{j=k}^{n}\binom{n}{j}F^j(y)\{1-F(y)\}^{n-j}$$
$$=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\ F^{j+l}(y),$$

respectively, for $k=1,2,3,...,n$.

Thus, the pdf and the cdf of k th order statistics of GTPLD are given by

$$f_Y(y)=\frac{n!\alpha\,\theta^2x^{\alpha-1}\left(\beta+x^\alpha\right)e^{-\theta x^\alpha}}{(\beta\theta+1)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}\times\left[1-\left(1+\frac{\theta x^\alpha}{\beta\theta+1}\right)e^{-\theta x^\alpha}\right]^{k+l-1}$$

and $F_Y(y)=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[1-\left(1+\frac{\theta x^\alpha}{\beta\theta+1}\right)e^{-\theta x^\alpha}\right]^{j+l}$

4.3. Renyi Entropy Measure

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy introduced by Renyi (1961). If X is a continuous random variable having probability density function $f(.)$, then Renyi entropy is defined as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int f^\gamma(x)\,dx\right\}$$

where $\gamma>0$ and $\gamma\neq 1$.

Thus, the Renyi entropy for GTPLD (2.1) can obtained as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\,\theta^2}{(\beta\theta+1)}\right)^\gamma\int_0^\infty x^{(\alpha-1)\gamma}\left(\beta+x^\alpha\right)^\gamma e^{-\theta\gamma x^\alpha}dx\right]$$
$$=\frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\,\theta^2}{(\beta\theta+1)}\right)^\gamma\int_0^\infty \beta^\gamma\left(1+\frac{x^\alpha}{\beta}\right)^\gamma x^{(\alpha-1)\gamma}e^{-\theta\gamma x^\alpha}dx\right]$$
$$=\frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\,\beta\,\theta^2}{(\beta\theta+1)}\right)^\gamma\int_0^\infty\sum_{j=0}^\infty\binom{\gamma}{j}\left(\frac{x^\alpha}{\beta}\right)^j x^{(\alpha-1)\gamma}e^{-\theta\gamma x^\alpha}dx\right]$$
$$=\frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\,\beta\,\theta^2}{(\beta\theta+1)}\right)^\gamma\sum_{j=0}^\infty\binom{\gamma}{j}\frac{1}{\beta^j}\int_0^\infty e^{-\theta\gamma x^\alpha}x^{\alpha(\gamma+j)-\gamma}dx\right]$$

Taking $y=x^\alpha$, which gives $x=(y)^{\frac{1}{\alpha}}$ and $dx=\frac{1}{\alpha}y^{\frac{1-\alpha}{\alpha}}dy$, we get

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\,\beta\,\theta^2}{(\beta\theta+1)}\right)^\gamma\sum_{j=0}^\infty\binom{\gamma}{j}\frac{1}{\beta^j}\int_0^\infty e^{-\theta\gamma y}y^{(\gamma+j)-\frac{\gamma}{\alpha}}\frac{y^{\frac{1-\alpha}{\alpha}}}{\alpha}dy\right]$$

$$= \frac{1}{1-\gamma} \log \left[\left(\frac{\alpha \beta \theta^2}{\alpha(\beta\theta+1)} \right)^{\gamma} \sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{1}{\beta^j} \int_0^{\infty} e^{-\theta\gamma y} y^{(\gamma+j)-\frac{(\gamma-1)}{\alpha}-1} dy \right]$$

$$= \frac{1}{1-\gamma} \log \left[\left(\frac{\alpha \beta \theta^2}{\alpha(\beta\theta+1)} \right)^{\gamma} (\theta\gamma)^{\frac{1}{\alpha}(\gamma-\alpha\gamma-1)} \sum_{j=0}^{\infty} \binom{\gamma}{j} \left(\frac{1}{\theta\beta\gamma} \right)^{j} \Gamma\left((\gamma+j) - \frac{(\gamma-1)}{\alpha} \right) \right].$$

5. Maximum Likelihood Estimation of Parameters

In this section the estimation of parameters of GTPLD using maximum likelihood estimation has been discussed. Assuming $(x_1, x_2, x_3, ..., x_n)$ a random sample from GTPLD(θ, α, β), the natural log-likelihood function, $\ln L$ of GTPLD can be expressed as

$$\ln L = n\left[\ln\alpha + 2\ln\theta - \ln(\beta\theta+1)\right] + \sum_{i=1}^{n} \ln\left(\beta + x_i^{\alpha}\right) + (\alpha-1)\sum_{i=1}^{n} \ln(x_i) - \theta \sum_{i=1}^{n} x_i^{\alpha} .$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ of parameters (θ, α, β) of GTPLD (2.1) is the solutions of the following natural log- likelihood equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} - \frac{n\beta}{\beta\theta+1} - \sum_{i=1}^{n} x_i^{\alpha} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha} \ln(x_i)}{\beta + x_i^{\alpha}} + \sum_{i=1}^{n} \ln(x_i) - \theta \sum_{i=1}^{n} x_i^{\alpha} \ln(x_i) = 0$$

$$\frac{\partial \ln L}{\partial \beta} = -\frac{n\theta}{\beta\theta+1} + \sum_{i=1}^{n} \frac{1}{\beta + x_i^{\alpha}} = 0$$

These three natural log- likelihood equations do not seem to be solved directly because they cannot be expressed in closed forms. However, the MLE's $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ of parameters (θ, α, β) can be obtained directly by solving the log likelihood equation using Newton-Raphson iteration method available in R –Software till sufficiently close estimates of $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$ are obtained.

6. Numerical Examples

The applications of GTPLD have been explained with two real lifetime datasets regarding failure times (in minutes) from Lawless (2003) pp. 204 and 263. The goodness of fit of GTPLD has been discussed along with the goodness of fit given by generalized Lindley distribution (GLD) introduced by Zakerzadeh and Dolati (2009), generalized gamma distribution introduced by Stacy (1962), two-parameter Lindley distribution of Shanker and Mishra (2013), generalized exponential distribution proposed by Gupta and Kundu (1999), power Lindley distribution introduced by Ghitany *et al* (2013), gamma distribution, Weibull distribution suggested by Weibull (1951), Lognormal distribution, Lindley distribution and exponential distribution. In table 1, the pdf and the cdf of fitted distributions has been presented. The goodness of fit of the fitted distribution has been presented in tables 2 and 3, respectively.

Dataset 1: The first set of data represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test and the data are

1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23.0, 30.6, 37.3, 46.3, 53.9, 59.8, and 66.2.

Dataset 2: The following data set represents the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level, Lawless (2003) pp. 204 and 263.

15 20 38 42 61 76 86 98 121 146 149 157 175 176

180 180 198 220 224 251 264 282 321 325 653

Table 1. The pdf and the cdf of the fitted distributions

Distributions	pdf and cdf	
GLD	pdf	$f(x;\theta,\alpha,\beta)=\frac{\theta^{\alpha+1}}{(\beta+\theta)}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha+\beta x)e^{-\theta x};x>0,\theta>0,\alpha>0,\beta>0$
	cdf	$F(x;\theta,\alpha,\beta)=1-\frac{\alpha(\beta+\theta)\Gamma(\alpha,\theta x)+\beta(\theta x)^{\alpha}e^{-\theta x}}{(\beta+\theta)\Gamma(\alpha+1)};x>0,\theta>0,\alpha>0,\beta>0$
GGD	pdf	$f(x;\theta,\alpha,\beta)=\frac{\beta\theta^{\alpha}}{\Gamma(\alpha)}x^{\beta\alpha-1}e^{-\theta x^{\beta}};\ x>0,\theta>0,\alpha>0,\beta>0$
	cdf	$F(x;\theta,\alpha,\beta,)=1-\frac{\Gamma\left(\alpha,\theta x^{\beta}\right)}{\Gamma(\alpha)};\ x>0,\theta>0,\alpha>0,\beta>0$
PLD	pdf	$f(x;\theta,\alpha)=\frac{\alpha\theta^{2}}{\theta+1}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}\ ;x>0,\theta>0,\alpha>0$
	cdf	$F(x;\theta,\alpha)=1-\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]e^{-\theta x^{\alpha}}\ ;x>0,\theta>0,\alpha>0$
GED	pdf	$f(x;\theta,\alpha)=\theta\alpha\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\theta x};x>0,\theta>0,\alpha>0$
	cdf	$F(x;\theta,\alpha)=\left(1-e^{-\theta x}\right)^{\alpha};x>0,\theta>0,\alpha>0$
Weibull	pdf	$f(x;\theta,\beta)=\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\ ;x>0,\theta>0,\beta>0$
	cdf	$F(x;\theta,\beta)=1-e^{-\theta x^{\beta}}\ ;x>0,\theta>0,\beta>0$
Gamma	pdf	$f(x;\theta,\alpha)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1};\ \ x>0,\theta>0,\alpha>0$
	cdf	$F(x;\theta,\alpha)=1-\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)};x>0,\theta>0,\alpha>0$
Lognormal	pdf	$f(x;\theta,\alpha)=\frac{1}{\sqrt{2\pi}\alpha x}e^{-\frac{1}{2}\left(\frac{\log x-\theta}{\alpha}\right)^{2}};x>0,\theta>0,\alpha>0$
	cdf	$F(x;\theta,\alpha)=\phi\left(\frac{\log x-\theta}{\alpha}\right);x>0,\theta>0,\alpha>0$

Table 2. ML estimates and summary of goodness of fit for dataset 1

Distributions	ML Estimates	Std Errors	$-2\log L$	K-S	p-value
GTPLD	$\hat{\theta}=0.02476$	0.03885	127.94	0.097	0.9962
	$\hat{\alpha}=1.20279$	0.33713			
	$\hat{\beta}=54.99303$	214.27357			
GLD	$\hat{\theta}=0.06415$	0.02132	128.16	0.095	0.9961
	$\hat{\alpha}=1.20258$	0.81310			
	$\hat{\beta}=0.08329$	0.27068			
GGD	$\hat{\theta}=0.008937$	0.014654	127.98	0.096	0.9960
	$\hat{\alpha}=0.912815$	0.417267			
	$\hat{\beta}=1.357973$	0.350075			
TPLD	$\hat{\theta}=0.062325$	0.017145	128.21	0.347	0.0397
	$\hat{\alpha}=6.343568$	12.14555			
GED	$\hat{\theta}=0.04529$	0.01372	128.47	0.108	0.9868
	$\hat{\alpha}=1.44347$	0.51301			
PLD	$\hat{\theta}=0.097651$	0.056373	128.47	0.098	0.9950
	$\hat{\alpha}=0.904312$	0.160673			
Gamma	$\hat{\theta}=0.05235$	0.02066	128.37	0.103	0.9920
	$\hat{\alpha}=1.44219$	0.47771			
Weibull	$\hat{\theta}=0.01190$	0.01124	128.04	0.098	0.9950
	$\hat{\alpha}=1.30586$	0.24925			
Lognormal	$\hat{\theta}=2.93059$	0.26472	131.23	0.312	0.045
	$\hat{\alpha}=1.02527$	0.18718			
Lindley	$\hat{\theta}=0.07022$	0.01283	128.81	0.110	0.9830
Exponential	$\hat{\theta}=0.03631$	0.00936	129.47	0.156	0.8061

Table 3. ML estimates and summary of goodness of fit for dataset 2

Distributions	ML Estimates	Std Errors	$-2\log L$	K-S	p-value
GTPLD	$\hat{\theta}=0.00853$	0.00751	304.89	0.128	0.8031
	$\hat{\alpha}=1.03725$	0.14432			
	$\hat{\beta}=19.02472$	61.32714			
GLD	$\hat{\theta}=0.01018$	0.00301	304.88	0.137	0.7370
	$\hat{\alpha}=0.81866$	0.48587			
	$\hat{\beta}=3.97404$	63.12878			
GGD	$\hat{\theta}=0.014831$	0.022530	304.92	0.139	0.718
	$\hat{\alpha}=1.989652$	0.865396			
	$\hat{\beta}=0.947350$	0.213317			
TPLD	$\hat{\theta}=0.010927$	0.001898	304.92	0.430	0.0001
	$\hat{\alpha}=5.100397$	21.23750			
GED	$\hat{\theta}=1.88641$	0.54466	304.98	0.178	0.2341
	$\hat{\alpha}=0.00917$	0.00176			
PLD	$\hat{\theta}=0.014157$	0.00190	304.93	0.137	0.728
	$\hat{\alpha}=0.955741$	0.00235			
Gamma	$\hat{\theta}=0.01008$	0.00294	304.87	0.138	0.721
	$\hat{\alpha}=1.79528$	0.45901			
Weibull	$\hat{\theta}=0.00256$	0.00068	306.57	0.931	0.0000
	$\hat{\alpha}=1.14807$	0.05897			
Lognormal	$\hat{\theta}=4.87956$	0.17468	308.16	0.178	0.2300
	$\hat{\alpha}=0.87433$	0.12351			
Lindley	$\hat{\theta}=0.01118$	0.00156	305.01	0.129	0.7980
Exponential	$\hat{\theta}=0.00565$	0.00109	309.18	0.202	0.2604

It is obvious from the goodness of fit of GTPLD that GTPLD is competing well with other lifetime distributions and hence can be considered an important lifetime distribution in statistics literature. The variance-covariance matrix of the estimated parameters of the GTPLD for datasets 1 and 2 has been given in tables 4 and 5, respectively.

Table 4. Variance-covariance matrix of GTPLD for dataset 1

$$\begin{array}{c} \\ \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\hat{\theta} \quad & \quad\hat{\alpha}\quad & \quad\hat{\beta}\quad \end{array} \\ \begin{bmatrix} 0.00176 & -0.01337 & -5.19676 \\ -0.01337 & 0.10684 & 36.88400 \\ -5.19676 & 36.88400 & 18207.39274 \end{bmatrix} \end{array}$$

Table 5. Variance-covariance matrix of GTPLD for dataset 2

$$\begin{array}{c} \\ \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \end{array} \begin{array}{c} \begin{array}{ccc} \hat{\theta} & \hat{\alpha} & \hat{\beta} \end{array} \\ \begin{bmatrix} 0.000056 & -0.001058 & -0.380728 \\ -0.001058 & 0.020828 & 6.632788 \\ -0.380728 & 6.632788 & 3761.01813 \end{bmatrix} \end{array}$$

7. Concluding Remarks

A generalization of two-parameter Lindley distribution (GTPLD), which includes one parameter exponential and Lindley distributions, two-parameter Lindley distribution (TPLD) of Shanker and Mishra (2013), Weibull distribution, gamma distribution, generalized gamma distribution of Stacy (1962) and power Lindley distribution of Ghitany *et al* (2013) as particular cases, has been proposed. Its raw moments, hazard rate function, mean residual life function, order statistic, Renyi entropy measure has been studied. Maximum likelihood estimation has been discussed for estimating its parameters. Applications of the distribution have been explained with two examples of observed real lifetime datasets from engineering and the goodness of fit is quite satisfactory over other one parameter, two-parameter and three-parameter lifetime distributions. .

ACKNOWLEDGEMENTS

Authors are grateful to the editor-in-chief and the anonymous reviewer whose comments improved the quality and the presentation of the paper.

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