

# Characterization of Benktander Type II Distribution via Truncated Moments and Order Statistics

N. M. Kilany<sup>1,\*</sup>, W. A. Hassanein<sup>2</sup>

<sup>1</sup>Faculty of Science, Menoufia University, Al Minufya, Egypt

<sup>2</sup>Faculty of Science, Tanta University, Tanta, Gharbia, Egypt

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**Abstract** Characterization of a specific probability distribution is an essential procedure to give the investigator familiarity about their model if it fits the requirements of the underlying probability distribution. A characterization results is uniquely specified the probability distribution. One of the important type of distributions, which used extensively in economic and actuarial science, is the size distributions. This paper is concerned with characterization of one of these distributions, which is Benktander type II distribution. The characterization procedure is studied via right, left truncated moments and truncated moments of order statistics. Finally, to give a validation practice that the obtained characterization results could precisely identify Benktander distribution, a simulation study for this procedure is presented.

**Keywords** Benktander type II distribution, Characterization of distributions, Truncated moments, Order statistics

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## 1. Introduction

Before a specific distribution model is applied to fit the real data, it is crucial to check whether the given distribution satisfies the requirements by its characterization. Recently, a developing concentration has been focused on characterization of probability distributions. Practically, the characterization results are applied widely in insurance, economic, quality control and queuing theory. It has attracted the attention of many researchers in many fields by different methods. For example, *Nanda* [10] studied the characterization of exponential distribution and Rayleigh distribution through failure rate and mean residual life functions. *Afify, et al.* [1] used the  $s^{\text{th}}$  conditional expectation of order statistics in terms of their failure rate to characterize the exponential and power function distributions. *Kilany* [8] introduced a new theorem for characterization of the continuous distribution based on truncated moments of order statistics and employ it to characterize Lindley distribution. Moreover, a characterization of Lindley distribution using the truncated moments of order statistics is presented. *Ahsanullah & Hamedani* [3] characterized beta of the first kind and power function distribution using order statistics. Many other papers in the literature dealing with characterization, see for instance, *Huang & Su* [7], *Su & Huang* [11] and *Ahsanullah et al.* ([2], [4], [5]).

This paper introduces the characterization of Benktander distribution type II by truncated moments and truncated moments of order statistics. The organization of the paper is; In Section 2, Benktander type II distribution and its main statistical properties are summarized. Section 3 includes the characterization of Benktander type II by left and right truncated moments. Moreover, in Section 4, the characterization through truncated moments of order statistics of Benktander type II is derived. Finally, a simulation study is given to enable the practitioner to identify the present distribution in Section 5.

## 2. Benktander Type II Distribution

One of the vital size distributions is called Benktander distribution. Starting from the observation that empirical mean excess functions point to distributions that are intermediate between Pareto and exponential distributions. In 1970, Gunnar Benktander discussed two types of distributions called Benktander Type I & Type II distributions (see *Kleiber & Kotz* [9]). These distributions used to model heavy tailed losses commonly found in non-life and causality actuarial science using

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\* Corresponding author:

neveenkilany@hotmail.com (N. M. Kilany)

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various forms of mean excess functions. The first type of Benktander is close to lognormal distribution and Benktander of the second type asymptotically resembles the mean excess function of Weibull distribution. Benktander type II (second type) is sometimes called Benktander Weibull distribution.

The main statistical functions concerning Benktander Type II distribution are defined as follows

- The probability density function (p.d.f) of Benktander Type II distribution is defined by

$$f(x) = e^{\frac{a(1-x^b)}{b}} x^{b-2} (ax^b - b + 1) \quad x \geq 1, a > 0, \quad 0 < b \leq 1 \quad (1)$$

- The cumulative distribution function (c.d.f)

$$F(x) = 1 - x^{b-1} e^{\frac{a(1-x^b)}{b}} \quad (2)$$

- The failure rate function

$$r(x) = \frac{ax^b - b + 1}{x} \quad (3)$$

- The reversed failure rate function

$$\psi(x) = \frac{e^{a/b} x^b (b-1-ax^b)}{e^{a/b} x^b - e^{\frac{ax^b}{b}} x} \quad (4)$$

### 3. Characterization of Benktander Type II Distribution by Truncated Moments

Henceforward, let  $X$  be an absolutely continuous random variable with c.d.f.  $F(x)$  and p.d.f  $f(x) = F'(x)$ . Assume  $\alpha = \inf \{x, s. t. F(x) > 0\}$  and  $\beta = \sup \{x, s. t. F(x) < 1\}$ . Define the failure rate function

$$r(x) = \frac{f(x)}{1 - F(x)}, \quad x \in (\alpha, \beta)$$

and the reversed failure rate function

$$\psi(x) = \frac{f(x)}{F(x)} \quad x \in (\alpha, \beta)$$

#### 3.1. Characterization of Benktander Type II by Left Truncated Moments

The following Lemma is needed to perform the characterization procedure by left truncated moments. See, *Ahsanullah et al.* [5].

##### Lemma 1.

Let  $g(x)$  be a continuous function and  $\alpha < E[g(X)] < \beta$ . If

$$E[g(X) | X \geq x] = h(x) r(x), \quad x \in (\alpha, \beta)$$

where  $h(x)$  is a differentiable function in  $x \in (\alpha, \beta)$ , then

$$f(x) = K \exp \left[ - \int_{\alpha}^x \frac{g(y) + h'(y)}{h(y)} dy \right] \quad x \in (\alpha, \beta)$$

where  $K > 0$  is a normalization constant.

##### Theorem 1.

A random variable,  $X$ , has Benktander Type II distribution with parameters  $a > 0$ ,  $0 < b \leq 1$  if and only if

$$E(X | X \geq x) = r(x) \cdot \frac{1 + ax^b}{ax^{b-2}(ax^b - b + 1)}; \quad x \geq 1, a > 0, 0 < b \leq 1$$

where,  $r(x)$  is the hazard function of Benktander Type II distribution.

##### Proof.

*Necessity:* Suppose  $X$  has Benktander Type II distribution with parameters  $a > 0$ ,  $0 < b \leq 1$ . Using Equations (1) and (3)

$$\int_x^{\infty} y f(y) dy = \frac{e^{\frac{a}{b}(1-x^b)}(1+ax^b)}{a}$$

$$\text{Hence, } E(X | X \geq x) = \frac{1}{1-F(x)} \int_x^{\infty} y f(y) dy = r(x) \cdot \frac{1+ax^b}{ax^{b-2}(ax^b-b+1)}$$

Sufficiency: Let  $g(x) = x$  and

$$h(x) = \frac{1+ax^b}{ax^{b-2}(ax^b-b+1)}$$

is a differentiable function in  $x \geq 1$ ,

$$h'(x) = \frac{x(-4a(-1+b) + (-2+b)[(-1+b)x^{-b} - a^2x^b])}{a(1-b+ax^b)^2}$$

After some simplifications,

$$\frac{g(x) + h'(x)}{h(x)} = \frac{2-2b+ax^b + \frac{b(b-1)}{b-ax^b-1}}{x}$$

Now for all  $x \geq 1$ ,

$$\int_1^x \frac{2-2b+ay^b + \frac{b(b-1)}{b-ay^b-1}}{y} dy = \frac{-a}{b} + \frac{ax^b}{b} + \ln(1+a-b) - (b-2)\ln(x) - \ln(1-b-ax^b)$$

By using Lemma 1, for all  $x \geq 1$ , we get

$$f(x) = K \exp \left[ - \int_1^x \frac{g(y) + h'(y)}{h(y)} dy \right] = K(b-a-1)e^{\frac{a(1-x^b)}{b}} x^{b-2}(ax^b-b+1)$$

where  $K > 0$  is the normalizing constant,  $K = \frac{1}{b-a-1}$ . Then, the proof is completed.

### 3.2. Characterization of Benktander Type II by Right Truncated Moments

The following Lemma is needed to obtain the characterization of Benktander Type II by right truncated moments. See, Ahsanullah *et al.* [5].

#### Lemma 2.

Let  $g(x)$  be a continuous function in  $(\alpha, \beta)$ , and  $\alpha < E(g(X)) < \beta$ .

If

$$E[g(X) | X \leq x] = c(x) \psi(x), \quad x \in (\alpha, \beta)$$

where,  $c(x)$  is a differentiable function in  $(\alpha, \beta)$  then

$$f(x) = K \exp \left[ - \int_{\alpha}^x \frac{c'(y) - g(y)}{c(y)} dy \right], \quad x \in (\alpha, \beta)$$

where,  $K > 0$  is a normalization constant.

#### Theorem 2.

A random variable,  $X$ , has Benktander Type II distribution with parameters  $a > 0$ ,  $0 < b \leq 1$  . if and only if

$$E(X | X \leq x) = \psi(x) \cdot \frac{e^{-\frac{a(1-x^b)}{b}} x^{2-b} \left( 1 + a - e^{\frac{a-ax^b}{b}} (1 + ax^b) \right)}{a(1-b+ax^b)} \quad x \geq 1, a > 0, 0 < b \leq 1$$

where  $\psi(x)$  is the Benktander Type II reversed hazard function.

#### Proof.

*Necessity.* Suppose  $X$  has a Benktander Type II distribution with parameters  $a > 0$ ,  $0 < b \leq 1$ . Then by using Equation (1) and (4)

$$\int_1^x y^n f(y) dy = \frac{1 + a - e^{\frac{a-ax^b}{b}}(1 + ax^b)}{a}$$

$$E(X/X \leq x) = \frac{1}{F(x)} \int_1^x y^n f(y) dy = \psi(x) \cdot \frac{e^{\frac{a(1-x^b)}{b}} x^{2-b} (1 + a - e^{\frac{a-ax^b}{b}}(1 + ax^b))}{a(1-b+ax^b)}$$

Sufficiency. Let  $g(x) = x$  and

$$c(x) = \frac{e^{\frac{a(1-x^b)}{b}} x^{2-b} (1 + a - e^{\frac{a-ax^b}{b}}(1 + ax^b))}{a(1-b+ax^b)}$$

is a differentiable function in  $x \geq 1$  and

$$c'(x) = \frac{e^{\frac{a}{b}x^{1-b}}((1+a)e^{\frac{ax^b}{b}}(2-3b+b^2-3a(-1+b)x^b+a^2x^{2b})+e^{a/b}(-b^2-2(1+ax^b)^2+b(1+ax^b)(3+ax^b)))}{a(1-b+ax^b)^2}$$

After some simplifications,

$$\frac{c'(x) - g(x)}{c(x)} = \frac{2-2b+ax^b + \frac{(b-1)b}{-1+b-ax^b}}{x}$$

For all  $x \geq 1$ ,

$$\int_1^x \frac{c'(y) - g(y)}{c(y)} dy = -\frac{a}{b} + \frac{ax^b}{b} + \ln(1+a-b) - (-2+b)\ln(x) - \ln(1-b+ax^b)$$

Then by using Lemma 2, the following result can be obtained

$$f(x) = K \exp\left[-\int_1^x \frac{c'(y) - g(y)}{c(y)} dy\right] = K(b-a-1)e^{\frac{a(1-x^b)}{b}} x^{b-2}(ax^b - b + 1)$$

where  $K > 0$  is a normalization constant,  $K = \frac{1}{b-a-1}$ . This complete the proof.

#### 4. Characterization of Benktander Type II Distribution by Truncated Moments of Order Statistics

Suppose  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from a continuous distribution with c.d.f.  $F(x)$ , p.d.f.  $f(x)$  and survival function  $S(x) = 1 - F(x)$ . Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the corresponding order statistics. Then the p.d.f. of  $X_{(r)}$  and the joint p.d.f. of  $X_{(r)}$  and  $X_{(r+1)}$  are given, respectively, as follows (Arnold *et al.* [6])

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \quad (5)$$

$$f_{X_{(r)}, X_{(r+1)}}(x, y) = \frac{n!}{(r-1)!(n-r-1)!} f(x)f(y)[F(x)]^{r-1}[1-F(x)]^{n-r-1} \quad (6)$$

From Equations (5) and (6) the conditional p.d.f. of  $X_{(r+1)}$  given  $X_{(r)} = x$  is given by

$$f_{X_{(r)}, X_{(r+1)}}(y|x) = \frac{f_{X_{(r)}, X_{(r+1)}}(x, y)}{f_{X_{(r)}}(x)} = (n-r) \frac{[1-F(y)]^{n-r-1}}{[1-F(x)]^{n-r}} f(y) \quad (7)$$

In this Section, Benktander type II distribution is to be characterized through truncated moments of order statistics given by

$$E(X_{(r+1)}^s | X_{(r)} = x) = \int_x^\infty y^s f_{X_{(r+1)}|X_{(r)}}(y|x) dy, s = 1, 2, 3, \dots, r = 1, 2, \dots, n-1. \quad (8)$$

##### 4.1. Characterization Results

The following theorem is presented for characterization of Benktander type II distribution based on the  $r^{\text{th}}$ -truncated moments of the order statistics.

**Theorem 3.**

Suppose  $X$  be a continuous random variable with distribution function  $F(x)$  and survival function  $S(x) = 1 - F(x)$ . Let  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  denote the order statistics of a random sample of size  $n$  from  $F(x)$ . Then  $X$  has the Benktander type II distribution with parameters  $a > 0, 0 < b \leq 1$  if and only if

$$E(X_{(r+1)}|X_{(r)} = x) = \frac{\left(\frac{b}{a(n-r)}\right)^{(n-r)-\frac{1}{b}(n-r-1)} e^{\frac{a(n-r)}{b}}}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^{(n-r)}} \left\{ \frac{(1-b)(n-r)}{b} \Gamma\left[(n-r) - \frac{1}{b}(n-r-1), \frac{a(n-r)}{b} x^b\right] + \Gamma\left[(n-r+1) - \frac{1}{b}(n-r-1), \frac{a(n-r)}{b} x^b\right] \right\} \quad (9)$$

**Proof.**

*Necessity.* Suppose  $X$  has Benktander Type II distribution with parameters  $a > 0, 0 < b \leq 1$ . Using Equations (2), (7) and (8), the following results are obtained

$$\begin{aligned} E(X_{(r+1)}|X_{(r)} = x) &= \frac{(n-r)}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^{n-r}} \int_x^\infty y^{b-1} \left(e^{\frac{a(1-y^b)}{b}} y^{b-1}\right)^{n-r-1} e^{\frac{a(1-y^b)}{b}} (ay^b - b + 1) dy \\ &= \frac{(n-r)}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^{n-r}} \int_x^\infty \left(e^{\frac{a(1-y^b)}{b}} y^{b-1}\right)^{n-r} (ay^b - b + 1) dy \\ &= \frac{(n-r)}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^{n-r}} \{I_1 + I_2\} \end{aligned} \quad (10)$$

where,

$$\begin{aligned} I_1 &= (1-b)e^{\frac{a(n-r)}{b}} \int_x^\infty y^{(b-1)(n-r)} e^{-\frac{a(n-r)}{b} y^b} dy, \\ I_2 &= ae^{\frac{a(n-r)}{b}} \int_x^\infty y^{(b-1)(n-r)} y^b e^{-\frac{a(n-r)}{b} y^b} dy. \end{aligned}$$

Substituting in integrations  $I_1$  and  $I_2$  by  $u = \frac{a(n-r)}{b} y^b$  and using the definition of incomplete gamma function,  $\Gamma[k, z] = \int_z^\infty t^{k-1} e^{-t} dt$ ,

$$\begin{aligned} I_1 &= \frac{(1-b)e^{\frac{a(n-r)}{b}}}{a(n-r)} \left(\frac{b}{a(n-r)}\right)^{(1-\frac{1}{b})(n-r-1)} \int_{\frac{a(n-r)}{b} x^b}^\infty u^{(1-\frac{1}{b})(n-r-1)} e^{-u} du \\ &= \frac{(1-b)e^{\frac{a(n-r)}{b}}}{a(n-r)} \left(\frac{b}{a(n-r)}\right)^{(1-\frac{1}{b})(n-r-1)} \Gamma\left[(n-r) - \frac{1}{b}(n-r-1), \frac{a(n-r)}{b} x^b\right] \end{aligned} \quad (11)$$

and

$$\begin{aligned} I_2 &= \frac{e^{\frac{a(n-r)}{b}}}{(n-r)} \left(\frac{b}{a(n-r)}\right)^{(n-r)-\frac{1}{b}(n-r-1)} \int_{\frac{a(n-r)}{b} x^b}^\infty u^{(n-r)-\frac{1}{b}(n-r-1)} e^{-u} du \\ &= \frac{e^{\frac{a(n-r)}{b}}}{(n-r)} \left(\frac{b}{a(n-r)}\right)^{(n-r)-\frac{1}{b}(n-r-1)} \Gamma\left[(n-r+1) - \frac{1}{b}(n-r-1), \frac{a(n-r)}{b} x^b\right] \end{aligned} \quad (12)$$

Substituting from Equations (11) and (12) into Equation (10), the result is

$$\begin{aligned} E(X_{(r+1)}|X_{(r)} = x) &= \frac{\left(\frac{b}{a(n-r)}\right)^{(n-r)-\frac{1}{b}(n-r-1)} e^{\frac{a(n-r)}{b}}}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^{(n-r)}} \\ &\quad \left\{ \frac{(1-b)(n-r)}{b} \Gamma\left[(n-r) - \frac{1}{b}(n-r-1), \frac{a(n-r)}{b} x^b\right] + \Gamma\left[(n-r+1) - \frac{1}{b}(n-r-1), \frac{a(n-r)}{b} x^b\right] \right\} \end{aligned}$$

*Sufficiency:* Let  $w(x)$  be the right hand side of Equation (9), i.e.

$$w(x) = \frac{\left(\frac{b}{ap}\right)^{p-\frac{1}{b}(p-1)} e^{\frac{ap}{b}}}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^p} \left\{ \frac{(1-b)p}{b} \Gamma\left[p - \frac{1}{b}(p-1), \frac{ap}{b} x^b\right] + \Gamma\left[(p+1) - \frac{1}{b}(p-1), \frac{ap}{b} x^b\right] \right\},$$

where  $= n - r$ . Then, Equation (9) can be rewritten as follows

$$\int_x^\infty pyf(y)[1 - F(x)]^{p-1} dy = [1 - F(x)]^p w(x) \quad (13)$$

Differentiating both sides of Equation (13) with respect to  $x$ , then

$$-pxf(x)[S(x)]^{p-1} = [S(x)]^p w'(x) - pf(x)[S(x)]^{p-1} w(x) \quad (14)$$

where  $S(x) = 1 - F(x)$  is the survival function.

Dividing both sides of Equation (14) by  $[S(x)]^{p-1}$ , then

$$pf(x)[w(x) - x] = S(x)w'(x)$$

Or equivalently,

$$\frac{f(x)}{S(x)} = \frac{w'(x)}{p[w(x) - x]} \quad (15)$$

From the fact that  $\Gamma[k+1, z] = k\Gamma[k, z] + z^k e^{-z}$ , then  $w(x)$  and  $w'(x)$  will be

$$w(x) = \frac{\left(\frac{b}{ap}\right)^{p-\frac{1}{b}(p-1)} \left(\frac{1}{b}\right) e^{\frac{ap}{b}}}{\left(e^{\frac{a(1-x^b)}{b}} x^{b-1}\right)^p} \Gamma\left[p - \frac{1}{b}(p-1), \frac{ap}{b} x^b\right] + x$$

and

$$w'(x) = - \frac{p \left(\frac{b}{ap}\right)^p e^{\frac{ap}{b}} (-1 + b - ax^b) \Gamma\left[p - \frac{1}{b}(p-1), \frac{ap}{b} x^b\right]}{bx \left(e^{\frac{a-ax^b}{b}} x^{b-1}\right)^p}$$

Hence, Equation (15) becomes

$$\frac{f(x)}{S(x)} = \frac{ax^b - b + 1}{x}$$

Now, for  $x \geq 1$ , integration both sides of the above Equation with respect to  $x$  gives

$$\ln S(x) = - \int_1^x \frac{ax^b - b + 1}{x} dx \frac{a(1-x^b)}{b} - (b-1) \ln x$$

Hence,

$$S(x) = x^{b-1} e^{\frac{a(1-x^b)}{b}}, \quad x \geq 1, a > 0, 0 < b \leq 1$$

which is the survival function of the Benktander type II distribution. This completes proof of the theorem.

## 4.2. Simulation Study

The purpose of the simulation study is to give a validation procedure for simple and easily checks using the simulated data. This enables the experimenter to test whether the available data follows a specified distribution or not, instead of going to sophisticated analysis of hypothesis test.

To validate the correctness of the mathematical results obtained for characterization of Benktander type II, a simulation study is showed by generating samples  $X_i, i = 1, 2, \dots, mn$ . They are randomly classified into  $m$  samples, each of containing  $n$  observations. To satisfy more accurate consequence with less variability and more stability in characterization results, the study is conducted with large sample size, for example;  $n = 100$ . Choosing the number of samples are  $m = 10, 15, 20, 50$  and different values of the order  $r$  with various choices of the parameter estimates  $a$  and  $b$ . Table (1) shows the L.H.S and R.H.S of Equation (9) and the absolute relative difference between them. The results indicate the closeness of both sides of Equation (9) which supporting the precision of the characterization results of Benktander distribution.

**Table 1.** Verification of the characterization results through truncated order statistics

$n$	$m$	$r$	$a$	$b$	$x$	L.H.S	R.H.S	$\left  \frac{\text{L. H. S} - \text{R. H. S}}{\text{R. H. S}} \right $
100	50	80	0.01	0.1	7.0534	7.4922	7.4624	0.00398
			0.02	0.2	5.39806	5.6965	5.74481	0.00848
			1.00	0.5	2.3008	2.35873	2.28336	0.03195
			2.00	0.7	1.9177	1.95948	1.9456	0.00708
			5.00	0.8	1.43024	1.44857	1.44069	0.00539
			0.50	0.1	1.22159	1.23205	1.26644	0.0279
			1.50	0.6	1.34548	1.35629	1.37652	0.0149
			0.10	0.4	2.00782	2.01149	2.15414	0.0709
			10	0.9	1.171504	1.18045	1.18008	0.000313
			50	0.01	1.04161	1.04263	1.04314	0.0000489
100	20	30	0.01	0.1	1.97365	2.0051	1.97548	0.01477
			0.02	0.2	1.90197	1.93558	1.95364	0.00933
			1.00	0.5	1.25771	1.26885	1.2669	0.00154
			2.00	0.7	1.52158	1.52888	1.54464	0.0103
			5.00	0.8	1.1139	1.11672	1.12711	0.0093
			0.50	0.1	2.15392	2.1755	2.18731	0.00543
			1.50	0.6	1.16718	1.17536	1.17701	0.00145
			0.10	0.4	1.55607	1.58756	1.57918	0.00528
			10	0.9	1.01716	1.01858	1.01797	0.000599
			50	0.01	1.01001	1.01029	1.01046	0.000168
100	10	50	0.01	0.1	3.06	3.12867	3.1536	0.00797
			0.02	0.2	2.6433	2.70902	2.75563	0.0172
			1.00	0.5	1.31979	1.33593	1.33334	0.00194
			2.00	0.7	2.98908	3.00208	3.30199	0.0999
			5.00	0.8	1.35458	1.36931	1.36518	0.00302
			0.50	0.1	1.47092	1.49193	1.48788	0.002715
			1.50	0.6	1.19623	1.20784	1.20403	0.21099
			0.10	0.4	4.23015	4.34148	4.36353	0.00508
			10	0.9	1.01914	1.02112	1.01028	0.0106
			50	0.01	1.02678	1.02718	1.02742	0.000234
100	15	20	0.01	0.1	1.24398	1.26764	1.2613	0.0050014
			0.02	0.2	1.48117	1.50413	1.50405	0.005319
			1.00	0.5	1.63294	1.6557	1.64447	0.00678
			2.00	0.7	1.02316	1.02884	1.02866	0.000175
			5.00	0.8	1.07569	1.07911	1.07814	0.000899
			0.50	0.1	1.01291	1.01913	1.02203	0.00285
			1.50	0.6	1.0384	2.37973	2.35837	0.00898
			0.10	0.4	2.31858	2.37973	2.35837	0.00897
			10	0.9	1.0375	1.03917	1.03874	0.000414
			50	0.01	1.02005	1.03282	1.0203	0.012122

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## REFERENCES

- [1] Afify, A., Nofal, Z. and Ahmed, A. (2013). "Characterization of Exponential and Power Function Distributions Using  $S^{\text{th}}$  Truncated Moments of Order Statistics". *Journal of Advances in Mathematics*, 4(3), 486-494.
- [2] Ahsanullah, M. and Aliev, F. (2008). "Some Characterizations of Exponential Distribution by Record Values". *Journal of Statistical Research*, 42(2), 41-46.
- [3] Ahsanullah, M. and Hamedani, G.G. (2007). "Certain Characterizations of Power Function and Beta Distributions Based on Order Statistics", *Journal of Statistical Theory and Applications*, 6, 220-226.
- [4] Ahsanullah, M. and Shakil, M. (2013). "Characterizations of Rayleigh Distribution Based on Order Statistics and Record Values". *Bulletin of the Malaysian Mathematical Sciences Society*, 36(3), 625-635.
- [5] Ahsanullah, M. Shakil, M. and Kibra, B.M. (2016) "Characterizations of Continuous Distributions by Truncated Moments". *Journal of Modern Applied Statistical Methods*, 15(1), 316-331.
- [6] Arnold, B.C., Balakrishnan, N. and Nagaraja, H. N. (1992). "A First Course in Order Statistics". John Wiley & Sons, New York.
- [7] Huang, W.J. and Su, N.C. (2012). "Characterizations of Distributions Based on Moments of Residual Life". *Communications in Statistics -Theory and Methods*, 41(15), 2750-2761.
- [8] Kilany, N. M. (2017). "Characterization of Lindley Distribution Based on Truncated Moments of Order Statistics. *Journal of Statistics Applications and Probability*, 6 (2), 1-6.
- [9] Kleiber, C. and Kotz, S. (2003), "Statistical Size Distributions in Economics and Actuarial Sciences". Wiley & Sons, Inc., Canada.
- [10] Nanda, A.K. (2010). "Characterization of Distributions through Failure Rate and Mean Residual Life Functions". *Statistics & Probability Letters*, 80(9-10), 752-755.
- [11] Su, N.C. and Huang, W.J. (2000). "Characterizations Based on Conditional Expectations". *Statist. Papers*, 41(4), 423-435.